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DETERMINISTIC IMPULSE CONTROL PROBLEMS*

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Abstract. We prove that the optimal cost function of a deterministic impulse control problem is the unique viscosity solution of a first-order Hamilton-Jacobi quasi-variational inequality in $\mathbb{R}^N$.

Key words. deterministic impulse control, dynamic programming principle, viscosity solution, first-order Hamilton-Jacobi equations, quasi-variational inequality

Introduction. Impulse control problems lead, via the dynamic programming principle, to the study of various kinds of quasi-variational inequalities (see for more details and for many examples A. Bensoussan and J. L. Lions [3]).

In this work, we are interested in deterministic impulse control in $\mathbb{R}^N$. More precisely, our main result states that the optimal cost function of a deterministic impulse control problem is the unique viscosity solution of a first-order Hamilton-Jacobi quasi-variational inequality in $\mathbb{R}^N$ of a particular form:

$$\text{(P)} \qquad \max\,(H(x, u, Du), u - Mu) = 0 \quad \text{in } \mathbb{R}^N,$$

where

$$H(x, t, p) = \sup_{v \in V} (b(x, v) \cdot p + \lambda t - f(x, v)),$$

$$M\varphi(x) = \inf_{\xi \in (\mathbb{R}^+)^N} (\varphi(x+\xi) + c(\xi));$$

V is a separable metric space, b and f are functions from $\mathbb{R}^N \times V$ into $\mathbb{R}^N$, c is a continuous positive function, $\lambda > 0$ (precise assumptions are detailed in § 2). Let us just mention that the state of the controlled system is given by the solution $y_x(t)$ of the following problem:

$$\frac{dy_x(t)}{dt} + b(y_x(t), v(t)) = 0, \qquad t \in]\theta_i, \theta_{i+1}[,$$

$$y_x(0) = x,$$

$$y_x(\theta_i + 0) = y_x(\theta_i - 0) + \xi_i,$$

where $\theta = (\theta_i)_{i \in \mathbb{N}}$ is a nondecreasing sequence of positive reals which satisfies: $\theta_n \to +\infty$ when $n \to +\infty$ and $\xi = (\xi_i)_{i \in N}$ is a sequence of elements of $(\mathbb{R}^+)^N$.

Finally, $v(t)$ is any measurable function which states its values in a compact subset of V. Finally, $K = (\theta, \xi, v(\cdot))$ is the *control*. The optimal cost function u is given by:

$$u(x) = \inf_K \left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \sum_{i \in \mathbb{N}} c(\xi_i) c^{-\lambda \theta_i} \right).$$

We first recall below the definition and main properties of the notion of viscosity solutions for problems like (P). These results are easy extensions of those obtained by M. G. Crandall and P. L. Lions [6], M. G. Crandall, L. C. Evans and P. L. Lions [5] and P. L. Lions [10] for first-order Hamilton-Jacobi equations. Let us just mention

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that the form of M creates some difficulties for stability results (see for more details G. Barles [1]).

In § 2, we introduce the deterministic impulse control problem and we prove various results concerning the optimal cost function u which are similar to those obtained in the classical deterministic case (see P. L. Lions [10]). An essential tool is the dynamic programming principle (we give two forms of this result). We indicate some properties of u (its regularity, its behavior at infinity, the fact that u is the maximal subsolution of the Q.V.I. in the distribution sense). Finally, we show that u is a viscosity solution of (P).

In the last section, we prove that problem (P) has a unique viscosity solution. In this context, the particular form of H and the well-known results of optimal stopping time problems enable us to adapt a proof due to B. Hanouzet and J. L. Joly [9] for the elliptic Q.V.I.

The methods we use combine elements from the deterministic control (see P. L. Lions [10]), from the first-order Hamilton-Jacobi equations (especially the methods due to the notion of viscosity solutions—see M. G. Crandall, L. C. Evans and P. L. Lions [5], M. G. Crandall and P. L. Lions [6] or P. L. Lions [10]) and from the theory of elliptic Q.V.I. (see A. Bensoussan and J. L. Lions [3]).

1. Viscosity solutions for first-order Hamilton–Jacobi quasi-variational inequalities. We just want to recall the main equivalent definitions and to mention the most important properties of the viscosity solutions of first-order Hamilton–Jacobi quasi-variational inequalities, without proofs. More details and complete proofs can be found in G. Barles [1]. The notion of viscosity solutions of first-order Hamilton–Jacobi equations was introduced by M. G. Crandall and P. L. Lions [6] and all the results mentioned below are straightforward extensions of those obtained by M. G. Crandall and P. L. Lions [6], M. G. Crandall, L. C. Evans and P. L. Lions [5] or P. L. Lions [10].

1.1. Main definitions. We denote by BUC $(\mathbb{R}^N)$, the space of bounded and uniformly continuous functions on $\mathbb{R}^N$.

We recall the following notions of sub- and superdifferential of continuous functions considered in [5] and [6]. Let $\varphi \in C(\mathbb{R}^N)$.

DEFINITION 1.1. (i) The superdifferential of φ at $x_0 \in \mathbb{R}^N$, denoted by $D^+\varphi(x_0)$, is the set (possibly empty) defined by

$$D^+\varphi(x_0) = \left\{ p \in \mathbb{R}^N, \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - (p|x - x_0)}{|x - x_0|} \leqq 0 \right\}. \tag{1}$$

(ii) The subdifferential of φ at $x_0 \in \mathbb{R}^N$, denoted by $D^-\varphi(x_0)$, is the set given by $D^-\varphi(x_0) = -D^+(-\varphi)(x_0)$, i.e.,

$$D^-\varphi(x_0) = \left\{ p \in \mathbb{R}^N, \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - (p|x - x_0)}{|x - x_0|} \geqq 0 \right\}. \tag{2}$$

Remark 1.1. For $x \in \mathbb{R}^N$, $D^+\varphi(x)$ (resp. $D^-\varphi(x)$) is a closed convex set in $\mathbb{R}^N$.

Remark 1.2. The notion of subdifferential considered in [6] has been introduced independently for different purposes, by E. de Giorgi, A. Marino and M. Tosques [7] and A. Marino and M. Tosques [12].

DEFINITION 1.2. $u \in$ BUC $(\mathbb{R}^N)$ said to be a viscosity solution of the problem (P) if we have:

$$\forall y \in \mathbb{R}^N, \forall p \in D^+u(y), \quad \max(H(y, u(y), p), u - Mu) \leqq 0, \tag{3}$$

$$(4) \qquad \forall y \in \mathbb{R}^N, \forall p \in D^- u(y), \quad \max(H(y, u(y), p), u - Mu) \geqq 0.$$

Let us give another equivalent definition which is more practical in particular to show uniqueness results.

PROPOSITION 1.1. *$u \in \mathrm{BUC}(\mathbb{R}^N)$ is a viscosity solution of the problem* (P) *if and only if the two following properties hold*:

(5) *$\forall \phi \in C^1(\mathbb{R}^N)$, at each local maximum point x_0 of $u - \phi$, we have*
$$\max(H(x_0, u(x_0), D\phi(x_0)), u(x_0) - Mu(x_0)) \leqq 0;$$

(6) *$\forall \phi \in C^1(\mathbb{R}^N)$, at each local minimum point x_0 of $u - \phi$, we have*
$$\max(H(x_0, u(x_0), D\phi(x_0)), u(x_0) - Mu(x_0)) \geqq 0.$$

Remark 1.3. The proof is exactly the same as for first-order Hamilton-Jacobi equations; we just have to consider the Hamiltonian:

$$\tilde{H}(x, t, p) = \max(H(x, t, p), t - Mu(x)).$$

The two definitions mean that u is a viscosity solution of the obstacle problem, with the implicit obstacle Mu. The following proposition shows how we can use the particular form of M.

PROPOSITION 1.2. *$u \in \mathrm{BUC}(\mathbb{R}^N)$ is a viscosity solution of the problem* (P) *if and only if the two following properties hold*: *for all $\varphi \in C_b^1(\mathbb{R}^N)$*:

(7) *at each <u>global</u> maximum point x_0 of $u - \varphi$, we have*
$$\max(H(x_0, u(x_0), D\varphi(x_0)), \varphi(x_0) - M\varphi(x_0)) \leqq 0;$$

(8) *at each <u>global</u> minimum point x_0 of $u - \varphi$, we have*
$$\max(H(x_0, u(x_0), D\varphi(x_0)), \varphi(x_0) - M\varphi(x_0)) \geqq 0.$$

Remark 1.4. This proposition can be extended to more general operators M. It suffices that M is nondecreasing and that M satisfies

$$\forall \phi \in \mathrm{BUC}(\mathbb{R}^N), \forall c \in \mathbb{R}, \quad M(\phi + c) = M(\phi) + c.$$

1.2. Main properties of viscosity solutions of problem (P). Most of the properties of the viscosity solutions of first-order Hamilton-Jacobi equations are still valid for the viscosity solutions of the problem (P). It is an easy consequence of Remark 1.3. Only the stability results need to be considered because of the nonlocal form of M. So, we give a stability result.

PROPOSITION 1.3. *Assume that u_ε is a viscosity solution of*

$$\max(H_\varepsilon(x, u^\varepsilon, Du^\varepsilon), u^\varepsilon - Mu^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N$$

and that H_ε converges to H uniformly on compact subsets of $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and u^ε converges to $u \in \mathrm{BUC}(\mathbb{R}^N)$ uniformly on compact subsets of $\mathbb{R}^N$ when ε goes to zero, with the following condition:

$$(9) \qquad \|(u^\varepsilon - u)^-\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{when } \varepsilon \to 0.$$

Then u is a viscosity solution of the problem (P).

Remark 1.5. With the same assumptions, the result is also valid for the vanishing viscosity method.

Remark 1.6. The condition (9) is quite optimal. One can find in G. Barles [1] a counterexample of the statement in Proposition 1.3 in the case when (9) is not satisfied.

More details and complete proofs can be found in G. Barles [1]; the proof of this result being essentially routine adaptations of M. G. Crandall and P. L. Lions [6].

2. The deterministic impulse control problem. Our purpose is to show that the optimal cost function of a deterministic impulse control problem is always a viscosity solution of a problem (P).

We also give regularity results and results concerning the behavior at infinity for this optimal cost function. All these results are analogous to those obtained in the classical deterministic control problem (see P. L. Lions [10]). These are obtained essentially by using the dynamic programming principle.

2.1. Setting of the problem. Through all this section, $\theta=(\theta_i)_{i\in\mathbb{N}}$ will be a non-decreasing sequence of positive reals satisfying:

$$\theta_n \to +\infty \quad \text{when } n \to +\infty \tag{10}$$

and $\xi=(\xi_i)_{i\in\mathbb{N}}$ will be a sequence of elements of $(\mathbb{R}^+)^N$.

Letting V be a separable metric space, we consider the functions $b_i (1\leqq i\leqq N)$ and f satisfying:

$$\begin{aligned} &\varphi\in C(\mathbb{R}^N xV),\\ &\forall v\in V \quad \varphi(\cdot, v)\in W^{1,\infty}(\mathbb{R}^N) \quad \text{and} \quad \sup_{v\in V}\|\varphi(\cdot, v)\|_{1,\infty}<\infty, \end{aligned} \tag{11}$$

where $\varphi=b_i\ (1\leqq i\leqq N), f$.

Finally, $v(\cdot)$ will be measurable function which takes its value in a compact subset of V. Then the collection $K=(\theta, \xi, v(\cdot))$ will be called the *control*.

Next, we consider a *system* whose *state* is given by the solution $y_x(t)$ of the following problem:

$$\begin{aligned} &\frac{dy_x(t)}{dt}+b(y_x(t), v(t))=0 \quad \text{for } t\in\,]\theta_i, \theta_{i+1}[\quad \text{for all } i\in\mathbb{N},\\ &y_x(0)=x,\\ &y_x(\theta_i+0)=y_x(\theta_i-0)+\xi_i. \end{aligned} \tag{12}$$

The assumptions (11) imply the existence and the uniqueness of a solution $y_x(t)$ of (12).

Then we define the cost function (or pay-off function) for each control K:

$$J(x, K)=\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt+\sum_{i\in\mathbb{N}} c(\xi_i)\, e^{-\lambda\theta_i}, \tag{13}$$

where $\lambda>0$ and c is a continuous function from $(\mathbb{R}^+)^N$ into $\mathbb{R}$ which satisfies

$$\begin{aligned} &c=k+c_0 \quad \text{where } k>0,\\ &c_0(\xi_1+\xi_2)\leqq c_0(\xi_1)+c_0(\xi_2) \quad \text{for all } \xi_1\xi_2\in(\mathbb{R}^+)^N,\\ &c_0(0)=0, \qquad c(\xi_1)\leqq c(\xi_2) \quad \text{if } (\xi_2-\xi_1)\in(\mathbb{R}^+)^N. \end{aligned} \tag{14}$$

The problem to solve is to minimize the cost function over all controls K, that is to find for all $x\in\mathbb{R}^N$:

$$u(x)=\inf_K (J(x, K)). \tag{15}$$

2.2. Dynamic programming principle. The dynamic programming principle is the essential tool to obtain the solution of our problem. We shall give two forms of this result; the first shows that u is the solution of an optimal stopping time problem, the second being more classical.

Remark 2.1. In the introduction of this part, we took $\theta_0 \leqq \theta_1 \cdots \leqq \theta_n \leqq \cdots$; in fact the assumption (14) implies that we may assume without loss of generality: $\theta_0 < \theta_1 \cdots < \theta_n < \cdots$ in (15).

THEOREM 2.1. *Under assumptions* (11), (14), *we have*

$$u(x) = \inf_{(v(\cdot),\theta_0)} \left(\int_0^{\theta_0} f(y_x(t), v(t))\, e^{-\lambda t}\, dt + Mu(y_x(\theta_0 - 0))\, e^{-\lambda\theta_0} \right). \tag{16}$$

Remark 2.2. The formula (16) means that u is the solution of an optimal stopping time problem as should be expected (for optimal stopping time problems, see for instance A. Bensoussan and J. L. Lions [3]). Remark also that if we take $\theta_0 = 0$, we have: $u(x) \leqq Mu(x)$.

THEOREM 2.2. *Under assumptions* (11), (14) *we have*

$$u(x) = \inf_K \left(\int_0^T f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \sum_{\theta_i < T} c(\xi_i)\, e^{-\lambda\theta_i} + e^{-\lambda T} u(y_x(T-0)) \right). \tag{17}$$

Remark 2.3. These two results are easy adaptations of classical results and we just recall briefly the ideas of their proofs (cf. also [10]).

Proof of Theorem 2.1. We call $\tilde{u}(x)$ the right-hand side of (16).

Step 1. $u(x) \leqq \tilde{u}(x)$.

$$\forall K \quad u(x) \leqq \int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \sum_{i \in \mathbb{N}} c(\xi_i)\, e^{-\lambda\theta_i}.$$

Then

$$\begin{aligned} u(x) \leqq{}& \int_0^{\theta_0} f(y_x(t), v(t))\, e^{-\lambda t}\, dt \\ &+ e^{-\lambda\theta_0} \left[\int_0^\infty f(y_x(t+\theta_0), v(t+\theta_0))\, e^{-\lambda t}\, dt + \sum_{i \geqq 1} c(\xi_i)\, e^{-\lambda(\theta_i - \theta_0)} + c(\xi_0) \right]. \end{aligned}$$

Taking the infimum in the bracket, we obtain easily

$$u(x) \leqq \int_0^{\theta_0} f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta_0}[u(y_x(\theta_0 - 0) + \xi_0) + c(\xi_0)].$$

As the left-hand side does not depend on K, we conclude easily by taking the infimum in the right-hand side.

Step 2. $\tilde{u}(x) \leqq u(x)$. Let $\varepsilon > 0$; we choose K such that $u(x) + \varepsilon \geqq J(x, K)$, where $K = (\theta, \xi, v)$. The same computation yields

$$u(x) + \varepsilon \geqq \int_0^{\theta_0} f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda_0}[u(y_x(\theta_0 - 0) + \xi_0) + c(\xi_0)].$$

Then

$$u(x) + \varepsilon \geqq \tilde{u}(x).$$

Since ε is arbitrary, we conclude easily.

Proof of Theorem 2.2. The proof is essentially the same as the proof of Theorem 2.1. We just take T instead of θ_0. In fact, we obtain

$$u(x)=\inf_K\left(\int_0^T f(y_x(t), v(t))\, e^{-\lambda t}\, dt+\sum_{\theta_i<T} c(\xi_i)\, e^{-\lambda\theta_i}\right.$$
$$\left.+e^{-\lambda T}\min\,(u(y_x(T-0)), Mu(y_x(T-0)))\right).$$

The result is then given by the inequality $u(x)\leqq Mu(x)$ in $\mathbb{R}^N$.

2.3. Properties of u. First, we give a result concerning the regularity of u: let

$$\lambda_0=\sup_{\substack{x\neq x'\\ v\in V}}\left(-\frac{(x-x'|b(x,v)-b(x',v))}{|x-x'|^2}\right).$$

PROPOSITION 2.1. *Under assumptions* (11), (14), *we have*: $u\in \mathrm{BUC}(\mathbb{R}^N)$. *More precisely, we have*:

- *if* $0<\lambda<\lambda_0$, $u\in C^{0,\delta}(\mathbb{R}^N)$ *with* $\delta=\lambda/\lambda_0$;
- *if* $\lambda=\lambda_0$, $u\in C^{0,\delta}(\mathbb{R}^N)$ *for all* $\delta\in[0,1[$;
- *if* $\lambda>\lambda_0^+$, $u\in W^{1,\infty}(\mathbb{R}^N)$.

Proof of Proposition 2.1. The proof is exactly the same as in the standard deterministic case (see P. L. Lions [10]).

We use (17) to obtain

$$u(x)-u(x')\leqq\sup_K\left(\int_0^T |f(y_x(t), v(t))-f(y_{x'}(t), v(t))|\, e^{-\lambda t}\, dt+2\, e^{-\lambda T}\|u\|_{L^\infty(\mathbb{R}^N)}\right).$$

(Recall that $\|u\|_{L^\infty(\mathbb{R}^N)}\leqq(\|f\|/\lambda)L^\infty(\mathbb{R}^N)$.)

For a given control K, we have (see [10]):

$$|y_x(t)-y_{x'}(t)|\leqq e^{\lambda_0 t}|x-x'|.$$

So we use (11) to obtain

$$u(x)-u(x')\leqq\int_0^T C\, e^{(\lambda_0-\lambda)t}|x-x'|\, dt+2\|u\|_{L^\infty(\mathbb{R}^N)}\, e^{-\lambda T}$$

where $C=\sup_{v\in V}\|f(\cdot, v)\|_{1,\infty}$. If $\lambda>\lambda_0^+$, we let $T\to\infty$.

In the other case, we may assume $|x-x'|<1$ and we choose T such that $e^{-\lambda_0 T}=|x-x'|$. Then one concludes easily.

Now, we can show the relation between the deterministic impulsive control and the first-order Hamilton-Jacobi quasi-variational inequalities.

THEOREM 2.3. *Under assumptions* (11), (14) *and if u is differentiable at x, we have*

$$\max\left(\sup_{v\in V}(b(x,v)\cdot Du(x)+\lambda u-f(x,v)), u(x)-Mu(x)\right)=0. \tag{18}$$

Remark 2.4. We shall use the notation

$$H(x,t,p)=\sup_{v\in V}(b(x,v)p+\lambda t-f(x,v)). \tag{19}$$

By the well-known Rademacher theorem, we have the following corollary:

COROLLARY 2.1. *Under assumptions* (11), (14) *and if* $\lambda > \lambda_0^+$, *then* $u \in W^{1,\infty}(\mathbb{R}^N)$ *and satisfies*

$$\max\left(\sup_{v\in V}(b(x,v)Du+\lambda u - f(x,v)),\ u - Mu\right) = 0 \quad a.e.\ in\ \mathbb{R}^N. \tag{20}$$

Proof of Theorem 2.3. It is again a more or less standard proof (see P. L. Lions [10]).

Step 1. $\max(H(x,u(x),Du(x)), u(x)-Mu(x)) \leqq 0$. We take the particular control $v(t) \equiv v \in V$ and $\theta_0 = +\infty$. For all T, we have by the dynamic programming principle

$$u(x) \leqq \int_0^T f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda T} u(y_x(T)).$$

Dividing by T and letting $T \to 0$, we obtain the result as in [10]:

$$H(x, u(x), Du(x)) \leqq 0.$$

As we know that $u(x) \leqq Mu(x)$, the proof of the first step is complete.

Step 2. $\max(H(x,u(x),Du(x)), u(x)-Mu(x)) \geqq 0$. If $u(x) = Mu(x)$, we have nothing to show.

If $u(x) < Mu(x)$, we must prove that $H(x,u(x),Du(x)) \geqq 0$. We need the following lemma:

LEMMA 2.1. *Let* $x \in \mathbb{R}^N$ *such that* $u(x) < Mu(x)$. *Then there exists* $\varepsilon > 0$ *such that*

$$u(x) = \inf_{\substack{K\\ \theta_0 \geqq \varepsilon}} \left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \sum_{i\in\mathbb{N}} c(\xi_i)\, e^{-\lambda\theta_i}\right).$$

Proof of Lemma 2.1. Let $K^n = (\theta^n, \xi^n, v^n)$ a sequence such that

$$J(x, K^n) \to u(x).$$

We denote by $\chi(n) = J(x, K^n) - u(x)$,

$$u(x) + \chi(n) = \int_0^\infty f(y_x^n(t), v^n(t))\, e^{-\lambda t}\, dt + \sum_{i\in\mathbb{N}} c(\xi_i^n)\, e^{-\lambda\theta_i^n}.$$

By the same computation as in the proof of Theorem 2.1, we obtain

$$u(x) + \chi(n) \geqq \int_0^{\theta_0^n} f(y_x^n(t), v^n(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta_0^n} Mu(y_x^n(\xi_0^n - 0)).$$

Since $|y_x^n(\theta_0^n - 0) - x| \leqq \|b\|_{L^\infty(\mathbb{R}^N \times V)}\theta_0^n$, if there exists a subsequence $\theta_0^{n'}$ which converges to 0, we deduce easily that

$$u(x) \geqq Mu(x).$$

(Recall that $Mu \in \mathrm{BUC}(\mathbb{R}^N)$ because $u \in \mathrm{BUC}(\mathbb{R}^N)$, so Mu is continuous at x.) So we obtain a contradiction which proves the lemma.

Using Lemma 2.1 and (17) with $T < \varepsilon$, we obtain:

$$\forall T < \varepsilon \quad u(x) = \inf_{v(\cdot)}\left(\int_0^T f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda T} u(y_X(T))\right)$$

y_x is defined by (12) with $\theta_0 \geqq \varepsilon$.

We conclude as in the standard case of continuous control only (see [10]).

THEOREM 2.4. *Under assumptions* (11), (14) *and* $\lambda > 0$, $u \in C^{0,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *for some* $\alpha \in]0, 1]$ *and satisfies*:

$$(21)\qquad \begin{aligned} &u(x) \leqq Mu(x), \\ &\forall v \in V \quad b(x, v)Du + \lambda u - f(x, v) \leqq 0 \quad \text{in } D'(\mathbb{R}^N). \end{aligned}$$

In addition, if $w \in \mathrm{BUC}(\mathbb{R}^N)$ *and satisfies*:

$$(22)\qquad \begin{aligned} &w \leqq Mw, \\ &\forall v \in V \quad b(x, v)Dw + \lambda w - f(x, v) \leqq 0 \quad \text{in } D'(\mathbb{R}^N), \end{aligned}$$

then $w \leqq u$.

Remark 2.5. The last property means that u is the maximum subsolution of (P) in $D'(\mathbb{R}^N)$ (for related results see P. L. Lions [10], R. Gonzalez and E. Rofman [8] or P. L. Lions and J. L. Menaldi [11]).

Proof of Theorem 2.4. We choose the particular control $\theta_0 = +\infty$, $v(t) \equiv v \in V$. Then

$$\forall t \geqq 0 \quad \frac{1}{t}(u(x) - u(y_x(t))\, e^{-\lambda t}) \leqq \frac{1}{t} \int_0^t f(y_x(t), v)\, e^{-\lambda t}\, dt.$$

We conclude in the same way as in P. L. Lions [10].

Remark 2.6. For the second part, in order to be clear we use the following remark. We can consider in the definition of u only the case of a finite number of impulses (i.e. a finite number of θ_i or $\theta_{n+1} = +\infty$).

Then

$$u(x) = \inf_{(\theta, \xi, v, n)} \left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \sum_{i=1}^n c(\xi_i)\, e^{-\lambda \theta_i} \right).$$

LEMMA 2.2. *Under the assumptions of Theorem* 2.4, *if* $w \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies*

$$(23)\qquad \begin{aligned} &w(x) \leqq Mw(x), \\ &\forall v \in V, \quad b(x, v)Dw(x) + \lambda w(x) \leqq f(x, v) + \delta \quad \text{in } \mathbb{R}^N \end{aligned}$$

for $\delta \geqq 0$, *then* $w \leqq u + \delta/\lambda$.

Proof of Lemma 2.2. Let $\theta = (\theta_1, \cdots, \theta_n)$, $\xi = (\xi_1, \cdots, \xi_n)$, $v(t)$ a control. We assume that $\theta_0 > 0$ Then, since $w \in C^1(\mathbb{R}^N)$:

$$\begin{aligned} \forall i \in \mathbb{N} \quad & w(y_x(\theta_i + 0))\, e^{-\lambda \theta_i} - w(y_x(\theta_{i+1} - 0))\, e^{-\lambda \theta_{i+1}} \\ &= \int_{\theta_i}^{\theta_{i+1}} \{Dw(y_x(s))b(y_x(s), v(s)) + \lambda w(y_x(s))\}\, e^{-\lambda s}\, ds, \end{aligned}$$

$$w(x) - w(y_x(\theta_0 - 0))\, e^{-\lambda \theta_0} = \int_0^{\theta_0} \{Dw(y_x(s))b(y_x(s), v(s)) + \lambda w(y_x(s))\}\, e^{-\lambda s}\, ds.$$

Since $Dw \cdot b + \lambda w \leqq f + \delta$, we obtain

$$\begin{aligned} w(x) - w(y_x(T))\, e^{-\lambda T} \leqq & \int_0^T f(y_x(s), v(s))\, e^{-\lambda s}\, ds \\ & + \sum_{i=1}^n e^{-\lambda \theta_i}(w(y_x(\theta_i - 0)) - w(y_x(\theta_i + 0))) + \int_0^T \delta\, e^{-\lambda s}\, ds. \end{aligned}$$

But $y_x(\theta_i + 0) = y_x(\theta_i - 0) + \xi_i$ and from (23)

$$w(y_x(\theta_i - 0)) - w(y_x(\theta_i + 0)) \leqq c(\xi_i).$$

Therefore, we finally obtain

$$w(x)-w(y_x(T))\,e^{-\lambda T}\leqq\int_0^T f(y_x(s),v(s))\,e^{-\lambda s}\,ds+\sum_{i=1}^n c(\xi_i)\,e^{-\lambda\theta_i}+\frac{\delta}{\lambda}.$$

Letting $T\to\infty$, we have the result (because w is bounded).

In order to prove the theorem, it is enough to build $w_\varepsilon\in C^\infty(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, which converges uniformly to w in $\mathbb{R}^N$ and such that:

$$\begin{aligned}&\forall v\in V\quad b(x,v)Dw_\varepsilon(x)+\lambda w_\varepsilon(x)\leqq f(x,v)+\delta(\varepsilon),\\&w_\varepsilon(x)\leqq Mw_\varepsilon(x)\end{aligned}\tag{24}$$

where $\delta(\varepsilon)\to 0$, as $\varepsilon\to 0^+$.

Let $\rho\in D^+(\mathbb{R}^N)$, supp $\rho\subset B_1$, $\int_{\mathbb{R}^N}\rho(x)\,dx=1$ and let ρ_ε defined by:

$$\rho_\varepsilon(x)=\frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right).$$

It is well known that $w_\varepsilon=w*\rho_\varepsilon=\int_{\mathbb{R}^N}w(y)\rho_\varepsilon(x-y)\,dy\in C^\infty(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, converges uniformly to w (actually $\|w_\varepsilon-w\|_{L^\infty(\mathbb{R}^N)}\leqq\sup_{|h|\leqq\varepsilon}\|w(x+h)-w(x)\|_{L^\infty(\mathbb{R}^N)}$). Moreover w_ε satisfies the first inequality in (24) (see P. L. Lions [10]).

To conclude, we just have to prove $w_\varepsilon\leqq Mw_\varepsilon$. Since $w(x)\leqq k+c_0(\xi)+w(x+\xi)$ and $\rho\geqq 0$:

$$w_\varepsilon(x)\leqq k+c_0(\xi)+w_\varepsilon(x+\xi).$$

Then, $w_\varepsilon\leqq Mw_\varepsilon$ and the proof is complete.

Let us finally give a result concerning the behavior of u at infinity.

PROPOSITION 2.2. *Under assumptions* (11), (14) *and if we assume*

$$c(\xi)\to+\infty\quad\textit{if } |\xi|\to+\infty\tag{25}$$

and

$$f(x,v)\to l\quad\textit{if } |x|\to+\infty \textit{ uniformly with respect to } v,\tag{26}$$

then

$$u(x)\to l/\lambda\quad\textit{if } |x|\to+\infty.\tag{27}$$

Remark 2.7. The boundedness of u implies that in the definition of u we can impose the following additional condition on the control:

$$\forall T>0\quad\sum_{\theta_i\leqq T}c(\xi_i)\leqq C\,e^{+\lambda T}.$$

But from assumption (14), we have

$$c\left(\sum_{\theta_i\leqq T}\xi_i\right)\leqq\sum_{\theta_i\leqq T}c(\xi_i)\leqq C\,e^{+\lambda T}.$$

Then, from (25), we obtain

$$\left|\sum_{\theta_i\leqq T}\xi_i\right|\leqq C(T).\tag{28}$$

Proof of Proposition 2.2. Let $\varepsilon>0$. We want to show that for $|x|$ large enough, we have: $|u(x)-l/\lambda|\leqq\varepsilon$.

Since f is bounded, we can choose T independent of x and of all controls (θ, ξ, v) such that

$$\left|\int_T^\infty f(y_x(s); v(s))\, e^{-\lambda s}\, ds\right| \leqq \frac{\varepsilon}{4}. \tag{29}$$

Now we fix T with the property (29) and such that

$$\left|\int_T^\infty l \cdot e^{-\lambda s}\, ds\right| \leqq \frac{\varepsilon}{4}. \tag{30}$$

We use the following inequalities:

$$\inf_{K_1}\left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt\right) \leqq u(x) \leqq \inf_{K_2}\left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt\right),$$

where $K_1 = \{K = (v, \theta, \xi)$ such that ξ satisfies (28)$\}$ and $K_2 = \{K = (v, \theta, \xi)/\theta_0 = +\infty\}$. Then, from (29) and (30),

$$\begin{aligned} &-\frac{\varepsilon}{2} + \inf_{K_1}\left(\int_0^\infty [f(y_x(t), v(t)) - l]\, e^{-\lambda t}\, dt\right) \\ &\leqq u(x) - \frac{l}{\lambda} \leqq \frac{\varepsilon}{2} + \inf_{K_2}\left(\int_0^\infty [f(y_x(t), v(t)) - l]\, e^{-\lambda t}\, dt\right). \end{aligned} \tag{31}$$

Considering Remark 2.7, we have for all $K \in K_1$ and for all $x \in \mathbb{R}^N$:

$$0 \leqq t \leqq T, \qquad |y_x(t) - x| \leqq \|b\|_\infty T + C(T),$$

and then

$$|y_x(t)| \geqq |x| - \|b\|_\infty T - C(T).$$

So, for $|x|$ large enough, we have $|f(y_x(t), v(t)) - l| \leqq \varepsilon\lambda/2$ for $t \in [0, T]$. Finally:

$$\int_0^T |f(y_x(t), v(t)) - l|\, e^{-\lambda t}\, dt \leqq \frac{\varepsilon}{2} \quad \text{for all } K \in K_1 \text{ and } K \in K_2; \tag{32}$$

(31) and (32) give the result.

Remark 2.8. The result of Proposition 2.2 is false without (25). Take in $\mathbb{R}$: $b \equiv 0$, $0 < \lambda < 1$, $c \equiv k < 1$; $f = 1$ if $|x| \geqq 2$, 0 if $|x| \leqq 1$ and $0 \leqq f \leqq 1$ if $|x| \in [1, 2]$. Then $u \to k$ if $x \to -\infty$.

2.4. The viscosity formulation of the dynamic programming principle.

THEOREM 2.5. *Let u be the optimal cost function defined by* (15). *Under assumptions* (11), (14), *then $u \in \mathrm{BUC}(\mathbb{R}^N)$ is a viscosity solution of*

$$\max(H(x, u, Du), u - Mu) = 0 \quad \text{in } \mathbb{R}^N. \tag{33}$$

Proof of Theorem 2.5. The proof is inspired from the corresponding one in P. L. Lions [10].

Let $\phi \in C^1(\mathbb{R}^N)$ and x_0 a local maximum point of $u - \phi$. We fix a control K such that $v(t) \equiv v \in V$ and $\theta_0 = +\infty$. Then for all $T > 0$ we have

$$u(x_0) \leqq \int_0^T f(y_{x_0}(t), v)\, e^{-\lambda t}\, dt + u(y_{x_0}(T))\, e^{-\lambda T}.$$

If T is small enough we have $u(y_{x_0}(T)) \leqq \phi(y_{x_0}(T)) + (u(x_0) - \phi(x_0))$ because

$|y_{x_0}(t)-x_0|\leqq Ct$. So we obtain easily

$$u(x_0)\frac{(1-e^{-\lambda T})}{T}\leqq\frac{1}{T}\int_0^T f(y_{x_0}(t),v)\,e^{-\lambda t}\,dt+e^{-\lambda T}\frac{[\phi(y_{x_0}(T))-\phi(x_0)]}{T}.$$

Letting $T\to 0$, we obtain

$$\forall v\in V\quad b(x_0,v)D\phi(x_0)+\lambda u(x_0)-f(x_0,v)\leqq 0,$$

which ends the first part of the proof.

Let $\phi\in C^1(\mathbb{R}^N)$ and x_0 a local minimum of $u-\phi$; then two cases are possible.

Case 1. $u(x_0)=Mu(x_0)$ and there is nothing to prove.

Case 2. $u(x_0)<Mu(x_0)$. We use Lemma 2.1 to claim, for $0<T<\varepsilon$,

$$u(x_0)=\inf_{\substack{v(\cdot)\\ \theta_0>T}}\left(\int_0^T f(y_x(t),v(t))\,e^{-\lambda t}\,dt+u(y_{x_0}(T))\,e^{-\lambda T}\right).$$

By assumption for T small enough $u(y_{x_0}(T))\geqq\phi(y_{x_0}(T))+(u(x_0)-\phi(x_0))$. So we have

$$u(x_0)\left(\frac{1-e^{-\lambda T}}{T}\right)\geqq\inf_{v(\cdot)}\left(\frac{1}{T}\int_0^T f(y_{x_0}(t),v(t))\,e^{-\lambda t}\,dt+e^{-\lambda T}\frac{[\phi(y_{x_0}(T))-\phi(x_0)]}{T}\right).$$

But

$$|f(y_{x_0}(t),v(t))-f(x_0,v(t))|\leqq C|y_{x_0}(t)-x_0|\leqq\tilde{C}t$$

and

$$\phi(y_{x_0}(T))-\phi(x_0)=\int_0^T-b(y_{x_0}(s),v(s))D\phi(y_x(s))\,ds,$$

$$|b(y_{x_0}(s),v(s))D\phi(y_x(s))-b(x_0,v(s))D\phi(x_0)|=C(s),$$

where $C(s)\to 0$ if $s\to 0^+$, using the fact that b is Lipschitz and $D\phi$ continuous at x_0.

We deduce easily:

$$u(x_0)\left(\frac{1-e^{-\lambda T}}{T}\right)\geqq\inf_{v(\cdot)}\left(\frac{1}{T}\int_0^T(f(x_0,v(t))-b(x_0,v(t))D\phi(x_0))\,dt-\varepsilon(T)\right),$$

where $\varepsilon(T)\to 0$ when $T\to 0$.

Now we remark that:

$$\frac{1}{T}\int_0^T(f(x_0,v(t))-b(x_0,v(t))D\phi(x_0))\,dt\geqq\inf_{v\in V}(f(x_0,v)-b(x_0,v)D\phi(x_0)).$$

Letting $T\to 0$, we obtain the result

$$\sup(b(x_0,v)D\phi(x_0)+\lambda u(x_0)-f(x_0,v))\geqq 0.$$

Remark 2.9. We have shown that u is a viscosity solution of the problem (P). In fact, this result will be completely satisfactory if we prove a uniqueness result for viscosity solutions of (33). This is the goal of the third part.

3. A uniqueness result for viscosity solutions of quasi-variational inequality (33).

THEOREM 3.1. *Under assumptions* (11), (14) *and* $\lambda>0$, *there exists a unique viscosity solution of the problem* (33) *in* BUC $(\mathbb{R}^N)$.

Remark 3.1. The idea of the proof consists in adapting the method due to B. Hanouzet and J. L. Joly [9] for elliptic Q.V.I. (see also A. Bensoussan [2] or A. Bensoussan and J. L. Lions [4]).

Remark 3.2. The proof uses in an essential way results concerning optimal stopping time problems (in the deterministic case) that we give briefly in the following proposition. For more details and complete proof of this result, see P. L. Lions [10], A. Bensoussan and J. L. Lions [3] or the part II which gives all the ideas necessary to the proof.

Let us define the optimal stopping time problem

$$\text{(34)}\qquad \begin{aligned} &\frac{dy_x}{dt}(t)+b(y_x(t), v(t))=0 \quad \text{for all } t \geqq 0,\\ &y_x(0)=x, \end{aligned}$$

$$\text{(35)}\qquad J(x, v, \theta)=\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt+\psi(y_x(\theta))\, e^{-\lambda\theta},$$

$$\text{(36)}\qquad u(x)=\inf_{(v,\theta)} J(x, v, \theta).$$

Now we have the following result:

PROPOSITION 3.1. *Under assumptions* (11) *and* $\lambda>0$, *and if* $\psi\in \mathrm{BUC}(\mathbb{R}^N)$, *then* u *given by* (36) *is the unique viscosity solution in* $\mathrm{BUC}(\mathbb{R}^N)$ *of*

$$\text{(37)}\qquad \max(H(x, u, Du), u-\psi)=0.$$

In addition, if $\psi\in W^{1,\infty}(\mathbb{R}^N)$ *and* $\lambda>\lambda_0^+$, *then* $u\in W^{1,\infty}(\mathbb{R}^N)$.

Proof of Theorem 3.1. Without loss of generality, we can assume that $f\geqq 0$ (if this is not the case, we add constants to u and f).

We define the operator T, for $w\in \mathrm{BUC}(\mathbb{R}^N)$, by:

$$\text{(38)}\qquad Tw(x)=\inf_{(v,\theta)}\left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt+Mw(y_x(\theta))\, e^{-\lambda\theta}\right),$$

where y_x is defined by (34).

We need the two following lemmas:

LEMMA 3.1. *T maps* $\mathrm{BUC}(\mathbb{R}^N)$ *into* $\mathrm{BUC}(\mathbb{R}^N)$, *is increasing and concave. Furthermore, Tw is the unique viscosity solution in* $\mathrm{BUC}(\mathbb{R}^N)$ *of*

$$\text{(39)}\qquad \max(H(x, z, Dz), z-Mw)=0 \quad \text{in } \mathbb{R}^N.$$

LEMMA 3.2. *Let* u_0 *defined by*

$$u_0(x)=\inf_{v(\cdot)}\left(\int_0^\infty f(y_x(t), v(t))\, e^{-\lambda t}\, dt\right),$$

with y_x *defined by* (34).

Let $\mu>0$ *be such that* $\mu\|u_0\|_{L^\infty(\mathbb{R}^N)}<k$ *and* $\mu<1$. *Let* z *and* $\tilde{z}$ *two positive functions of* $\mathrm{BUC}(\mathbb{R}^N)$ *satisfying*

$$\text{(40)}\qquad z-\tilde{z}\leqq \gamma z \quad \text{for one } \gamma\in[0, 1].$$

Then

$$\text{(41)}\qquad Tz-T\tilde{z}\leqq \gamma(1-\mu)Tz.$$

Remark 3.3. The formulation of Lemma 3.2 is very much akin to the lemma due to B. Hanouzet and J. L. Joly [9] used in elliptic Q.V.I.

Remark 3.4. Let us just recall that u_0 defined in Lemma 3.2 is the unique viscosity solution in $\mathrm{BUC}(\mathbb{R}^N)$ of: $H(x, u, Du)=0$ in $\mathbb{R}^N$ (cf. P. L. Lions [10]).

Moreover, by uniqueness result in $\mathbb{R}^N$ (see M. G. Crandall and P. L. Lions [6] or P. L. Lions [10]) or directly, we have $Tw \leqq u_0$ for all $w \in \mathrm{BUC}(\mathbb{R}^N)$. (This inequality can be obtained by (38).)

We first show the theorem using the two lemmas.

If we consider two viscosity solutions u and v of (33), we can assume that they are positive: it suffices to add $\max(\|u\|_\infty, \|v\|_\infty)$ to u and v and so f is changed in $f + \lambda \max(\|u\|_\infty, \|v\|_\infty)$.

Besides, by Lemma 3.1, u and v are fixed points of T. (It is a consequence of the uniqueness for the obstacle problem in $\mathrm{BUC}(\mathbb{R}^N)$ for Mu and Mv).

To conclude, we just have to use the Lemma 3.2: since $v \geqq 0$, $u - v \leqq u$, then:

$$Tu - Tv \leqq (1-\mu)u,$$

but $u = Tu$ and $v = Tv$; we obtain by induction

$$\forall n \in \mathbb{N} \quad u - v \leqq (1-\mu)^n u.$$

Since u is bounded, letting $n \to \infty$, we have $u \leqq v$. Changing u and v, we have the result.

Next, we prove the two lemmas.

Proof of Lemma 3.1. T maps $\mathrm{BUC}(\mathbb{R}^N)$ in $\mathrm{BUC}(\mathbb{R}^N)$ is an easy consequence of the Proposition 3.1 because M maps $\mathrm{BUC}(\mathbb{R}^N)$ in $\mathrm{BUC}(\mathbb{R}^n)$. The fact that T is increasing is obvious. Let us prove the concavity.

Let w_1 and $w_2 \in \mathrm{BUC}(\mathbb{R}^N)$ and $\mu \in [0, 1]$.

$$T(\mu w_1 + (1-\mu) w_2)(x) = \inf_{(v(\cdot),\theta)} \left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta} M(\mu w_1 + (1-\mu) w_2)(y_x(\theta)) \right).$$

But M is concave:

$$T(\mu w_1 + (1-\mu) w_2)(x) \geqq \inf_{(v(\cdot),\theta)} \left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta} M(\mu M w_1(y_x(\theta)) + (1-\mu) M w_2)(y_x(\theta)) \right).$$

We conclude easily using that:

$$\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta}(Mw_1(y_x(\theta)) + (1-\mu) M w_2(y_x(\theta)))$$
$$= \mu\left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + e^{-\lambda\theta} M w_1(y_x(\theta)) \right)$$
$$+ (1-\mu)\left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + M w_2(y_x(\theta)) \right).$$

Then

$$T(\mu w_1 + (1-\mu) w_2) \geqq \mu T w_1 + (1-\mu) T w_2, \qquad T \text{ is concave.}$$

Finally, we prove that Tw is the *unique* viscosity solution of (39) in $\mathrm{BUC}(\mathbb{R}^N)$. In fact, it is an easy consequence of uniqueness results for first-order Hamilton–Jacobi equations in $\mathbb{R}^N$. It suffices to consider the Hamiltonian

$$\tilde{H}(x, t, p) = \max(H(x, t, p), t - Mw(x))$$

(see M. G. Crandall and P. L. Lions [6] or P. L. Lions [10]). This ends the proof of Lemma 3.1].

Proof of Lemma 3.2. According to the monotonicity of T, (38) implies that

$$T\tilde{z} \geqq T((1-\gamma)z).$$

Using the concavity gives $T\tilde{z} \geqq (1-\gamma)Tz + \gamma T(0)$.

We first prove that $\mu u_0 \leqq T(0)$:

$$T(0)(x) = \inf_{(v(\cdot),\theta)} \left(\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + k\, e^{-\lambda\theta} \right).$$

Now we use that $k \geqq \mu u_0(y_x(\theta))$ and that $\mu f \leqq f$ $(f \geqq 0)$:

$$T(0)(x) \geqq \inf_{(v(\cdot),\theta)} \left(\int_0^\theta \mu f(y_x(t), v(t))\, e^{-\lambda t}\, dt + \mu u_0(y_x(\theta))\, e^{-\lambda\theta} \right).$$

But, for all $\theta > 0$, using the dynamic programming principle for standard deterministic control problems, we have

$$\mu \left[\int_0^\theta f(y_x(t), v(t))\, e^{-\lambda t}\, dt + u_0(y_x(\theta))\, e^{-\theta} \right] \geqq \mu u_0(x).$$

This last inequality gives the result we need.

Now using Remark 3.4 gives $u_0 \geqq Tz$. Then

$$T\tilde{z} \geqq (1-\gamma)Tz + \gamma\mu Tz,$$

and we easily deduce the result:

$$Tz - T\tilde{z} \leqq \gamma(1-\mu)Tz.$$

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