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USING GAP SEQUENCES TO ESTIMATE GAP ACCEPTANCE FUNCTIONS

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Abstract—Traditional gap acceptance functions have been estimated based on the first gap observed. In this paper we show that the critical gap of drivers is decreasing on the average, as they are waiting for an acceptable gap. Our gap acceptance function is based on a probit model which assumes a normal distribution of gaps across gaps and drivers.

The concept of gap acceptance is used in modelling drivers' behavior when crossing (or merging with) a traffic stream of higher priority. Such models are used to study delays and capacity at unsignalized intersections, pedestrian crossings, freeway merging and lane changing maneuvers. To model such situations it is assumed that each driver (or pedestrian) has some "critical gap." A driver would accept (i.e. proceed with his intended maneuver) a gap in the traffic stream if its duration is longer than his critical gap.

Clearly, the critical gap for a certain maneuver would vary across drivers (and gaps), and is therefore modelled as a random variable. Several probability density functions have been used to described the distribution of the critical gap. Drew *et al.* (1967), Cohen *et al.* (1955) and Solberg *et al.* (1966) used the lognormal distribution; Miller (1972) and Daganzo (1979) used the normal distribution; Blunden *et al.* (1962) used the gamma distribution; and Herman and Weiss (1961) used the exponential distribution.

Given a distribution of critical gaps in the population, one can define gap acceptance functions. Such functions relate the probability that a randomly chosen driver would accept a certain gap to the characteristics of this gap. Naturally the most important characteristic of the gap is its length (duration). However, in this paper we estimate additional gap acceptance parameters related to the whole string of gaps considered by a driver up to the accepted one. We therefore explicitly address the functional form and the estimation of model parameters for the situation recognized by Weiss and Maradudin (1962) as that of the "impatient driver" for whom the critical gap decreases with each passing gap.

As noted by Miller (1972) most estimators of gap acceptance suffer from either bias or statistical inefficiency. The bias is due to the over-representation of cautious drivers who reject many gaps before final acceptance. The standard solution for this problem is to consider only the first gap, a procedure which is bound to introduce inefficiency since it involves discarding data. Miller suggested a normal distribution based maximum likelihood estimator using all the observed gaps. This was the basis for the estimation technique used in this paper. However, as noted by Miller, his estimates did not account for the dependence of critical gaps on previously rejected gaps as well as for the duration of each gap. (This phenomenon can be recognized by observing drivers who reject a gap which is longer than the one eventually accepted).

Recently, Daganzo (1979) extended Miller's method by using a multinomial probit model to estimate the parameters of the multivariate normal distribution of critical gaps. This multivariate approach accounted for the variation among gaps for a given driver as well as variation among drivers. Unfortunately, estimability problems prevented Daganzo from actually distinguishing between the two above-mentioned components of stochastic variation, even though they were assumed independent.

In this paper, we propose a model of gap acceptance behavior that captures variations in individual driver behavior among gaps and across drivers. It is a relatively straightforward extension of Miller's suggestions as the stochastic variation in the critical gap is not explicitly decomposed into its "within" drivers and "across" drivers† components. However, we try to explain some of the variability between gaps (i.e. "within drivers") by modeling it in the systematic component of our model, i.e. by explicitly specifying the dependence of the mean critical gap on the number of rejected gaps.

In the following sections we discuss the model and its specifications, compare the estimation results for several models and conclude with a discussion of applications and further research. Throughout the paper we refer to drivers at unsignalized intersections but the content is applicable to other gap acceptance situations as well.

THE MODEL

In our model, the gap acceptance process is viewed as a series of binary decisions. The first gap that is longer than the critical one is accepted. Our hypothesis is that the length of the critical gap, for a given randomly chosen driver, is not constant but rather a function of the number of gaps rejected up to the gap under consideration. Mathematically the model can be specified as follows:

$$T_{cr}(i) = \bar{T}_{cr} + f(i-1) + \epsilon_i, \tag{1}$$

where $T_{cr}(i)$ is the critical gap of a driver randomly chosen from the population when facing the i-th gap in a sequence; $\bar{T}_{cr}$ is the mean critical gap when facing the first gap, i.e. for $i = 1$; $f(i-1)$ is a function of the number of rejected gaps up to gap i (by definition $f(0) \equiv 0$); and ϵ_i is a disturbance term which varies across drivers and gaps (by assumption, $\epsilon_i \sim N(0, \sigma^2)$ and the gaps' disturbance terms are assumed independent). In this study we assume that the form of $f(\cdot)$ is $f(i-1) = \beta(i-1)^\delta$, where β and δ are parameters to be estimated. Note that this formulation cannot distinguish between the distribution across drivers and across gaps. The unit of observation is the i-th gap of the j-th driver and all the normal variates are assumed independent and identically distributed.

The gap acceptance function is given by the probability that a certain driver would accept a given gap. The gaps are characterized by their sequential number, i, and their duration, t_i. Thus the gap acceptance function is given by the probit function:

$$\begin{aligned} Pr(\text{accept gap } i \mid \bar{T}_{cr}, \beta, \delta, i, t_i) &= Pr[t_i \geq T_{cr}(i)] \\ &= Pr(t_i \geq \bar{T}_{cr} + \beta(i-1)^\delta + \epsilon_i) \\ &= \Phi\left(\frac{t_i - \bar{T}_{cr} - \beta(i-1)^\delta}{\sigma}\right) \end{aligned} \tag{2}$$

where $\Phi(\cdot)$ denotes the standard cumulative normal curve. In order to construct a proper likelihood function for this model let the superscript j refer to an observation (a driver) from a simple random sample of size N. Let k_j denote the sequential number of the gap accepted by driver j; $j = 1, 2, \ldots, N$. The probability that the j-th driver would accept the k-th gap, p_k^j, is given by:

$$P_k^j = Pr[t_k^j \geq T_{cr}^j(k)] \cdot \prod_{i=1}^{k-1} Pr[t_i^j < T_{cr}^j(i)], \tag{3}$$

since accepting the k-th gap means that all preceding ones were rejected. The likelihood function of this model for a given sample is:

$$L(\bar{T}_{cr}, \beta, \delta, \sigma \mid t_\cdot, k_\cdot) = \prod_{j=1}^{N} P_{k_j}^j$$

† Empirical work by Bottom and Ashworth (1978) (see also Ashworth and Bottom (1977)) suggested that over 85% of the variance in gap acceptance behavior may be due to variations among gaps for given drivers ("within drivers" variability).

and using eqn (2), the log-likelihood function is:

$$\log L(\cdot) = \sum_{j=1}^{N} \left[\log \Phi\left(\frac{t_{k_j}^j - T_{cr}(k_j)}{\sigma}\right) + \sum_{i=1}^{k_j - 1} \log \Phi\left(\frac{T_{cr}(i) - t_i^j}{\sigma}\right)\right]. \quad (4)$$

This model has been estimated using CHOMP (Choice Modelling Program), the probit estimation program developed by Daganzo and Schoenfeld (1978). However, the functional form of the abovementioned dependence of the critical gap on the gap's sequential number, i.e. $f(i-1) = \beta(i-1)^{\delta}$, is not the only one used in this study. A second specification was tried by constraining $\delta = 1$, imposing linearity of T_{cr} in $(i-1)$. This constraint has been motivated by the results of the first model where δ was not significantly different from 1, as we show in the next section. A third model assumes that drivers' critical gap is influenced by the length of the waiting time rather than the number of gaps up to the one accepted. This assumption is consistent with the empirical findings of Bottom and Ashworth (1978) who found that the proportion of gaps accepted increased with the waiting time. In this case

$$f(i-1) = \beta \cdot \sum_{l=1}^{i-1} t_l$$

where t_l is the duration of the l-th gap. In addition, in order to compare this model to the one suggested by Miller (1972), his model was estimated as well using the same data set. Miller's model can be specified, in our notation, as $f(i-1) \equiv 0$, in other words $T_{cr} = \bar{T}_{cr} + \epsilon$. The estimation results are presented in the next section.

MODEL ESTIMATION

The data set used for the estimation process was the same one used and reported by Daganzo (1979). The data includes 203 observations (drivers) collected from roadside observations at various intersections in Berkeley, including all rejected gaps and the accepted one for each observation (406 records were thus used for the estimation).

Table 1 shows the estimation results for Miller's model, where the mean critical gap is assumed independent of the series of rejected gaps. The mean critical gap is estimated to be 6.3 sec and the variance of the critical gap across the population is estimated to be 7.6 sec^2. Table 2 depicts the estimates for the first model mentioned in the preceding section, i.e. $T_{cr}(i) = \bar{T}_{cr} + \beta(i-1)^{\delta} + \epsilon_i$. In this case the mean critical time is estimated to be $7.3 - 1.4(i-1)^{0.7}$ seconds with a variance of 5.2. Note that the sign of β is negative — as expected, the critical gap decreases as more gaps are passing by.† However the assumption that the power coefficient, δ, equals unity cannot be rejected at any reasonable confidence level (the value of the t-statistic for this test is 0.9). This leads us naturally to estimate a third model with the restriction $\delta = 1$.

The estimates for the restricted model are depicted in Table 3. Here $E[\hat{T}_{cr}(i)] = 7.2 - 0.94(i-1)$ seconds and $\hat{\sigma}^2 = 5.4$ sec^2. Note that even though this model involves fewer parameters than the previous one, the likelihood ratio index, which is an index of goodness of fit, is virtually unchanged. This comparison is particularly revealing when the last model is compared with Miller's model. The definite improvement in the goodness of fit between the last (3 parameter) model and Miller's model seems to suggest that the critical gaps do decrease as drivers are waiting. However a comparison with the four-parameter model (Table 2) seems to suggest that the reduction in the critical gap is approximately linear (over the estimated range) with the number of gaps already rejected. It should be noted though that many drivers in the sample did not wait more than two gaps. Thus all observations where the accepted gap was either the first or the second did not contribute to the estimate of δ which is thus based on a relatively small sample.

† This result cannot be extrapolated beyond the range observed in the data (the expected critical time becomes negative for long sequences of gaps). This, however, should not present a problem in practical considerations in which the number of rejected gaps may be no more than 3 or 4. A formula applicable for a wider range of gap sequences may be estimated with a suitable data set.

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Table 1. Parameter estimates for Miller's model

Model: $T_{cr} \sim N\{\bar{T}_{cr}, \sigma^2\}$		
Parameter	Estimate	Standard Error of Estimate
$\bar{T}_{cr}$	6.3	0.30
σ^2	7.6	1.35
Log likelihood at zero = −288		
Log likelihood at convergence = −103		
Likelihood Ratio Index† = 0.64		

† The log likelihood at zero is the value of the log likelihood function for the model under consideration when both choices are equally likely (in this case, when $\bar{T}_{cr} \equiv 0$); the log likelihood at convergence is the value of this function when the arguments are the maximum likelihood estimators (in this case, the $\hat{T} = 6.3$ and $\hat{\sigma}^2 = 7.6$); the likelihood ratio index is the absolute difference between these quantities divided by the likelihood at zero.

The parameter estimates for the last model specified are given in Table 4. In this model the critical gap is specified as a function of the delay up to the accepted gap. As evident from Table 4, this model does not fit the data as well as the model depicted in Table 3.

Note that all three specifications support our basic hypothesis that the critical gap is decreasing as drivers wait for an acceptable gap. In other words the coefficient β is always negative and the hypothesis that it is zero can be rejected at any reasonable confidence level.

Table 2. Parameter estimates for model with non linear influence of the number of rejected gaps

Model: $T_{cr}(i) \sim N\{[\bar{T}_{cr} + \beta(i-1)^{\delta}], \sigma^2\}$		
Parameter	Estimate	Standard Error of Estimate
$\bar{T}_{cr}$	7.3	0.35
β	−1.4	1.18
δ	0.7	0.84
σ^2	5.2	2.28
Log likelihood at zero = −288		
Log likelihood at convergence = −89.6		
Likelihood Ratio Index = 0.69		

Clearly, the model that best fits the data is the one depicted in Table 3, where

$$T_{cr}(i) = \bar{T}_{cr} + \beta(i-1) + \epsilon$$

and

$$\epsilon_i \sim N(0, \sigma^2).$$

It is interesting to compare this model to the one suggested by Miller, the estimation results of which are depicted in Table 1. Since the specification for Miller's model is:

$$T_{cr} = \bar{T}_{cr} + \epsilon$$

and

$$\epsilon \sim N(0, \sigma^2),$$

the latter model is a linear restriction on our model ($\beta = 0$) and since in both cases we used maximum likelihood to estimate the models, the likelihood ratio test applies. From Tables 1 and 3, $-2(\log L_m - \log L) = 27.6$ (where L_m refers to the value of the likelihood

Table 3. Parameter estimates for model with linear influence of the number of rejected gaps

Model: $T_{cr}(i) \sim N\{[\bar{T}_{cr} + \beta(i-1)], \sigma^2\}$

Parameter	Estimate	Standard Error of Estimate
$\bar{T}_{cr}$	7.2	0.32
β	−0.94	0.17
σ^2	5.4	1.0

Log likelihood at zero = −288
Log likelihood at convergence = −89.8
Likelihood Ratio Index = 0.69

function for Miller's model and L refers to the value of the likelihood function for our model). This value is substantially higher than the boundary of the rejection region which, for a confidence level of 0.005 is $\chi^2_{0.005,1} = 7.9$, indicating that the hypothesis $\beta = 0$ implied by Miller's model can be rejected. Another way of reaching the same conclusion is to note that the t-statistic for the parameter β in our model (for testing the hypothesis $\beta = 0$) is -5.47, a value substantially larger than $t_{0.005} = 2.58$, again, indicating a rejection of the hypothesis that $\beta = 0$.

Table 4. Parameter estimates for model with delay dependent critical gap

Model: $T_{cr}(i) \sim N\left\{\left[\bar{T}_{cr} + \beta \sum_{l=1}^{i-1} t_l\right], \sigma^2\right\}$

Parameter	Estimate	Standard Error of Estimate
$\bar{T}_{cr}$	6.9	0.32
β	−0.2	0.05
σ^2	6.1	1.14

Log likelihood at zero = −288
Log likelihood at convergence = −96.9
Likelihood Ratio Index = 0.66

In comparing the estimates for our model with the estimates of Miller's model, it is also interesting to note that the estimated variance of the distribution of the critical gaps in our model is smaller ($\hat{\sigma}^2 = 5.4$ vs $\hat{\sigma}^2 = 7.6$ for Miller's model). This seems to imply that the introduction of the dependence on the rejected gaps results in an estimate of a tighter distribution of the critical gap. In other words the dispersion in the distribution of the critical gap in Miller's model may be attributed to the omission of the dependence on the number of rejected gaps.

CLOSURE

One of the main uses of gap acceptance functions is in developing expressions for delays in crossing or merging situations. Using the gap acceptance model estimated in this paper, the delay problem may be solved (at least numerically) within the framework laid out by Weiss and Maradudin (1962) (see also Blumenfeld and Weiss, 1970). However, the difficulty of obtaining simple closed form analytical delay expressions should not detract from the applicability of our model, given the increased use of micro level traffic simulation models (e.g. NETSIM—see Peat, Marwick and Mitchell, 1973).

The integration of our gap acceptance function in existing traffic simulation models is straightforward and in light of the results of this paper, seems warranted.

The probit-based model presented in this paper verified our hypothesis that the mean duration of the critical gap is a decreasing function of the number of rejected gaps. Our model can be viewed as a generalization of Miller's (1972) model or as a particular case of a general multinomial probit model. Note however, that unlike the case with Daganzo's model ours is sequential in nature and therefore the number of gaps considered does not have to be defined *a priori* (Daganzo restricted his model to seven gaps). Furthermore our approach is more amenable to integration in a traffic simulation model for the same reason.

Further empirical research would be needed in order to establish values or a range of values applicable for our model in various situations. Such values can be used as a default option (or just as input) in the aforementioned traffic simulation programs.

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