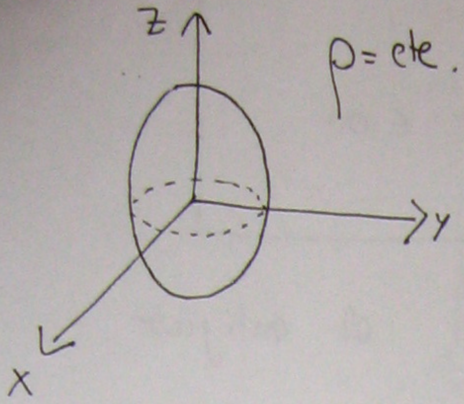
Problema 3 : Elipsoide

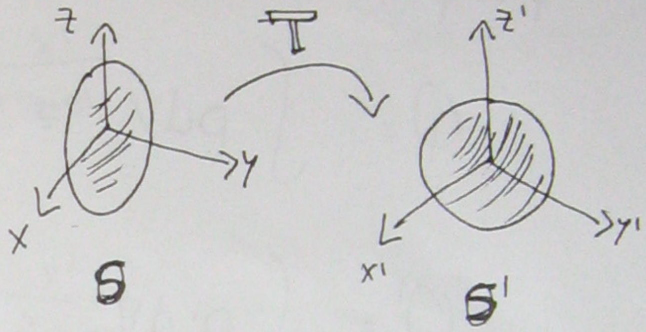


$\rho = cte.$

$$x^2 + y^2 + \left(\frac{z}{\lambda}\right)^2 = 1$$

Aplicamos un cambio (transf.) de coordenadas para pasar de una elipsoide a una esfera.

$$\underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\text{sistema nuevo}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}}_{\text{transformación}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{sistema antiguo}}$$



$\Rightarrow \quad x' = x, \quad y' = y, \quad z' = \frac{z}{\lambda} \quad \Rightarrow$

$$\boxed{x^2 + y^2 + \left(\frac{z}{\lambda}\right)^2 = 1} \quad \text{elipsoide}$$

$$\downarrow T$$

$$\boxed{x'^2 + y'^2 + z'^2 = 1} \quad \text{esfera.}$$

$$S : \quad dV = dx\,dy\,dz \overset{T}{=} dx'\,dy'(\lambda dz') = \lambda\, dV'$$

$$S' = \quad dV' = dx'\,dy'\,dz' = dx\,dy\,\frac{dz}{\lambda} = \frac{1}{\lambda}\, dV$$

$\therefore \quad \boxed{dV = \lambda\, dV'}$ relación entre los elementos de volumen de los sistemas.

Por otro lado, escribimos el campo eléctrico de una esfera maciza en todo el espacio de radio a', con coordenadas primas.

$$\boxed{\vec{E}' = \begin{cases} \rho' \dfrac{a'^3}{3\varepsilon_0} \dfrac{\vec{r}'}{r'^3} & r' > a' \\ \rho' \dfrac{\vec{r}'}{3\varepsilon_0} & r' \leq a' \end{cases}}$$

Ahora debemos pasar esta cantidad al antiguo sistema coordenado

Tenemos que la carga eléctrica se conserva entonces

$Q = Q'$.

$$Q = \int \rho dV = \int \rho \lambda dV'$$

$$Q' = \int \rho' dV' =$$

$$\int \rho dV = \int \rho' dV'$$

$$\int \rho \lambda dV' = \int \rho' dV'$$

$$\rho \lambda = \rho'$$

$$\rho = \frac{\rho'}{\lambda}$$

$$dV = \lambda dV'$$

$$\rho' = \rho \lambda$$

$$\int \rho dV = \int \rho' dV'$$

$$\int \rho \lambda dV' = \int \rho \lambda dV'$$

$$\int \frac{\rho'}{\lambda} dV = \int \rho' \frac{dV}{\lambda}$$

si $Q = Q' \Rightarrow \rho' \int dV' = \rho \lambda \int dV'$

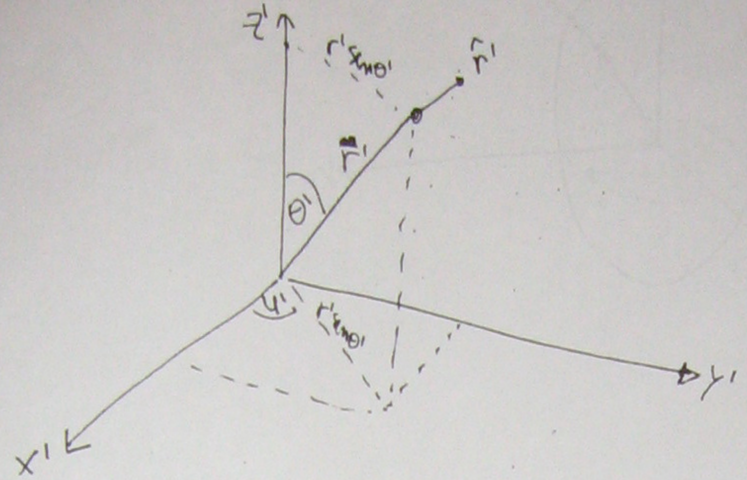
$\Rightarrow \boxed{\rho' = \rho \lambda}$

Por otro lado para el sistema nuevo

$$r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + \left(\frac{z}{\lambda}\right)^2$$

$$r^2 = x^2 + \cancel{z}^2 + z^2 = x'^2 + y'^2 + (\lambda z')^2$$

$$\hat{r}' = \operatorname{sen}\theta'\cos\varphi'\,\hat{x}' + \operatorname{sen}\theta'\operatorname{sen}\varphi'\,\hat{y} + \cos\theta'\,\hat{z}$$



$$\operatorname{sen}\theta' = \frac{\sqrt{x'^2+y'^2}}{r'} \quad ; \quad \cos\varphi' = \frac{x'}{r'\operatorname{sen}\theta'}$$

$$\cos\theta' = \frac{z'}{r'} \quad ; \quad \operatorname{sen}\varphi' = \frac{y'}{r'\operatorname{sen}\theta'}$$

$$\hat{r}' = \cancel{\operatorname{sen}\theta'}\,\frac{x'}{r'\cancel{\operatorname{sen}\theta'}}\,\hat{x} + \cancel{\operatorname{sen}\theta'}\,\frac{y'}{r'\cancel{\operatorname{sen}\theta'}}\,\hat{y} + \frac{z'}{r'}\,\hat{z}$$

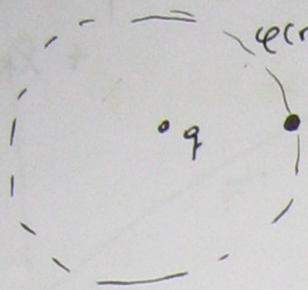
$$r'\hat{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$r'\hat{r}' = x\hat{x} + y\hat{y} + \frac{z}{\lambda}\hat{z} = \vec{r}'$$

$\therefore$ Pasamos la $\vec{E}' \xrightarrow{T} \vec{E}$ ($a=1$

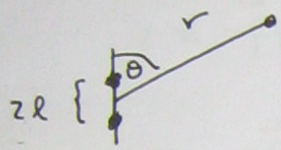
$$\boxed{\vec{E} = \begin{cases} \dfrac{\rho\lambda}{3\varepsilon_0}\left\{ \dfrac{\left(x\hat{x} + y\hat{y} + \left(\frac{z}{\lambda}\right)\hat{z}\right)}{\left(x^2 + y^2 + \left(\frac{z}{\lambda}\right)^2\right)^{3/2}} \right. & r' > 1 \\[2ex] \dfrac{\rho\lambda}{3\varepsilon_0}\left\{ \left(x\hat{x} + y\hat{y} + \left(\frac{z}{\lambda}\right)\hat{z}\right) \right. & r' \leq 1 \end{cases}}$$

Energía de interacción entre dipolos



$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$U = q'\varphi(r) = \frac{qq'}{4\pi\varepsilon_0}\frac{1}{r}$$



$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{2\vec{l}\cos\theta}{r^2}$$

$$\vec{p} = q(2\vec{l})$$

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$\boxed{\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Notación r_i

$$\varphi(\overset{x_i}{x_k}) = \frac{1}{4\pi\varepsilon_0}\frac{p_i x_i}{|x_k|^3} \qquad \varphi(x_i) = \frac{1}{4\pi\varepsilon_0}\frac{p_i x_i}{|x_k|^3}$$

distancia: no depende del índice

$$\vec{E} = -\nabla\varphi$$

$$E_i = -\frac{\partial\varphi}{\partial x_i} = -\partial_i\varphi$$

$$E_i = -\partial_i\varphi(x_j)$$

$$E_i = \frac{-1}{4\pi\varepsilon_0} p_j\,\partial_i\left\{\frac{x_j}{|x_k|^3}\right\}$$

$$\partial_i\{x_j|x_k|^{-3}\} = (\partial_i x_j)|x_k|^{-3} - 3x_j|x_k|^{-4}\,\partial_j|x_k|$$

pero

$$|x_k| = \sqrt{x_k x_k}$$

$$\partial_i |x_k| = \partial_i (x_k x_k)^{1/2} = \frac{1}{2} (x_k x_k)^{-1/2} \left\{ (\partial_i x_k) x_k + x_k (\partial_i x_k) \right\}$$

Notemos que:

$$\frac{\partial x_k}{\partial x_i} = \delta_{ik}$$

$$= \frac{1}{2} \frac{1}{\sqrt{x_k x_k}} \left\{ \delta_{ik} x_k + x_k \delta_{ik} \right\}$$

$$\partial_i |x_k| = \frac{\delta_{ik} x_k}{\sqrt{x_k x_k}} = \frac{x_i}{|x_k|}$$

$$\therefore \partial_i \left\{ x_j |x_k|^3 \right\} = \frac{\delta_{ij}}{|x_k|^3} - \frac{3 x_j}{|x_k|^4} \cdot \frac{x_i}{|x_k|}$$

$$E_i = \frac{-p_j}{4\pi\varepsilon_0} \frac{1}{|x_k|^3} \left\{ \delta_{ij} - 3 \frac{x_i x_j}{|x_k|^2} \right\} \quad / \quad i = j$$

$$E_i = \frac{-p_i}{4\pi\varepsilon_0} \frac{1}{|x_k|^3} \left\{ 1 - \frac{3 x_i x_i}{|x_k|^2} \right\}$$

$$E_i = \frac{1}{4\pi\varepsilon_0} \frac{1}{|x_k|^3} \left\{ 3 \frac{x_i p_i}{|x_k|^2} x_i - p_i \right\}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left\{ 3 \frac{(\vec{p} \cdot \vec{r})}{r^2} \vec{r} - \vec{p} \right\}}$$

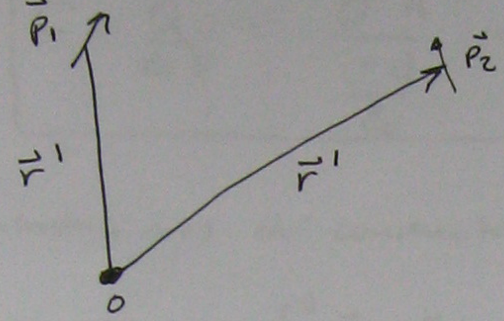
en el origen el dipolo

si no $\vec{r} \to \vec{r} - \vec{r}\,'$

$\vec{r}\,'$: pos del dipolo

$$U(r) = \vec{p} \cdot \nabla \varphi_{ext}$$

$$U(r) = -\vec{p}\ \vec{E}_{ext.}$$

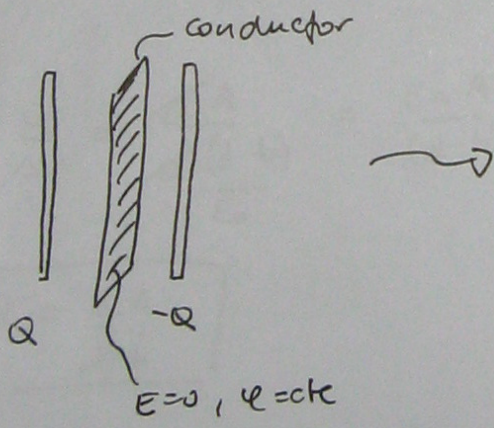


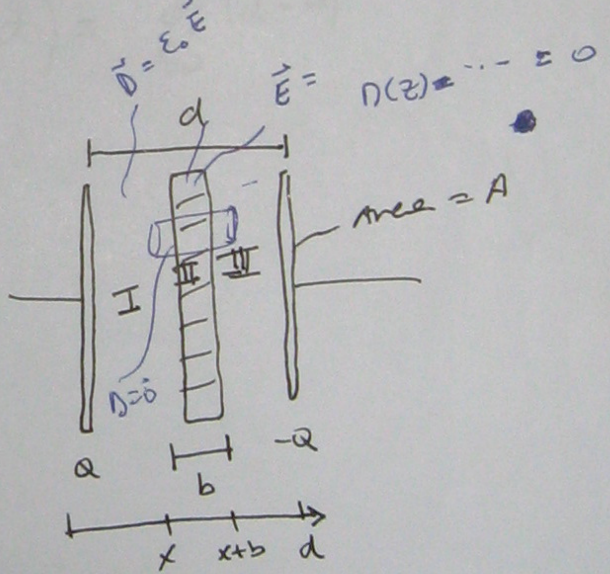
$$U = -\vec{p}_2 \vec{E}_1$$

$$U = \frac{-\vec{p}_2}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}\,'|^3} \left\{ \frac{3\vec{p}_1(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}(\vec{r}-\vec{r}\,') - \vec{p}_1 \right\}$$

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}\,'|^3} \left\{ \vec{p}_1\vec{p}_2 - 3\frac{[\vec{p}_1(\vec{r}-\vec{r}\,')][\vec{p}_2\cdot(\vec{r}-\vec{r}\,')]}{|\vec{r}-\vec{r}\,'|^2}\right.$$

<u>Problema</u> <u>Capacitancia</u>





Antes de introducir ~~el~~ el conductor la capacitancia es:

$$\varphi = +\int_{\infty}^{\vec{r}} \vec{E}\, d\vec{r}$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x} \quad ; \quad \sigma = \frac{Q}{A}$$

$$\Delta V = -\int_{\infty}^{r+} \vec{E}\cdot d\vec{l} \qquad \int_{V-}^{V+} dV = -\int_{d}^{0} \frac{\sigma}{\varepsilon_0}\hat{x}\, dx\, \hat{x}$$

$$\Delta V = -\int_{a}^{b} \vec{E}\cdot d\vec{l} \qquad (V_+ - V_-) = +\frac{\sigma}{\varepsilon_0}\int_0^d dx = \frac{\sigma d}{\varepsilon_0}$$

$$V_+ - V_- = -\frac{\sigma}{\varepsilon_0}(0-d)$$

$$\boxed{V_+ - V_- = \frac{\sigma d}{\varepsilon_0}}$$

$$\boxed{C = \frac{Q}{\Delta V} = \frac{\sigma A}{\frac{\sigma d}{\varepsilon_0}} = \frac{\varepsilon_0 A}{d}}$$

© Introducimos el conductor de espesor $\underline{\underline{b}}$

$$V_+ - V_- = -\int_d^0 \vec{E}\cdot d\vec{x}$$

$$= -\int_x^0 \vec{E}_I \cdot d\vec{x} - \cancel{\int_{x+b}^{x} \vec{E}_{II}\, d\vec{x}} - \int_d^{x+b} \vec{E}_{III}\cdot d\vec{x}$$

$$= \frac{\sigma}{\varepsilon_0}\int_x^0 dx - \frac{\sigma}{\varepsilon_0}\int_d^{x+b} dx$$

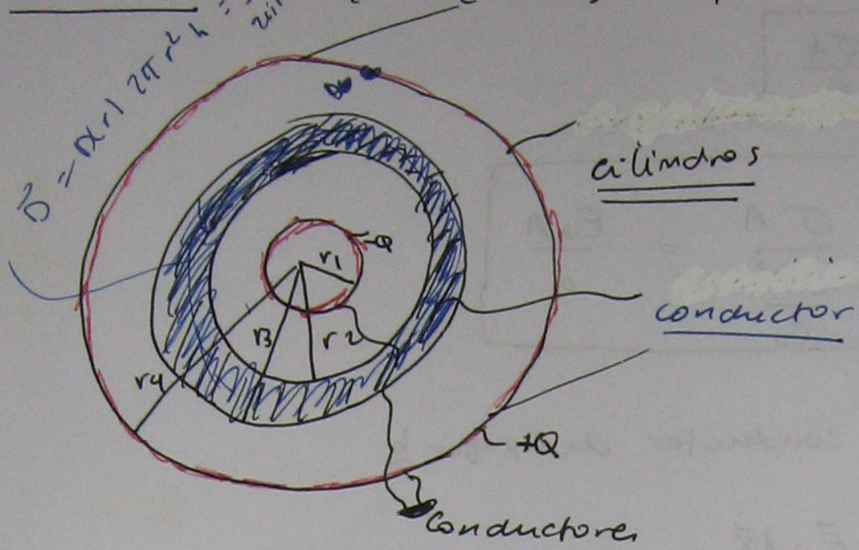
$$= \frac{\sigma}{\varepsilon_0}x - \frac{\sigma}{\varepsilon_0}(x+b-d)$$

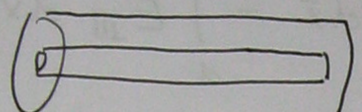
$$\Delta V = V_+ - V_- = \frac{\sigma}{\varepsilon_0}(\cancel{x-x}-b+d) = \frac{\sigma}{\varepsilon_0}(d-b)$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma}(d-b)}{\varepsilon_0}} = \frac{\varepsilon_0 A}{(d-b)}$$

$$\boxed{C = \frac{\varepsilon_0 A}{d-b}}$$

- **Problema** $r_1 < r_2 < r_3 < r_4$





* Gauss

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_T}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{Q}{\varepsilon_0}$$

- $$\vec{E} = \frac{q}{2\pi \varepsilon_0 L} \frac{(-\hat{r})}{r} \quad ; \quad q = Q$$

$$V_+ - V_- = -\int_{r_4}^{r_1} \vec{E} \cdot d\vec{r}$$

$$= -\int_{r_2}^{r_1} \frac{Q}{2\pi\varepsilon_0 L} \frac{(-\hat{r} \cdot \hat{r} \cdot dr)}{r} - \frac{Q}{2\pi\varepsilon_0 L} \int_{r_4}^{r_3} \frac{-\hat{r}\,\hat{r}\,dr}{r}$$

$$= \frac{+Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{r_1}{r_2}\right) + \ln\left(\frac{r_3}{r_4}\right)\right]$$

$$\boxed{\Delta V = \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{r_1 r_3}{r_2 r_4}\right)}$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{r_1 r_3}{r_4 r_2}\right)}$$

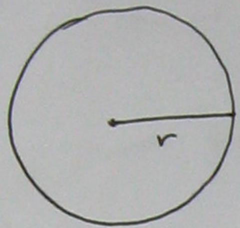
$$\boxed{\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{r_1 r_3}{r_2 r_4}\right)}}$$

• Cascarón uniformemente cargado de carga Q y radio R

$$U = \frac{1}{2} \int_V \rho(r)\, \varphi(r)\, dv + \frac{1}{2} \int_S \sigma(r)\, \varphi(r)\, ds$$

$$U = \frac{1}{2} \int \vec{E} \cdot \vec{D}\, dv = \frac{\varepsilon_0}{2} \int E^2\, dv$$

medios isotropos lineales.



$$U = \frac{1}{2} \int \sigma_0\, \varphi(r)\, ds$$

$$\varphi(R) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R}$$

$$U = \frac{1}{2} \int \sigma_0 \frac{q}{4\pi\varepsilon_0} \frac{1}{R}\, ds$$

$$= \frac{q}{8\pi\varepsilon_0} \frac{1}{R} \int \sigma_0\, ds$$

$$\boxed{U = \frac{q^2}{8\pi\varepsilon_0 R}}$$

$$\underset{r>R}{\vec{E}}(r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r}$$

$$E^2 = \frac{q^2}{(4\pi\varepsilon_0)^2} \frac{1}{r^2}$$

$$U = \frac{\varepsilon_0}{2} \int_0^\infty E^2\, dV$$

$$= \frac{\varepsilon_0}{2} \left(\int_0^R \cancelto{0}{E^2\, dV} + \int_R^\infty E^2\, dv \right)$$

$$= \frac{\varepsilon_0}{2} \frac{q^2}{(4\pi\varepsilon_0)^2} \int_R^\infty \frac{1}{r^4} \underbrace{4\pi r^2\, dr}_{d\Omega}$$

$$U = \frac{\cancel{(4\pi\varepsilon_0)}\, q^2}{2(4\pi\varepsilon_0)^{\cancel{2}}} \int_R^{\infty} \frac{1}{r^2}\, dr = \frac{q^2}{8\pi\varepsilon_0} \left\{ \left. -\frac{1}{r} \right|_R^{\infty} \right\}$$

$$\boxed{U = \frac{q^2}{8\pi\varepsilon_0 R}}$$

lo mismo con una esfera sólida.

$$U = \frac{1}{2} \int_V \rho_0\, \varphi(r)\, dV$$

De una clase anterior

$$\varphi(r) = -\frac{\rho_0 r^2}{6\varepsilon_0} + \frac{\rho_0 R^2}{2\varepsilon_0}$$

$$U = \frac{\rho_0}{2} \int_0^R \left\{ -\frac{\rho_0 r^2}{6\varepsilon_0} + \frac{\rho_0 R^2}{2\varepsilon_0} \right\} 4\pi r^2\, dr$$

$$U = -\frac{4\pi \rho_0^2}{12\varepsilon_0} \int_0^R r^4\, dr + \frac{4\pi \rho_0^2 R^2}{4\varepsilon_0} \int_0^R r^2\, dr$$

$$U = -\frac{4\pi \rho_0^2 R^2}{R\varepsilon_0 \cdot 5} + \frac{4\pi \rho_0^2 R^2}{4\varepsilon_0} \frac{R^3}{3}$$

$$U = \frac{4\pi \rho^2 R^5}{R\varepsilon_0} \left\{ 1 - \frac{1}{5} \right\}$$

$$U = \frac{4\pi \rho_0^2 R^5}{12\varepsilon_0} \frac{5-1}{5} = \boxed{\frac{4\pi \rho_0^2 R^5}{15\varepsilon_0}}$$

$$\vec{E}(r) = \begin{cases} \dfrac{\rho_0 R^3}{3\varepsilon_0} \dfrac{1}{r^2} \hat{r} & r > R \\[2ex] \dfrac{\rho_0 \vec{r}}{3\varepsilon_0} & r \leq R. \end{cases}$$

$$E^2_{r \leq R} = \frac{\rho_0^2 r^2}{9\varepsilon_0^2} \quad ; \quad E^2_{r \geq R} = \frac{\rho_0^2 R^6}{9\varepsilon_0^2 r^4}$$

$$U = \frac{\varepsilon_0}{2} \int_{\text{todo el espacio}} E^2\, dV = \frac{\varepsilon_0}{2} \int_0^R E^2_{r \leq R}\, dV + \frac{\varepsilon_0}{2} \int_R^{\infty} E^2_{r \geq R}\, dV = ¿?$$

debe dar lo mismo !!