UNIVERSIDAD DE CHILE

CÁLCULO EN VARIAS VARIABLES
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Guía Regla de la Cadena(1^{er} Orden)

1. Sean $f : \mathbb{R}^2 \to \mathbb{R}$ y $g : \mathbb{R}^2 \to \mathbb{R}$ dos funciones diferenciables.
   Se define $z(u,v) = e^{g(u,v)f^2(u,v)}$.

   a) Determine las derivadas parciales de z. Es z diferenciable?.

   b) Si $f(u,v) = \sqrt{uv}$ y $g(u,v) = \frac{1}{v}$. Compruebe la fórmula obtenida en la parte anterior.

   Solución

   a) $\frac{\partial z(u,v)}{\partial u} = e^{g(u,v)f^2(u,v)}\left(\frac{\partial g(u,v)}{\partial u} f^2(u,v) + 2g(u,v)f(u,v)\frac{\partial f(u,v)}{\partial u}\right)$
   $\frac{\partial z(u,v)}{\partial v} = e^{g(u,v)f^2(u,v)}\left(\frac{\partial g(u,v)}{\partial v} f^2(u,v) + 2g(u,v)f(u,v)\frac{\partial f(u,v)}{\partial v}\right)$
   Notemos que $z(u,v)$ es composición de funciones diferenciables, y por lo tanto será diferenciable.

   b) Notemos que $f^2(u,v) = uv$ y $g(u,v) = \frac{1}{v}$. Por lo tanto:
   $\frac{\partial f(u,v)}{\partial v} = \frac{u}{2\sqrt{uv}}$, $\frac{\partial f(u,v)}{\partial u} = \frac{v}{2\sqrt{uv}}$
   $\frac{\partial g(u,v)}{\partial u} = 0$, $\frac{\partial g(u,v)}{\partial v} = -\frac{1}{v^2}$
   Con esto $z(u,v) = e^u$, con $\frac{\partial z(u,v)}{\partial u} = e^u$ y $\frac{\partial z(u,v)}{\partial v} = 0$. (esto sin utlizar la fórmula)
   Con la fórmula:
   $\frac{\partial z(u,v)}{\partial u} = e^u\left(0 \cdot uv + 2\frac{1}{v} \cdot \sqrt{uv} \cdot \frac{v}{2\sqrt{uv}}\right) = e^u$
   $\frac{\partial z(u,v)}{\partial v} = e^u\left(-\frac{1}{v^2} \cdot uv + 2\frac{1}{v} \cdot \sqrt{uv} \cdot \frac{u}{2\sqrt{uv}}\right) = 0$

   Con esto comprobamos la fórmula. □

2. Sean f y g dos funciones reales de variable real, derivables en $\mathbb{R}$. Se define la función:

$$z(x,y) = x^2yf(\frac{x}{y}) + xy^2g(\frac{y}{x})$$

   Encuentre el valor de E dado por la siguiente ecuación:

$$E = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

   Solución

   $\frac{\partial z(x,y)}{\partial x} = 2xyf(\frac{x}{y}) + x^2yf'(\frac{x}{y})\frac{1}{y} + y^2g(\frac{y}{x}) + xy^2g'(\frac{y}{x})\frac{-y}{x^2}$
   $\frac{\partial z(x,y)}{\partial y} = x^2f(\frac{x}{y}) + x^2yf'(\frac{x}{y})\frac{-x}{y^2} + 2xyg(\frac{y}{x}) + xy^2g'(\frac{y}{x})\frac{1}{x}$

Por lo tanto,
$E = x\cdot(2xyf(\frac{x}{y})+x^2yf'(\frac{x}{y})\frac{1}{y}+y^2g(\frac{y}{x})+xy^2g'(\frac{y}{x})\frac{-y}{x^2})+y\cdot(x^2f(\frac{x}{y})+x^2yf'(\frac{x}{y})\frac{-x}{y^2}+2xyg(\frac{y}{x})+xy^2g'(\frac{y}{x})\frac{1}{x})$
$E = 2x^2yf(\frac{x}{y})+x^3f'(\frac{x}{y})+xy^2g(\frac{y}{x})-y^3g'(\frac{y}{x})+x^2yf(\frac{x}{y})-x^3f'(\frac{x}{y})+2xy^2g(\frac{y}{x})+y^3g'(\frac{y}{x})$
$E = 3x^2yf(\frac{x}{y})+3xy^2g(\frac{y}{x})$
$E = 3z(x,y)$

Por lo tanto, $E = x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 3z(x,y)$. $\square$

3. Sean $g:\mathbb{R}\to\mathbb{R}$ y $h:\mathbb{R}^2\to\mathbb{R}$ funciones diferenciables. Se define la función:
$f(x,y) = x^2g(\frac{x}{y})+xyh(\frac{x}{x+y},\frac{x^2}{y^2})$.
Demuestre que:
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = 2f$$

Solución

$\frac{\partial f}{\partial x} = 2xg(\frac{x}{y})+\frac{x^2}{y}g'(\frac{x}{y})+yh(\frac{x}{x+y},\frac{x^2}{y^2})+\frac{xy^2}{(x+y)^2}\frac{\partial h}{\partial x}(\frac{x}{x+y},\frac{x^2}{y^2})+2\frac{x^2}{y}\frac{\partial h}{\partial y}(\frac{x}{x+y},\frac{x^2}{y^2})$
$\frac{\partial f}{\partial y} = \frac{-x^3}{y^2}g'(\frac{x}{y})+xh(\frac{x}{x+y},\frac{x^2}{y^2})-\frac{x^2y}{(x+y)^2}\frac{\partial h}{\partial x}(\frac{x}{x+y},\frac{x^2}{y^2})-2\frac{x^3}{y^2}\frac{\partial h}{\partial y}(\frac{x}{x+y},\frac{x^2}{y^2})$

Evaluando la expresión, se obtiene el resultado pedido. $\square$

4. Sea $f(\rho,\theta)$ una función de $\mathbb{R}^2$ en $\mathbb{R}$ diferenciable. Se define $F:\mathbb{R}^2\to\mathbb{R}$ de la siguiente manera:
$$F(x,y) = f(\rho,\theta) = f(\sqrt{x^2+y^2},\arctan\left(\frac{y}{x}\right))$$
con $x\neq 0$.

a) Demuestre que se cumple
$$\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$
Indique donde están evaluadas las derivadas parciales.

b) Compruebe la fórmula anterior para $f(\rho,\theta) = \rho\theta$

Solución

a) $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$, $\frac{\partial F}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$
$\frac{\partial \rho}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial \rho}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$
$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot -\frac{y}{x^2} = \frac{-y}{x^2+y^2}$, $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$
Luego,
$\left(\frac{\partial F}{\partial x}\right)^2 = \left(\frac{\partial f}{\partial \rho}\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial f}{\partial \theta}\cdot\frac{-y}{x^2+y^2}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2\cdot\frac{x^2}{x^2+y^2}+\left(\frac{\partial f}{\partial \theta}\right)^2\cdot\frac{y^2}{(x^2+y^2)^2}-2\frac{\partial f}{\partial \rho}\frac{\partial f}{\partial \theta}\frac{x}{\sqrt{x^2+y^2}}\frac{-y}{x^2+y^2}$

$\left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\cdot\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\cdot\frac{x}{x^2+y^2}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2\cdot\frac{y^2}{x^2+y^2}+\left(\frac{\partial f}{\partial \theta}\right)^2\cdot\frac{x^2}{(x^2+y^2)^2}+2\frac{\partial f}{\partial \rho}\frac{\partial f}{\partial \theta}\frac{y}{\sqrt{x^2+y^2}}\frac{x}{x^2+y^2}$

Sumando ambos términos:
$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2\cdot\frac{x^2+y^2}{x^2+y^2} + \left(\frac{\partial f}{\partial \theta}\right)^2\cdot\frac{y^2+x^2}{(x^2+y^2)^2}$
$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2\cdot\frac{1}{x^2+y^2}$

Pero, notemos que $\rho^2 = x^2+y^2$, por lo tanto obtenemos:

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Las derivadas parciales están evaluadas como sigue:
$\frac{\partial F}{\partial x}$ y $\frac{\partial F}{\partial y}$ en (x,y).
$\frac{\partial f}{\partial \rho}$ y $\frac{\partial f}{\partial \theta}$ en $(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right))$

□

b) Propuesto

5. Sean $f:\mathbb{R}^2\to\mathbb{R}$ y $h:\mathbb{R}^2\to\mathbb{R}$ diferenciables. Se define la función $g:\mathbb{R}\to\mathbb{R}$ por:

$$g(x) = f(x, h(x,x))$$

a) Encuentre $g'(x)$

b) Compruebe la fórmula anterior para $f(x,y) = x+2y$ y $h(x,y) = 2xy$

Solución

a) $g'(x) = \frac{\partial f}{\partial x}(x,h(x,x)) + \frac{\partial f}{\partial y}(x,h(x,x))(\frac{\partial h}{\partial x}(x,x) + \frac{\partial h}{\partial y}(x,x))$

b) $g(x) = x+4x^2 \implies g'(x) = 1+8x$.
Evaluando la fórmula se tiene $g'(x) = 1+2(2x+2x) = 1+8x$

□

6. Sean $f:\mathbb{R}^3\to\mathbb{R}$ y $h:\mathbb{R}^2\to\mathbb{R}$ dos funciones diferenciables.
Si $h(x,y) = f(x,y,h(x,y))$, encuentre $\frac{\partial h}{\partial x}$ y $\frac{\partial h}{\partial y}$.

Solución

$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}(x,y,h(x,y)) + \frac{\partial f}{\partial z}(x,y,h(x,y))\frac{\partial h}{\partial x}$
$\implies \frac{\partial h}{\partial x} = \frac{\frac{\partial f}{\partial x}(x,y,h(x,y))}{1-\frac{\partial f}{\partial z}(x,y,h(x,y))}$

$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial y}(x,y,h(x,y)) + \frac{\partial f}{\partial z}(x,y,h(x,y))\frac{\partial h}{\partial y}$
$\implies \frac{\partial h}{\partial y} = \frac{\frac{\partial f}{\partial y}(x,y,h(x,y))}{1-\frac{\partial f}{\partial z}(x,y,h(x,y))}$

□

7. Sean $f : \mathbb{R} \to \mathbb{R}$ y $h : \mathbb{R}^2 \to \mathbb{R}$ dos funciones diferenciables tales que $g(x,y) = xyf(\frac{x+y}{xy})$. Demuestre que

$$x^2\frac{\partial g}{\partial x} - y^2\frac{\partial g}{\partial y} = G(x,y)g(x,y)$$

Encuentre explícitamente a $G(x,y)$.
Solución

$\frac{\partial g(x,y)}{\partial x} = yf(\frac{x+y}{xy}) + xyf'(\frac{x+y}{xy}) \cdot \frac{xy-y(x+y)}{x^2y^2} = yf(\frac{x+y}{xy}) - f'(\frac{x+y}{xy}) \cdot \frac{y}{x}$
$\frac{\partial g(x,y)}{\partial y} = xf(\frac{x+y}{xy}) + xyf'(\frac{x+y}{xy}) \cdot \frac{xy-x(x+y)}{x^2y^2} = xf(\frac{x+y}{xy}) - f'(\frac{x+y}{xy}) \cdot \frac{x}{y}$

Luego,
$x^2\frac{\partial g}{\partial x} - y^2\frac{\partial g}{\partial y} = x^2yf(\frac{x+y}{xy}) - xyf'(\frac{x+y}{xy}) - xy^2f(\frac{x+y}{xy}) + xyf'(\frac{x+y}{xy})$
$x^2\frac{\partial g}{\partial x} - y^2\frac{\partial g}{\partial y} = x^2yf(\frac{x+y}{xy}) - xy^2f(\frac{x+y}{xy}) = (x-y)xyf(\frac{x+y}{xy}) = G(x,y)g(x,y)$

De donde podemos identificar $G(x,y) = x - y$.

$\square$

8. Sea $h : D \subset \mathbb{R}^2 \to \mathbb{R}$ una función definida por:

$$z(x,y) = xy\tan\left(\frac{y}{x}\right)$$

Determine, para $(x,y) \neq (0,0)$ (en términos de z) el valor de la siguiente expresión:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

Explique, además, que sucede en $(0,0)$.

Solución
$\frac{\partial z}{\partial x} = y\tan(\frac{y}{x}) - \frac{y^2}{x}sec^2(\frac{y}{x})$
$\frac{\partial z}{\partial y} = x\tan(\frac{y}{x}) + ysec^2(\frac{y}{x})$
Evaluando la expresión se tiene:
$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xy\tan(\frac{y}{x}) = 2z(x,y)$
En el origen, la función $z(x,y)$ no está definida, por lo tanto no tiene derivadas parciales.

$\square$

9. Se define la función $f(x,y,z) = \left(\frac{x-y+z}{x+y-z}\right)^n$.
Determine el valor de

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

si $(x,y,z) \neq (0,0,0)$
Solución

$\frac{\partial f(x,y,z)}{\partial x} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{x+y-z-(x-y+z)}{(x+y-z)^2} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{2y-2z}{(x+y-z)^2}$

$\frac{\partial f(x,y,z)}{\partial y} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{-(x+y-z)-(x-y+z)}{(x+y-z)^2} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{-2x}{(x+y-z)^2}$
$\frac{\partial f(x,y,z)}{\partial z} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{x+y-z+(x-y+z)}{(x+y-z)^2} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{2x}{(x+y-z)^2}$

Luego, obtenemos:
$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{2xy-2xz}{(x+y-z)^2} + n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{-2xy}{(x+y-z)^2} + n\left(\frac{x-y+z}{x+y-z}\right)^{n-1} \cdot \frac{2xz}{(x+y-z)^2}$

Por lo tanto,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0$$

□

10. Se define la función $G : \mathbb{R}^3 \to \mathbb{R}$ por $G(x,y,z) = z\sin\left(\frac{y}{x}\right)$. Si hacemos el cambio de variable $x = 3r^2 + 2s$, $y = 4r - 2s^3$ y $z = 2r^2 - 3s^2$, entonces G se transforma en una función $\widetilde{G} : \mathbb{R}^2 \to \mathbb{R}$.
Encuentre $\frac{\partial \widetilde{G}}{\partial r}$ y $\frac{\partial \widetilde{G}}{\partial s}$ evaluadas en r y s, pero en función de x, y, z.

Solución
Lo que se indica en el enunciado, es que mediante el cambio de variable se forma una función $\widetilde{G}(r,s) = G(3r^2 + 2s, 4r - 2s^3, 2r^2 - 3s^2)$.
Luego,
$\frac{\partial \widetilde{G}}{\partial r} = \frac{\partial G}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial r}$
$\frac{\partial \widetilde{G}}{\partial s} = \frac{\partial G}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial s}$.

Recordemos que las derivadas parciales están evaluadas de la siguiente forma:
$\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ y $\frac{\partial G}{\partial z}$ en $(3r^2 + 2s, 4r - 2s^3, 2r^2 - 3s^2)$.
$\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}$ y $\frac{\partial z}{\partial r}$ en (r,s).

Calculemos las derivadas parciales que necesitamos:
$\frac{\partial G}{\partial x} = z\cos\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2}$, $\frac{\partial G}{\partial y} = z\cos\left(\frac{y}{x}\right) \cdot \frac{1}{x}$, $\frac{\partial G}{\partial z} = \sin\left(\frac{y}{x}\right)$
$\frac{\partial x}{\partial r} = 6r$, $\frac{\partial y}{\partial r} = 4$, $\frac{\partial z}{\partial r} = 4r$
$\frac{\partial x}{\partial s} = 2$, $\frac{\partial y}{\partial s} = -6s^2$, $\frac{\partial z}{\partial s} = -6s$
Con lo recién calculado, estamos listos con el problema. Ahora simplemente debemos reemplazar. La continuación queda propuesta.

□

11. Pruebe que la función $F : \mathbb{R}^2 \to \mathbb{R}$ definida por $F(u,v) = f(uv, \frac{u^2-v^2}{2})$ donde $f : \mathbb{R}^2 \to \mathbb{R}$ es una función diferenciable, verifica la siguiente igualdad:

$$\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = (u^2 + v^2)\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)$$

Indique donde están evaluadas las derivadas parciales.

Solución

$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$, $\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

Notemos, además que:
$\frac{\partial x}{\partial v} = u$, $\frac{\partial x}{\partial u} = v$
$\frac{\partial y}{\partial v} = -v$, $\frac{\partial y}{\partial u} = u$

Luego,
$\left(\frac{\partial F}{\partial u}\right)^2 = \left(v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y}\right)^2 = v^2\left(\frac{\partial f}{\partial x}\right)^2 + u^2\left(\frac{\partial f}{\partial y}\right)^2 + 2uv\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$
$\left(\frac{\partial F}{\partial v}\right)^2 = \left(u\frac{\partial f}{\partial x} - v\frac{\partial f}{\partial y}\right)^2 = u^2\left(\frac{\partial f}{\partial x}\right)^2 + v^2\left(\frac{\partial f}{\partial y}\right)^2 - 2uv\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$

Finalmente, sumando ambos términos obtenemos que:

$$\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = (u^2+v^2)\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)$$

□

12. Sea $F:\mathbb{R}^3 \to \mathbb{R}^3$ definida por:

$$F(u,v,w) = \begin{pmatrix} f_1(u,v,w) \\ f_2(u,v,w) \\ f_3(u,v,w) \end{pmatrix}$$

donde $f_i:\mathbb{R}^3 \to \mathbb{R}$ $i=1,2,3$.
Se define la *divergencia* de F como:

$$divF = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Por otro lado, sea $G:\mathbb{R}^3 \to \mathbb{R}^3$ definida por:

$$G(r,\theta,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Se definen además los vectores unitarios $\widehat{r}$, $\widehat{\theta}$ y $\widehat{z}$ como sigue:

$$\widehat{r} = \frac{\frac{\partial G}{\partial r}}{\|\frac{\partial G}{\partial r}\|}, \widehat{\theta} = \frac{\frac{\partial G}{\partial \theta}}{\|\frac{\partial G}{\partial \theta}\|}, \widehat{z} = \frac{\frac{\partial G}{\partial z}}{\|\frac{\partial G}{\partial z}\|}$$

donde $\frac{\partial G}{\partial r}$, $\frac{\partial G}{\partial \theta}$ y $\frac{\partial G}{\partial z}$ son vectores que se obtienen derivando parcialmente a G componente a componente.
Considere, por último, $H:\mathbb{R}^3 \to \mathbb{R}^3$ definida como:

$$H(r,\theta,z) = h_1(r,\theta,z)\widehat{r} + h_2(r,\theta,z)\widehat{\theta} + h_3(r,\theta,z)\widehat{z}$$

a) Encuentre los vectores unitarios $\widehat{r}$, $\widehat{\theta}$ y $\widehat{z}$.

b) Deduzca que H se puede escribir como:

$$H(r,\theta,z) = A_1(r,\theta,z)\widehat{i} + A_2(r,\theta,z)\widehat{j} + A_3(r,\theta,z)\widehat{z}$$

donde $\widehat{i}$, $\widehat{j}$ y $\widehat{z}$ representan los vectores unitarios canónicos.
Encuentre explícitamente A_1, A_2 y A_3.

c) Demuestre que

$$divH = \frac{1}{r}\left(\frac{\partial(rh_1)}{\partial r} + \frac{\partial h_2}{\partial \theta}\right) + \frac{\partial h_3}{\partial z}$$

Solución

a) $\frac{\partial G}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ y $\|\frac{\partial G}{\partial r}\| = 1 \Rightarrow \widehat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

$\frac{\partial G}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$ y $\|\frac{\partial G}{\partial \theta}\| = r \Rightarrow \widehat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

$\frac{\partial G}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ y $\|\frac{\partial G}{\partial z}\| = 1 \Rightarrow \widehat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

b) Utilizando la parte anterior, tenemos que H se puede escribir como:
$H(r,\theta,z) = (h_1\cos\theta - h_2\sin\theta)\widehat{i} + (h_1\sin\theta + h_2\cos\theta)\widehat{j} + h_3\widehat{z}$

De donde,
$A_1 = h_1\cos\theta - h_2\sin\theta$
$A_2 = h_1\sin\theta + h_2\cos\theta$
$A_3 = h_3$

c) Notemos que de la definición de "divergencia" tenemos que:
$divH = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$.

Notemos además que $r = \sqrt{x^2+y^2}$ y que $\theta = \arctan\left(\frac{y}{x}\right)$, entonces:
$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$
$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$, $\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$

Calculemos las derivadas:
$\frac{\partial A_1}{\partial x} = \left(\frac{\partial h_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\cos\theta - h_1\sin\theta\frac{\partial \theta}{\partial x} - \left(\frac{\partial h_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_2}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\sin\theta - h_2\cos\theta\frac{\partial \theta}{\partial x}$
$\frac{\partial A_2}{\partial y} = \left(\frac{\partial h_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\sin\theta + h_1\cos\theta\frac{\partial \theta}{\partial y} + \left(\frac{\partial h_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_2}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\cos\theta - h_2\sin\theta\frac{\partial \theta}{\partial y}$
$\frac{\partial A_3}{\partial z} = \frac{\partial h_3}{\partial z}$

Reemplacemos ahora, las derivadas parciales:
$\frac{\partial A_1}{\partial x} = \left(\frac{\partial h_1}{\partial r}\cos\theta + \frac{\partial h_1}{\partial \theta}\frac{-\sin\theta}{r}\right)\cos\theta - h_1\sin\theta\frac{-\sin\theta}{r} - \left(\frac{\partial h_2}{\partial r}\cos\theta + \frac{\partial h_2}{\partial \theta}\frac{-\sin\theta}{r}\right)\sin\theta - h_2\cos\theta\frac{-\sin\theta}{r}$
$\frac{\partial A_2}{\partial y} = \left(\frac{\partial h_1}{\partial r}\sin\theta + \frac{\partial h_1}{\partial \theta}\frac{\cos\theta}{r}\right)\sin\theta + h_1\cos\theta\frac{\cos\theta}{r} + \left(\frac{\partial h_2}{\partial r}\sin\theta + \frac{\partial h_2}{\partial \theta}\frac{\cos\theta}{r}\right)\cos\theta - h_2\sin\theta\frac{\cos\theta}{r}$
$\frac{\partial A_3}{\partial z} = \frac{\partial h_3}{\partial z}$

Finalmente sumando las 3 ecuaciones anteriores se obtiene:
$divH = \frac{\partial h_1}{\partial r} + \frac{h_1}{r} + \frac{1}{r}\frac{\partial h_2}{\partial \theta} + \frac{\partial h_3}{\partial z}$

Lo cual nos conduce a:

$$divH = \frac{1}{r}\left(\frac{\partial (rh_1)}{\partial r} + \frac{\partial h_2}{\partial \theta}\right) + \frac{\partial h_3}{\partial z}$$

□

13. Sean $f, g : \mathbb{R} \to \mathbb{R}$ tales que f'' y g'' existen. Sea $F(x,y) = f(x+g(y))$ Calcular $\frac{\partial^2 F}{\partial x \partial y}$ y $\frac{\partial^2 F}{\partial y^2}$ ¿Dónde están evaluadas?.
Solución

$\frac{\partial F}{\partial y} = f'(x+g(y))g'(y)$
$\frac{\partial^2 F}{\partial y^2} = f''(x+g(y))[g'(y)]^2 + f'(x+g(y))g''(y)$
$\frac{\partial^2 F}{\partial x \partial y} = f''(x+g(y))g'(y)$ □

14. Sea $u : \mathbb{R}^2 \to \mathbb{R}$ de clase $C^2(\mathbb{R}^2)$. Demuestre que la ecuación

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

toma la forma

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s^2} = 0$$

bajo el cambio de variables $x = e^r$ $y = e^s$

Solución
$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}$
$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial r^2}$
$\Longrightarrow \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}x^2 + \frac{\partial u}{\partial x}x$
Por simetría se tiene que:
$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial y^2}y^2 + \frac{\partial u}{\partial y}y$
Sumando ambas expresiones se obtiene el resultado pedido.

□

15. Sea $z:\mathbb{R}^2\to\mathbb{R}$ de clase $C^2(\mathbb{R}^2)$. Demuestre que la ecuación

$$2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$$

se transforma en

$$3\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial z}{\partial u}=0$$

bajo el cambio de variables $u=x+2y+2 \quad v=x-y-1$

Solución

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$
$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial y}\frac{\partial v}{\partial y}=2\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$

$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)=\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v\partial u}\right)+\left(\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$
$\Longrightarrow \frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$

$\frac{\partial^2 z}{\partial x\partial y}=2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)=2\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v\partial u}\right)-\left(\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$
$\Longrightarrow \frac{\partial^2 z}{\partial x\partial y}=2\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial^2 z}{\partial v^2}$

$\frac{\partial^2 z}{\partial y^2}=2\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)=2\left(2\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v\partial u}\right)-\left(2\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial^2 z}{\partial v^2}\right)$
$\Longrightarrow \frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial u^2}-4\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$

Reemplazando en la expresión se obtiene el resultado pedido.

$\square$