Given: Turbulent boundary layer with velocity profile, $\frac{u}{U} = \eta^{1/6}$; $\eta = \frac{y}{\delta}$.

Find: Expressions for δ/x, C_f, using momentum integral equation. Compare with results of "1/7-power" profile, Section 9-5.2.

Solution: The momentum integral equation is

Computing equation: $$-\delta \overbrace{\frac{\partial p}{\partial x}}^{=0(1)} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\rho u\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy$$

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) δ is a function of x only; $\delta = 0$ at $x = 0$
(3) Incompressible flow
(4) $\tau_w = 0.0233\,\rho U^2 (\nu/U\delta)^{1/4}$

Then
$$\tau_w = U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy - \int_0^\delta u\rho u\,dy = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)\,dy$$
or
$$\tau_w = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \rho U^2 \frac{d\delta}{dx}\beta$$

Evaluating β,
$$\beta = \int_0^1 \eta^{1/6}(1-\eta^{1/6})\,d\eta = \left[\frac{6}{7}\eta^{7/6} - \frac{6}{8}\eta^{8/6}\right]_0^1 = \frac{6}{56}$$

Substituting
$$0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \rho U^2\frac{d\delta}{dx}\beta \quad \text{or} \quad \delta^{1/4}d\delta = \frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}dx$$

Integrating $\quad \frac{4}{5}\delta^{5/4} = \frac{0.0225}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x + c$, but $c=0$, since $\delta = 0$ at $x = 0$.

Thus $$\delta = \left[\frac{5}{4}\,\frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x\right]^{4/5} = 0.353\left(\frac{\nu}{U}\right)^{\frac{1}{5}}x^{\frac{4}{5}}$$

and
$$\frac{\delta}{x} = 0.353\left(\frac{\nu}{Ux}\right)^{\frac{1}{5}} = \frac{0.353}{(Re_x)^{1/5}}$$ $\frac{\delta}{x}$

Also
$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}}}{\frac{1}{2}\rho U^2} = 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{x}{\delta}\right)^{\frac{1}{4}}$$
$$= 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{1}{0.353}\right)^{\frac{1}{4}}\left[\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}\right]^{\frac{1}{4}}$$

$$C_f = \frac{0.0605}{(Re_x)^{1/5}}$$ C_f

Comparing:	$\delta/x\,(Re_x)^{1/5}$	$C_f\,(Re_x)^{1/5}$
1/6-power	0.353	0.0605
1/7-power	0.382	0.0594