where ε is the void fraction defined as the volumetric fraction of the material occupied by the interstitial gas. Strictly equation (1.1) should be written

$$\rho_b = \rho_s(1 - \varepsilon) + \rho_g \varepsilon \tag{1.2}$$

where ρ_g is the gas density. However, since the gas density is typically one-thousandth of that of the solid, equation (1.1) is sufficiently accurate.

Whilst the particles themselves may be compressible, the change in solid density over the range of stresses normally encountered is usually small, so that ρ_s is effectively a constant for a given material. On the other hand, the bulk density is found to vary significantly with applied stress, mainly as a result of rearrangement of the particles. Unfortunately on reduction of the stress, the material does not necessarily expand and as a result the bulk density depends not only on the current stress in the material but also on its stress history. Thus for a given material ρ_s may be treated as a constant but the value of ρ_b will depend on the present *and* past treatment of the material.

When considering the flow pattern within a discharging bunker, it is usual to distinguish between mass and core flow. In a mass flow hopper, all the material is in motion as illustrated in figure 1.1(*a*). In such a hopper the first material to be loaded is the first to be discharged, giving the 'first in, first out' flow pattern. However, mass flow can only occur in comparatively narrow hoppers. If the hopper half-angle α is large the flow will be confined to a narrow core surrounded by stagnant material as illustrated in figures 1.1(*b*) and 1.1(*c*). If the core is narrower than the width of the silo, as in figure 1.1(*c*), the material near the top will cascade down the top surface into the flowing core and will be discharged before material at a lower level, giving the 'first in, last out' pattern. However, the width of the flowing core normally increases with height and for a tall, narrow silo the flowing core will reach the upper parts of the walls. The Draft British Code of Practice (BMHB, 1987) subdivides what is usually known as core flow, i.e. the patterns illustrated by figures 1.1(*b*) and 1.1(*c*), into core flow in the strict sense in which the core reaches the upper parts of the walls, as in figure 1.1(*b*), and internal flow in which the flowing core never reaches the wall, as in figure 1.1(*c*). In this work we will use the older definitions of mass flow, in which all the material is moving, and core flow, in which some of the material is stagnant.

2

The analysis of stress and strain rate

2.1 Introduction

In this chapter we will develop relationships for the analysis of stress and rates of strain which will be familiar to many readers from their knowledge of fluid mechanics or elasticity. Such readers may wish to proceed directly to chapter 3, but their attention is drawn to §2.3 and §2.5 in which the sign conventions used in this book are defined, since these differ from those commonly used in fluid mechanics.

The nature of forces and stresses is discussed in §2.2 and in particular we note that force is a vector but that stress is a somewhat more complicated quantity and cannot therefore be resolved by the familiar techniques of vector resolution. The simplest method for determining the stress components on a particular plane is known as Mohr's circle and this is derived in §2.3 and compared with alternative methods in appendix 1.

Forces are generated as a result of stress gradients and these are related to the acceleration of the material by Euler's equation which is derived in §2.4. Finally in §2.5, we define the strain rate in terms of the velocity gradients and note that strain rates, like stresses, can be analysed by means of a Mohr's circle.

In an attempt to reduce the tedium of this chapter, most of the derivations are presented only for Cartesian co-ordinates and the results for other co-ordinate systems are given, without derivation, in the appendices.

2.2 Force, stress and pressure

It is assumed that the reader is fully familiar with the concept of force and with the fact that force is a vector. As a consequence of its vectorial nature, a force F can be expressed in terms of its components F_x, F_y and F_z parallel to the three co-ordinate directions and by convention these components are taken to be positive when acting in the direction of the co-ordinate increasing. Forces can be resolved in any chosen direction by the techniques of vector algebra but it is more usual to rely on a graphical construction known as the triangle of forces. This construction is, however, so simple that it is often possible to write down the answer by inspection without the necessity of drawing the diagram itself. In particular the component of a force F in a direction inclined to it by the angle θ is $F \cos \theta$, a result sometimes known as the cosine law of vector resolution.

The concept of stress is less familiar and is best illustrated by considering an elementary cuboid with edges parallel to the co-ordinate directions as shown in figure 2.1. It is usual to name the faces of such a cuboid according to the directions of their normals and there are therefore two x-faces as shown in the figure. On each face there may be a force and we will denote that acting on one of the x-faces by F_x. Since the cuboid is of infinitesimal size the force on the other x-face will not differ significantly. The force F_x will not necessarily be normal to the x-face and we can resolve it into its components in the three co-ordinate directions, F_{xx}, F_{xy} and F_{xz}. Dividing by the area of the x-face, A_x, we obtain the stresses on that face and it is usual to distinguish between the normal stress σ_{xx}, obtained from F_{xx}, and the other two stresses which are called shear stresses and denoted by τ_{xy} and τ_{xz}.

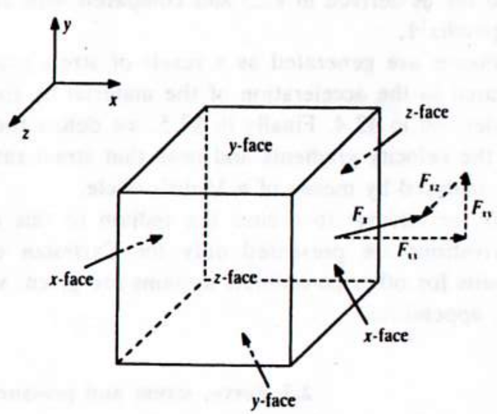

Figure 2.1 Components of force acting on the face of an elementary cuboid.

There are similarly three stress components on each of the two remaining pairs of faces, so that in total we have nine stress components which may be written in the form,

$$\begin{matrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{matrix}$$

It should be noted that in this formulation, the first subscript refers to the face on which the stress acts and the second subscript to the direction in which the associated force acts. A vector has three components in three-dimensional space and it is therefore clear that a stress, having nine components, cannot be a vector.

The components of a stress in any other set of co-ordinate directions can be obtained by matrix manipulations or by the techniques of tensor analysis. Fortunately most of the problems with which we are concerned are essentially two-dimensional; for such systems the very much simpler device known as Mohr's circle can be used and this has particular advantages as it fits conveniently with the basic relationships governing the behaviour of granular materials. Mohr's circle is considered in the next section and the matrix methods of co-ordinate transformation are given in appendix 1.

Circumstances can occur in which all three normal stresses are equal and all the shear stresses are zero. This is more common in the field of fluid mechanics and is known as a state of isotropic pressure. Pressure, which is a scalar since it acts equally in all directions, is therefore a particular case of a stress. Unfortunately, the words 'pressure' and 'stress' tend to be used indiscriminately. For example the Draft British Code of Practice (BMHB, 1987) recommends the use of 'stress' within the material but denotes the stresses exerted on the containing walls as 'pressures'. This usage is contrary to that commonly found in mechanics and will be avoided in this book. We will use the word 'stress' to apply to both internal and external stresses and reserve the word 'pressure' to the cases when the stress state is isotropic or when we need to consider the motion of the interstitial medium, which is usually air.

2.3 Two-dimensional stress analysis – Mohr's circle

Many of the problems of industrial importance have sufficient symmetry, either planar or cylindrical, to make a two-dimensional analysis realistic.

This is a great convenience as the manipulation of stresses in two-dimensional systems is very much easier than in three dimensions. The rather more complicated analysis of stress in three-dimensional systems is outlined in appendix 1.

The method we will use is known as Mohr's circle. This method does, however, have one disadvantage in that it requires a different sign convention from that required for matrix or tensorial manipulation of stresses. Some authors have attempted to combine the sign conventions by unsatisfactory devices such as reversing the signs of shear stresses if the subscripts are in alphabetical order. It is the opinion of the present author that it should be acknowledged that different sign conventions are necessary and that one should keep to the one appropriate for the technique in use.

Since granular materials can only rarely take tension, it is convenient to take compressive stresses as positive and, having selected this convention, the use of Mohr's circle requires that shear stresses should be taken as positive when acting on the element in an anticlockwise direction. Recalling that Mohr's circle is applicable only to two-dimensional situations, we can illustrate our sign convention by means of figure 2.2. The directions in which the stresses acting on the element are numerically positive are shown by the arrows in this figure.

If we take moments about an axis normal to the paper we find that, for stability,

$$\tau_{xy} = -\tau_{yx} \tag{2.3.1}$$

and thus the shear stresses occur as a complementary pair.

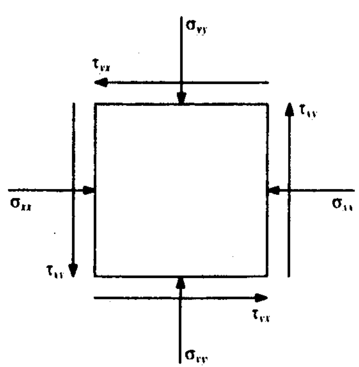

Figure 2.2 Definition of stresses.

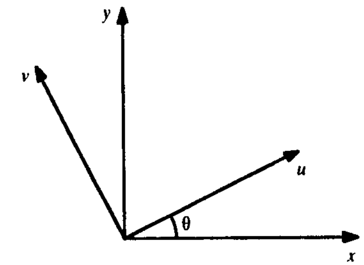

Figure 2.3 Definition of co-ordinate axes.

Let us consider a set of Cartesian axes (x,y) and a second set (u,v) where the u-axis is inclined at angle θ anticlockwise from the x-axis as shown in figure 2.3. Our objective is to predict the stress components σ_{uu}, τ_{uv}, σ_{vv} and τ_{vu} from the known values of σ_{xx}, τ_{xy}, σ_{yy} and τ_{yx}. We can do this by considering a wedge-shaped element of unit depth normal to the paper having faces parallel to the x and y co-ordinates and a face inclined at angle θ anticlockwise from the x-face as shown in figure 2.4. The stresses on the x- and y-faces are (σ_{xx},τ_{xy}) and (σ_{yy}, τ_{yx}) and are positive in the directions shown. The stresses on the remaining face are strictly σ_{uu} and τ_{uv} but they will be denoted simply by σ and τ for convenience.

If we take the area of the hypotenuse plane to be unity, the area of the x-face will be $\cos\theta$ and that of the y-face will be $\sin\theta$. Thus the forces on the x-face are $\sigma_{xx}\cos\theta$ and $\tau_{xy}\cos\theta$ and those on the y-face are $\sigma_{yy}\sin\theta$ and $\tau_{yx}\sin\theta$.

Resolving the forces in the direction of σ we have

$$\sigma = \sigma_{xx}\cos\theta\cos\theta - \tau_{xy}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\sin\theta + \tau_{yx}\sin\theta\cos\theta \tag{2.3.2}$$

and resolving in the direction of τ gives

$$\tau = \sigma_{xx}\cos\theta\sin\theta + \tau_{xy}\cos\theta\cos\theta - \sigma_{yy}\sin\theta\cos\theta + \tau_{yx}\sin\theta\sin\theta \tag{2.3.3}$$

Substituting from equation (2.3.1) and recalling that

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{2.3.4}$$

and that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{2.3.5}$$

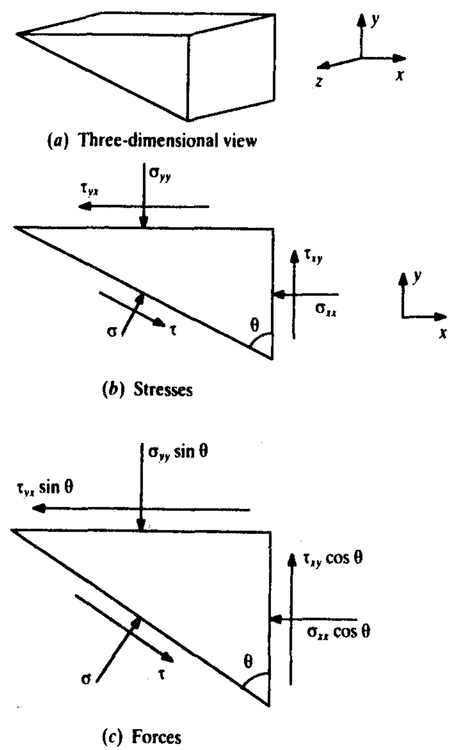

Figure 2.4 Stresses and forces on a wedge-shaped element.

we have

$$\sigma = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (2.3.6)$$

and

$$\tau = \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2.3.7)$$

Defining the symbols p, R and λ by

$$p = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) \qquad (2.3.8)$$

$$R^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2 \qquad (2.3.9)$$

$$\tan 2\lambda = \frac{2\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})} \qquad (2.3.10)$$

equations (2.3.6) and (2.3.7) take the form

$$\sigma = p + R\cos(2\theta + 2\lambda) \qquad (2.3.11)$$

$$\tau = R\sin(2\theta + 2\lambda) \qquad (2.3.12)$$

It can be seen that these two equations define a circle on (σ,τ) axes which is known as Mohr's circle. The circle has its centre at the point $(p,0)$ and its radius is R as shown in figure 2.5. Every point on the circle represents the combination of σ and τ on some plane and in particular the stresses on the x- and y-planes, (σ_{xx},τ_{xy}) and (σ_{yy},τ_{yx}) are marked by the points X and Y respectively. From equations (2.3.11) and (2.3.12) we see that the stresses on a plane inclined at θ anticlockwise from the x-plane are given by the end of the radius inclined at 2θ anticlockwise from the radius to the point X. In particular the stresses on the y-plane, for which θ is 90° are therefore given by the other end of the diameter from the point X. Inevitably therefore, $\tau_{xy} = -\tau_{yx}$ and it is this result that necessitates the use of a sign convention in which complementary shear stresses are equal and opposite and which prohibits the use of the sign convention required for matrix or tensor manipulation.

The stresses on the u-plane (σ,τ), or more correctly (σ_{uu},τ_{uv}), are given by the point U on figure 2.5. We see from equation (2.3.11) and (2.3.12) that the radius to the point U is inclined at an angle 2θ to the radius to point X. Thus, we move round Mohr's circle in the same direction as we rotate our axes but through *twice* the angle. If we had

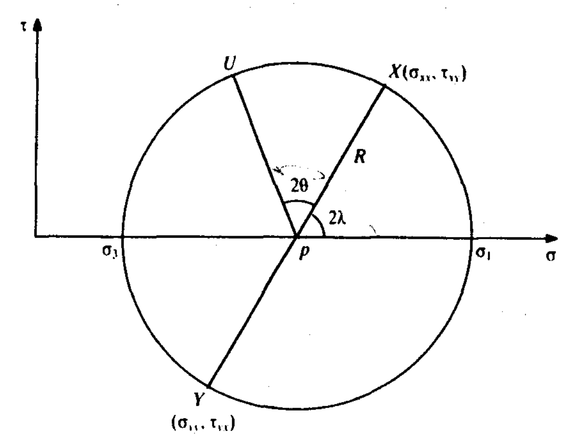

Figure 2.5 Mohr's circle for stresses.

taken as a sign convention that normal stresses were positive when compressive and that shear stresses were positive when clockwise, we would have found that rotation in Mohr's circle was in the opposite direction to that in physical space. This is clearly less convenient than the sign convention we have adopted.

There are two planes of particular interest, namely those on which the shear stress is zero. These are indicated in figure 2.5 and are known as the principal planes. The corresponding stresses σ_1 and σ_3 are called the major and minor principal stresses. From the figure it can be seen that the major principal plane lies at an angle λ clockwise from the x-plane and that the minor principal plane lies at an angle $90 - \lambda$ anticlockwise from the x-plane. Since the principal planes lie at opposite ends of a diameter, they are inevitably at right-angles to each other.

We can illustrate this analysis with a simple example. Let us consider a situation in which $\sigma_{xx} = 12$ kN m^{-2}, $\sigma_{yy} = 4$ kN m^{-2} and $\tau_{xy} = 3$ kN m^{-2}. We can identify the points X and Y since they have co-ordinates (12,3) and (4,−3) and plot them on a Mohr's diagram as in figure 2.6. These two points lie on opposite ends of the diameter, which can therefore be drawn. The centre of the circle can be found by inspection or from equation (2.3.8) as the point (8,0). Consideration of the triangle OXA or equation (2.3.9) gives $R = 5$ kN m^{-2}. Thus the principal stresses are 8 ± 5 kN m^{-2} i.e. $\sigma_1 = 13$ kN m^{-2} and $\sigma_3 = 3$ kN m^{-2}. Also by inspection $\tan 2\lambda = 3/4$ or $\lambda = 18.43°$. Thus the major principal stress acts in a direction inclined at 18.43° clockwise from the x-axis. We can find the stresses on a plane at 30° anticlockwise from the x-plane by constructing the radius at $2 \times 30° = 60°$ anticlock-

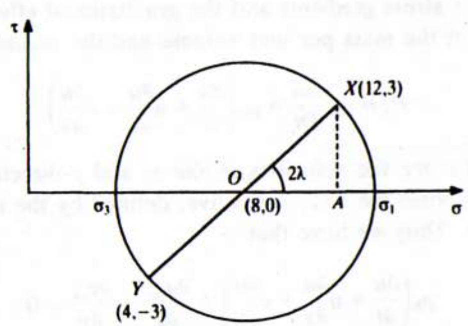

Figure 2.6 Example of the use of Mohr's circle.

wise from the radius to point X. The co-ordinates of the end of this radius, (σ_{uu}, τ_{uv}) are clearly

$$\sigma_{uu} = p + R\cos(2\theta + 2\lambda) = 8 + 5\cos(60 + 36.86) = 7.40 \text{ kN m}^{-2}$$

$$\tau_{uv} = R\sin(2\theta + 2\lambda) = 4.96 \text{ kN m}^{-2}.$$

The stresses on the perpendicular plane are

$$\sigma_{vv} = 8 + 5\cos(36.86 + 60 + 180) = 8.60 \text{ kN m}^{-2}$$

$$\tau_{vu} = 5\sin(36.86 + 60 + 180) = -4.96 \text{ kN m}^{-2}.$$

It should be noted that Mohr's circle can be used as a graphical construction, in which case the results are inevitably approximate, or can be used as the basis for a geometrical analysis, as above, in which case the results are exact.

It can be seen that this two-dimensional analysis will be valid in real, three-dimensional, space if the force on the z-plane of figure 2.4 has no component in the x- or the y-direction. Thus the shear stresses τ_{zx} and τ_{zy} must be zero and consequently the stress σ_{zz} must be a principal stress. Therefore, in general we can draw a set of three nesting Mohr's circles as shown in figure 2.7. Each circle represents rotation about one of the three principal axes. The three principal stresses are given by the intersections of the circles with the σ axis and are known (despite the rules of English grammar) as the major, the intermediate and the minor principal stresses and are conventionally denoted by σ_1, σ_2 and σ_3 where $\sigma_1 > \sigma_2 > \sigma_3$. In almost all the analyses in this book we will be concerned only with the Mohr's circle containing both the major and minor principal stresses, as it is found

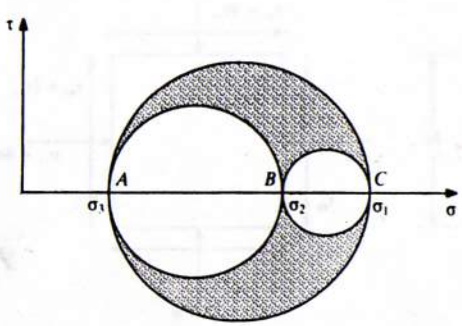

Figure 2.7 Mohr's circles for the rotation about the three principal axes.

experimentally that the value of the intermediate principal stress is irrelevant in a great many of the situations of interest. Thus, as a result of the nature of granular materials we can normally work in two dimensions and ignore the intermediate principal stress and its associated Mohr's circles.

The sign convention defined above cannot, in general, be used for three-dimensional systems, but can be adapted for systems of axial symmetry. Here we may use the convention as defined above in the positive quadrant, but in the negative quadrant we must work in mirror image as this quadrant becomes the positive quadrant when viewed from behind the paper.

2.4 The stress gradient and Euler's equation

If we consider the two-dimensional infinitesimal element shown in figure 2.8, we can note that the stresses on opposite sides of the element will differ if there is a stress gradient. We will denote these stress differences by $\delta\sigma_{xx}$ etc.

Recalling that the lengths of the sides are δx and δy and taking unit distance normal to the paper, we can evaluate the force per unit volume in the x-direction, P_x, by

$$P_x\delta x\delta y = \sigma_{xx}\delta y + \tau_{yx}\delta x - (\sigma_{xx} + \delta\sigma_{xx})\delta y - (\tau_{yx} + \delta\tau_{yx})\delta x \quad (2.4.1)$$

or

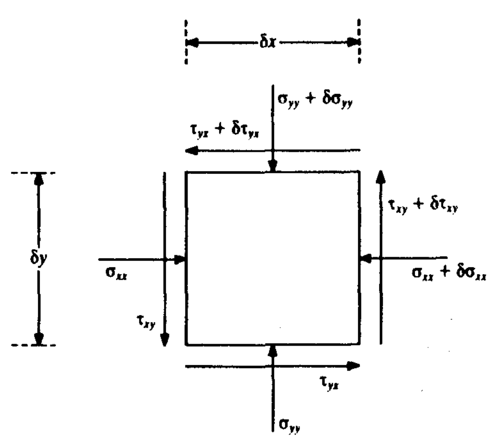

Figure 2.8 Stresses on an infinitesimal element.

$$P_x = -\frac{\partial\sigma_{xx}}{\partial x} - \frac{\partial\tau_{yx}}{\partial y} \quad (2.4.2)$$

Similarly there is a force in the y-direction given by

$$P_y = -\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} \quad (2.4.3)$$

It should be noted that in these equations all stresses appear as their derivatives with respect to their first subscript, and that a particular component of P is obtained by summing all the terms with the appropriate second subscript. The difference in sign in equations (2.4.2) and (2.4.3) results solely from the sign convention imposed upon us by the use of Mohr's circle.

In most of the analyses of chapters 3 to 7, we will be considering the statics of a mass of material and we will take a set of Cartesian axes with x measured horizontally to the left and y measured vertically downwards. Thus the forces P_x and P_y are respectively 0 and $-\rho_b g$ where ρ_b is the bulk density. Thus, from (2.4.2) we have

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0 \quad (2.4.4)$$

and from (2.4.3)

$$\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yx}}{\partial x} = \rho_b g \quad (2.4.5)$$

where we have made the substitution, $\tau_{xy} = -\tau_{yx}$ for subsequent convenience.

If the material is in motion, the force per unit volume, resulting from both the stress gradients and the gravitational effects, will equal the product of the mass per unit volume and the acceleration. Thus,

$$P_x = \rho_b\frac{Du}{Dt} = \rho_b\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \quad (2.4.6)$$

where u and v are the velocities in the x- and y-directions, t is time and Du/Dt denotes the total derivative, defined by the second part of this equation. Thus we have that

$$\rho_b\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0 \quad (2.4.7)$$

and similarly in the y-direction