AUXILIAR # 2 - IN540 - ECONOMETRIA

Pregunta 1

P3, AUX # 2, PRIM 2008

$$(1) \rightarrow Y_t = \beta_1 + \beta_2 X_t + \mu_t$$

$$(2) \begin{cases} Y_t^* = d Y_t \ , \ X_t^* = c X_t & \text{con} \quad d, c > 0 \\ Y_t^* = \beta_1^* + \beta_2^* X_t^* + \mu_t^* \end{cases}$$

a) **forma 1**

$$\hat{\beta}_{MCO} = (X'X)^{-1} X'Y$$

$$\hat{\beta}^*_{2\,MCO} = \frac{\sum_{i=1}^{T} (X_t - \bar{X})(Y_t - \bar{Y})}{\sum_{i=1}^{T} (X_t - \bar{X})^2}$$

$$\begin{aligned} Y_t &= \beta_1 + \beta_2 X_t + \mu_t \\ \bar{Y} &= \beta_1 + \beta_2 \bar{X} \end{aligned} \Big\downarrow \ominus$$

$$(Y_t - \bar{Y}) = \beta_2 (X_t - \bar{X}) + \mu_t$$

$$\hat{\beta}^*_{2\,MCO} = \frac{\sum (X_t^* - \bar{X}^*)(Y_t^* - \bar{Y}^*)}{\sum (X_t^* - \bar{X})^2} =$$

$$= \frac{\sum (c X_t - c \bar{X}^*)(d Y_t - d \bar{Y})}{\sum (c X_t - c \bar{X})^2} =$$

$$= \frac{\cancel{c} \cdot d}{c^{\cancel{2}}} \frac{\sum (X_t - \bar{X})(Y_t - \bar{Y})}{\sum (X_t - \bar{X})^2} = \frac{d}{c} \hat{\beta}_{2\,MCO}$$

$$\hat{\beta}^*_{1\,MCO} = \bar{Y}^* - \hat{\beta}_2^* \bar{X}^* = d \bar{Y} - \frac{d}{\cancel{c}} \hat{\beta}_2 \cancel{c} \bar{X} = d(\bar{Y} - \hat{\beta}_2 \bar{X}) = d \hat{\beta}_{1\,MCO}$$

forma 2

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_T \end{pmatrix} = \begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_T \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_T \end{pmatrix}$$

$$\hat{\beta}_{MCO} = (X'X)^{-1}(X'Y)$$

$$= \Bigg[\underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_T \end{pmatrix}}_{2 \times T} \underbrace{\begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_T \end{pmatrix}}_{T \times 2} \Bigg]^{-1} \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_T \end{pmatrix}}_{2 \times T} \underbrace{\begin{pmatrix} Y_1 \\ \vdots \\ Y_T \end{pmatrix}}_{T \times 1}$$

$$= \begin{pmatrix} T & \sum X_t \\ \sum X_t & \sum X_t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum Y_t \\ \sum X_t Y_t \end{pmatrix}$$

$$= \frac{1}{T \sum X_t^2 - (\sum X_t)^2} \begin{pmatrix} \sum X_t^2 & -\sum X_t \\ -\sum X_t & T \end{pmatrix} \begin{pmatrix} \sum Y_t \\ \sum X_t Y_t \end{pmatrix}$$

$$= \frac{1}{T \sum X_t^2 - (\sum X_t)^2} \begin{pmatrix} \sum X_t^2 \sum Y_t - \sum X_t \sum X_t Y_t \\ -\sum X_t \sum Y_t + T(\sum X_t Y_t) \end{pmatrix}.$$

entonces

$$\hat{\beta}^*_{1\,MCO} = \frac{\sum X_t^{*2} \sum Y_t^* - \sum X_t^* \sum X_t^* Y_t^*}{T \sum X_t^{*2} - (\sum X_t^*)^2}$$

$$= \frac{\sum c^2 X_t^2 \sum d Y_t - \sum c X_t \sum c X_t d Y_t}{T \sum c^2 X_t^2 - (\sum c X_t)^2}$$

$$= \frac{\cancel{c^2} d}{\cancel{c^2}} \cdot \frac{\sum X_t^2 \sum Y_t - \sum X_t \sum X_t Y_t}{T \sum X_t^2 - (\sum X_t)^2} = d\, \hat{\beta}_{1\,MCO}$$

$$\hat{\beta}^*_{2\,MCO} = \frac{-\sum X_t^* \sum Y_t^* + T \cdot \sum X_t^* Y_t^*}{T \sum X_t^{*2} - \left(\sum X_t^*\right)^2}$$

$$= \frac{\cancel{c}d}{c^{\cancel{2}}} \cdot \frac{-\sum X_t \sum Y_t + T \sum X_t Y_t}{T \sum X_t^2 - \left(\sum X_t\right)^2} = \frac{d}{c} \hat{\beta}_{2\,MCO}$$

b) $u_t^* = Y_t^* - \hat{Y}_t^* = d\left(Y_t - \hat{Y}_t\right) = d\, u_t$

c) $SST = \sum \left(Y_t - \bar{Y}\right)^2$

$SSR = \sum u_t^2$

$SSE = \sum \left(\hat{Y}_t - \bar{Y}\right)^2$

$SSR = \hat{\varepsilon}'\hat{\varepsilon}$

$SSE = \hat{\beta}' X' X \hat{\beta}$

$$R^{2*} = 1 - \frac{SSR^*}{SST^*} = 1 - \frac{\sum u_t^{*2}}{\sum \left(Y_t^* - \bar{Y}^*\right)^2} = 1 - \frac{\cancel{d^2} \sum u_t^2}{\cancel{d^2} \sum \left(Y_t - \bar{Y}\right)^2} =$$

$$= 1 - \frac{SSR}{SST} = R^2$$

d) $\hat{\beta}_2^* = \frac{d}{c} \hat{\beta}_2$

$\hookrightarrow Var\left(\hat{\beta}_2^*\right) = \left(\frac{d}{c}\right)^2 Var\left(\hat{\beta}_2\right)$

$\hat{\beta}_1^* = d\, \hat{\beta}_1$

$\hookrightarrow Var\left(\hat{\beta}_1^*\right) = d^2\, Var\left(\hat{\beta}_1\right)$

$$\begin{aligned}
Var(\hat{\beta}_{MCO}) &= Var\left[(X'X)^{-1}X'Y\right] \\
&= Var\left[(X'X)^{-1}X'(X\beta + \mu)\right] \\
&= Var\left[\beta + (X'X)^{-1}X'\mu\right] \\
&= Var\left[(X'X)^{-1}X'\mu\right] \\
&= (X'X)^{-1}X' \; Var(\mu) \; X(X'X)^{-1} \\
&= \sigma^2 (X'X)^{-1} \cancel{X'X} \cancel{(X'X)^{-1}} \\
&= \sigma^2 (X'X)^{-1}
\end{aligned}$$

MCO

$$Y = X\beta + \mu$$

$$\hat{Y} = X\hat{\beta} \qquad / X'$$

$$X'Y = X'X\hat{\beta} \qquad / (X'X)^{-1}$$

$$(X'X)^{-1}X'Y = \cancel{(X'X)^{-1}} \cancel{X'X} \hat{\beta}$$

$$\hookrightarrow \hat{\beta}_{MCO} = (X'X)^{-1}X'Y$$

$$A = (x_1 \; — \; x_T) \qquad (A')' = A$$

$$A' = \begin{pmatrix} x_1 \\ | \\ x_T \end{pmatrix} \qquad (AB)' = B'A'$$

$$(A+B)' = A' + B'$$

(1)

Pregunta 2

(P1, Aux # 3, prim 2008)

$$P_t = \beta_1 + \beta_2 D_t + u_t \quad \text{para } t = 1, \ldots, T$$

a) forma 1

$$\hat{\beta}_{2\,MCO} = \frac{\sum (D_t - \bar{D})(P_t - \bar{P})}{\sum (D_t - \bar{D})^2}$$

$$= \frac{\sum D_t P_t - \sum D_t \bar{P} - \sum \bar{D} P_t + \sum \bar{D}\bar{P}}{\sum D_t^2 - 2\sum \bar{D} D_t + \sum \bar{D}^2}$$

$$= \frac{\sum D_t P_t - \bar{P}\sum D_t - \bar{D}\sum P_t + \bar{D}\bar{P}\cdot T}{\sum D_t^2 - 2\bar{D}\sum D_t + \bar{D}^2 \cdot T}$$

$$= \frac{7500 - \frac{350}{T}500 - \frac{500}{T}350 + \frac{500}{T}\cdot\frac{350}{\cancel{T}}\cdot\cancel{T}}{10000 - 2\cdot\frac{500}{T}500 + \frac{500^2}{T^{\cancel{2}}}\cdot\cancel{T}}$$

$$= \frac{7500 - 350\cdot 500/T}{10000 - 500\cdot 500/T} \underset{T=30}{=} \frac{7500 - 5833,\bar{3}}{10000 - 8333,\bar{3}} = 1$$

$$\hat{\beta}_{1\,MCO} = \bar{P} - \hat{\beta}_2 \bar{D} = \frac{350}{T} - 1\cdot\frac{500}{T} = -5$$

forma 2

$$P_t = \beta_1 + \beta_2 D_t + u_t \qquad t = 1, \dots, T$$

$$\underset{[T\times 1]}{\begin{pmatrix} P_1 \\ | \\ P_T \end{pmatrix}} = \underset{[T\times 2]}{\begin{pmatrix} 1 & D_1 \\ | & | \\ 1 & D_T \end{pmatrix}} \underset{[2\times 1]}{\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}} + \underset{[T\times 1]}{\begin{pmatrix} u_1 \\ | \\ u_T \end{pmatrix}}$$

$$\begin{matrix} \| & & '' \\ Y & = & X\,\beta + u \end{matrix}$$

$$\hat{\beta}_{MCO} = (X'X)^{-1} X'Y$$

$$= \left[\begin{pmatrix} 1 & — & 1 \\ D_1 & — & D_T \end{pmatrix} \begin{pmatrix} 1 & D_1 \\ | & | \\ 1 & D_T \end{pmatrix}\right]^{-1} \left(\begin{pmatrix} 1 & — & 1 \\ D_1 & — & D_T \end{pmatrix} \begin{pmatrix} P_1 \\ | \\ P_T \end{pmatrix}\right)$$

$$= \begin{pmatrix} T & \Sigma D_t \\ \Sigma D_t & \Sigma D_t^2 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma P_t \\ \Sigma D_t P_t \end{pmatrix}$$

$$= \begin{pmatrix} 30 & 500 \\ 500 & 10000 \end{pmatrix}^{-1} \begin{pmatrix} 350 \\ 7500 \end{pmatrix}$$

$$= \frac{1}{300.000 - 250.000} \begin{pmatrix} 10000 & -500 \\ -500 & 30 \end{pmatrix} \begin{pmatrix} 350 \\ 7500 \end{pmatrix}$$

$$= \frac{1}{50000} \begin{pmatrix} 3.500.000 - 500 \cdot 7500 \\ -500 \cdot 350 + 30 \cdot 7500 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

Pregunta 2

la matriz de varianza-covarianza estimada será:

$$\widehat{Var}(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$$

donde $\hat{\sigma}^2 = \dfrac{e'e}{T-k}$ $\qquad$ k: nº regresores

$$Y = X\hat{\beta} + \mu$$

$$\mu'\mu = (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$= (Y' - \hat{\beta}'X')(Y - X\hat{\beta})$$

$$= Y'Y - Y'X\hat{\beta} - \hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$$= \sum P_t^2 - \begin{pmatrix} \sum P_t & \sum P_t D_t \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} - \begin{pmatrix} \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix} \begin{pmatrix} \sum P_t \\ \sum D_t P_t \end{pmatrix} +$$

$$+ \begin{pmatrix} \hat{\beta}_1 & \hat{\beta}_2 \end{pmatrix} \begin{pmatrix} T & \sum D_t \\ \sum D_t & \sum D_t^2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$$

$$= \sum P_t^2 - \hat{\beta}_1 \sum P_t - \hat{\beta}_2 \sum P_t D_t - \hat{\beta}_1 \sum P_t - \hat{\beta}_2 \sum D_t P_t +$$

$$+ (\hat{\beta}_1 (T\hat{\beta}_1 + \hat{\beta}_2 \sum D_t)) + (\hat{\beta}_2 (\hat{\beta}_1 \sum D_t + \hat{\beta}_2 \sum D_t^2))$$

$$= \sum P_t^2 - 2\hat{\beta}_1 \sum P_t - 2\hat{\beta}_2 \sum P_t D_t + \hat{\beta}_1^2 T + 2\hat{\beta}_1\hat{\beta}_2 \sum D_t +$$

$$+ \beta_2^2 \sum D_t^2$$

$$= 6000 - 2 \cdot (-5) \cdot 350 - 2 \cdot 1 \cdot 7500 + 25 \cdot 30 +$$

$$+ 2 \cdot (-5) \cdot 1 \cdot 500 + 1 \cdot 10000$$

$$= 250$$

luego

$$\hat{\sigma}^2 = \frac{u'u}{T-k} = \frac{250}{30-2} \approx 8,93$$

y por lo tanto la matriz de var-covar será:

$$\widehat{Var}(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$$

$$= 8,93 \begin{pmatrix} 30 & 500 \\ 500 & 10000 \end{pmatrix}^{-1}$$

$$= 8,93 \begin{pmatrix} 1/5 & -1/100 \\ -1/100 & 3/5000 \end{pmatrix}$$

$$= \begin{pmatrix} 1,786 & -0,0893 \\ -0,0893 & 0,005358 \end{pmatrix} //$$

b) - No, porque bajo las hipótesis del Modelo Lineal General el estimador MCO es el estimador insesgado lineal de mínima varianza (óptimo) (teorema Gauss-Markov)

- Se pueden encontrar otros estimadores con una varianza menor, pero serán sesgados o no lineales.