MA2001-2: pauta pregunta n°1, control n°2

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Pregunta 1

Considere el espacio vectorial normado $E = \mathcal{C}^1([0,1];\mathbb{R}) = \{f : [0,1] \longrightarrow \mathbb{R} : f' \textit{ es continua}\}$ dotado de la norma $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. Considere la función:

$$\begin{array}{rccc} J: & E & \longrightarrow & \mathbb{R} \\ & f & \longrightarrow & J(f) = \frac{1}{2}\int_0^1 \left[(f(x))^2 + (f'(x))^2\right] dx \end{array}$$

1. Calcule la derivada parcial $DJ(f;h)$, para $f, h \in E$.
2. Demuestre que $DJ(f;\cdot) : E \longrightarrow \mathbb{R}$ es lineal y continua.
3. Demuestre que la función J es diferenciable.

Solución

1. (2.0 ptos.)

$$\begin{aligned} DJ(f;h) &= \lim_{t\to 0} \frac{J(f+th) - J(f)}{t} \\ &= \lim_{t\to 0} \frac{1}{t}\left\{ \frac{1}{2}\int_0^1 \left[(f(x) + th(x))^2 + (f'(x) + th'(x))^2\right] dx \right. \\ &\quad \left. -\frac{1}{2}\int_0^1 \left[(f(x))^2 + (f'(x))^2\right] dx \right\} \\ &= \lim_{t\to 0} \frac{1}{t}\left\{ \frac{1}{2}\int_0^1 \left[(f(x))^2 + 2tf(x)h(x) + (th(x))^2 + (f'(x))^2 \right.\right. \\ &\quad \left.\left. +2tf'(x)h'(x) + (th'(x))^2 - (f(x))^2 - (f'(x))^2\right] dx \right\} \end{aligned}$$

$$= \lim_{t\to 0} \frac{1}{t} \left\{ \frac{1}{2} \int_0^1 \left[2tf(x)\,h(x) + t^2 (h(x))^2 + 2tf'(x)\,h'(x) + t^2 (h'(x))^2 \right] dx \right\}$$

$$= \int_0^1 [f(x)\,h(x) + f'(x)\,h'(x)]\,dx + \frac{1}{2} \lim_{t\to 0} t \int_0^1 \left[(h(x))^2 + (h'(x))^2 \right] dx$$

$$= \int_0^1 [f(x)\,h(x) + f'(x)\,h'(x)]\,dx + 0$$

Esto último se tiene pues: $\int_0^1 \left[(h(x))^2 + (h'(x))^2 \right] dx \leq \|h\|_\infty^2 + \|h'\|_\infty^2 < \infty$, ya que h, h' son continuas sobre $[0,1]$ ($h \in E$). Por lo tanto,

$$DJ(f;h) = \int_0^1 [f(x)\,h(x) + f'(x)\,h'(x)]\,dx$$

2.

- (1.0 pto.) $DJ(f;\cdot)$ es lineal: sean $h \in E$ y $\lambda \in \mathbb{R}$,

$$\begin{aligned} DJ(f; g+\lambda h) &= \int_0^1 \left[f(x)\,(g(x) + \lambda h(x)) + f'(x)\,(g(x) + \lambda h(x))' \right] dx \\ &= \int_0^1 [f(x)\,g(x) + \lambda f(x)\,h(x) + f'(x)\,g'(x) + \lambda f'(x)\,h'(x)]\,dx \\ &= \int_0^1 [f(x)\,g(x) + f'(x)\,g'(x)]\,dx + \lambda \int_0^1 [f(x)\,h(x) + f'(x)\,h'(x)]\,dx \\ &= DJ(f;g) + \lambda DJ(f;h) \end{aligned}$$

- (1.0 pto.) $DJ(f;\cdot)$ es continua: sea $h \in E$,

$$\begin{aligned} |DJ(f;h)| &= \left| \int_0^1 [f(x)\,h(x) + f'(x)\,h'(x)]\,dx \right| \\ &\leq \int_0^1 |f(x)\,h(x) + f'(x)\,h'(x)|\,dx \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \left[|f(x)\,h(x)| + |f'(x)\,h'(x)| \right] dx \\
&\le \|f\|_\infty \|h\|_\infty + \|f'\|_\infty \|h'\|_\infty \\
&\le \max\{\|f\|_\infty, \|f'\|_\infty\} \left(\|h\|_\infty + \|h'\|_\infty\right) \\
&= Cte \cdot \|h\|_1
\end{aligned}
$$

3. (2.0 ptos.) Veamos que $DJ(f) \equiv DJ(f;\cdot)$ es el diferencial de J en f.

$$
\begin{aligned}
\lim_{h\to 0} \frac{|J(f+h) - J(f) - DJ(f)(h)|}{\|h\|_1} &= \lim_{h\to 0} \frac{1}{\|h\|_1} \Bigg\{ \frac{1}{2} \int_0^1 \left[(f(x) + h(x))^2 + (f'(x) + h'(x))^2 \right] dx \\
&\quad - \frac{1}{2} \int_0^1 \left[(f(x))^2 + (f'(x))^2 \right] dx \\
&\quad - \int_0^1 \left[f(x)\,h(x) + f'(x)\,h'(x) \right] dx \Bigg\} \\
&= \lim_{h\to 0} \frac{1}{\|h\|_1} \Bigg\{ \frac{1}{2} \int_0^1 \Big[(f(x))^2 + 2f(x)\,h(x) + (h(x))^2 \\
&\quad + (f'(x))^2 + 2f'(x)\,h'(x) + (h'(x))^2 - (f(x))^2 \\
&\quad - (f'(x))^2 \Big] dx - \int_0^1 \left[f(x)\,h(x) + f'(x)\,h'(x) \right] dx \Bigg\} \\
&= \lim_{h\to 0} \frac{1}{\|h\|_1} \left\{ \frac{1}{2} \int_0^1 \left[(h(x))^2 + (h'(x))^2 \right] dx \right\} \\
&\le \lim_{h\to 0} \frac{1}{2\|h\|_1} \left\{ \|h\|_\infty^2 + \|h'\|_\infty^2 \right\} \\
&\le \lim_{h\to 0} \frac{1}{\|h\|_1} \left\{ \|h\|_\infty^2 + 2\|h\|_\infty \|h'\|_\infty + \|h'\|_\infty^2 \right\} \\
&= \lim_{h\to 0} \frac{1}{\|h\|_1} \left(\|h\|_\infty + \|h'\|_\infty\right)^2 \\
&= \lim_{h\to 0} \frac{1}{\|h\|_1} \|h\|_1^2 \\
&= \lim_{h\to 0} \|h\|_1 \\
&= 0
\end{aligned}
$$