Aux. 20/11
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P1) $\Omega \subseteq \mathbb{R}^N$ acotado de frontera regular, $T>0$.
Definamos $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = (\bar{\Omega}\times\{0\}) \cup \{\partial\Omega \times (0,T]\}$
Suponga que $u \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\bar{\Omega}_T)$ satisface

$$u_t \geq \Delta u + C(x,t)\cdot u$$

con $c \in L^\infty(\Omega_T)$. Demuestre que si $u \geq 0$ en $\Gamma_T \Rightarrow u \geq 0$ en Ω_T

Sol: veamos primero el caso $c<0$ en Ω_T, supongamos $\exists (x^*, t^*) \in \Omega_T$ tal que $u(x^*,t^*)<0$.

Por compacidad y continuidad $\exists (\bar{x},\bar{t})$ en Ω_T mínimo de u, se tiene que $u(\bar{x},\bar{t})<0$.

$$\left.\begin{array}{ll} \text{Si } \bar{t}=T \Rightarrow & \frac{\partial u}{\partial t}(\bar{x},\bar{t}) \leq 0 \\ \bar{t}<T \Rightarrow & D_{x,t}u(\bar{x},\bar{t})=0 \Rightarrow \frac{\partial u}{\partial t}(\bar{x},\bar{t})=0 \end{array}\right\} \Rightarrow \frac{\partial u}{\partial t}(\bar{x},\bar{t})=0$$

Como $\bar{x}$ es mínimo $\Rightarrow \Delta u(\bar{x},\bar{t}) \geq 0$ $\quad (\operatorname{tr} H_x(u(\bar{x},\bar{t})) \geq 0)$

$$\underbrace{\Delta u(\bar{x},\bar{t})}_{\not<} + C(\bar{x},\bar{t})\cdot u(\bar{x},\bar{t}) \leq u_t(\bar{x},\bar{t}) \leq 0$$

$$\Rightarrow \underbrace{C(\bar{x},\bar{t})}_{<0}\cdot u(\bar{x},\bar{t}) \leq 0 \Rightarrow u(\bar{x},\bar{t}) \geq 0$$

Para el caso general, consideramos $\omega(x,t) = e^{\alpha t}u(x,t)$

$$\Rightarrow \frac{\partial \omega}{\partial t} = \alpha e^{\alpha t}u(x,t) + e^{\alpha t}\frac{\partial u}{\partial t}(x,t) = \alpha\cdot\omega + e^{\alpha t}\frac{\partial u}{\partial t}(x,t)$$

$$\geq \alpha\cdot\omega + e^{\alpha t}(\Delta u + c(x,t)\cdot u)$$

$$= \alpha\omega + c(x,t)\cdot\omega + \Delta(e^{\alpha t}u)$$

$$= \omega\underbrace{(\alpha + c(x,t))}_{\tilde{c}(x,t)} + \Delta(\omega) \qquad c \in L^\infty(\Omega_T) \Rightarrow \tilde{c} \in L^\infty(\Omega_T)$$

como $\omega \geq 0$ en $\Gamma_T \Rightarrow \omega = e^{\alpha t}u > 0$ en $\Omega_T \Rightarrow u(x,t) \geq 0$ en Ω_T

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P2 Sea $f:\mathbb{R}\to\mathbb{R}$ Lipschitz. Sean $u,v\in \mathcal{C}^{2,1}(\Omega_T)\cap\mathcal{C}(\bar\Omega_T)$ que satisfacen

$$u_t \geq \Delta u + f(u) \quad \text{en } \Omega_T$$
$$v_t \leq \Delta v + f(v)$$

Pruebe que si $u\geq v$ en $\Gamma_T \Rightarrow u\geq v$ en Ω_T

Sol: sea $w = u - v \Rightarrow w_t = u_t - v_t$

$$\geq \Delta u + f(u) - \Delta v - f(v)$$
$$= \Delta w + c(x,t)\cdot w$$

donde
$$c(x,t) = \begin{cases} \dfrac{f(u(x,t)) - f(v(x,t))}{u(x,t)-v(x,t)} & u\neq v \text{ en } \Omega_T \\ \pi & \sim \end{cases}$$

f Lipschitz $\Rightarrow |c(x,t)|\leq K$ si $u\neq v$, luego, $M = \max\{\pi, K\}$ y se concluye //

P3 $\Omega\subseteq\mathbb{R}^n$ acotado de frontera regular, donde los ptos. de $\partial\Omega$ satisfacen la prop. de la bola interior.

Sea $u\in \mathcal{C}^{2,1}(\Omega\times(0,\infty))\cap\mathcal{C}^{1,0}(\bar\Omega\times(0,\infty))\cap\mathcal{C}(\bar\Omega\times[0,\infty))$ una solución de:

$$\begin{cases} u_t - \Delta u + u^{1/2} = 0 & \text{en } \Omega\times(0,\infty) \\ 0\leq u\leq M & \text{en } \bar\Omega\times[0,\infty) \\ \dfrac{\partial u}{\partial n} = 0 & \text{en } \partial\Omega\times(0,\infty) \end{cases}$$

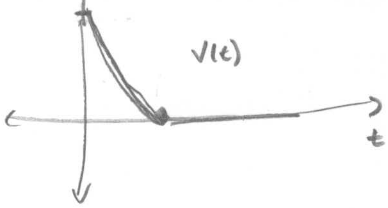
Queremos probar que $\exists\, T = T(M)$ tq $u\equiv 0$ en $\Omega\times(T,\infty)$.

a) Pruebe que la ec. $\dfrac{dv}{dt} + v^{1/2} = 0$ admite solución $\mathcal{C}^1$ y tal que $\exists\, T = T(M)$ $v(t) = 0$ $\forall t > T(M)$.

Sol: $w = v^{1/2} \Rightarrow w^2 = v \Rightarrow 2w\dfrac{dw}{dt} = \dfrac{dv}{dt} = -v^{1/2} = -w$

$\Rightarrow \dfrac{dw}{dt} + \dfrac{1}{2} = 0 \Rightarrow w(t) = -\dfrac{1}{2}t + M^{1/2}$ en $[0, 2M^{1/2})$

$\Rightarrow v(t) = (M^{1/2} - 1/2\, t)^2$



b) Considere $w = v - u$ y concluya.

Sol: Veamos que si $w \geq 0$ en $\Omega \times (0,\infty) \Rightarrow u \equiv 0$ en $\Omega \times (T,\infty)$

$$w \geq 0 \iff v \geq u \quad \text{en} \quad \Omega \times (0,\infty)$$

$$\Rightarrow u \leq 0 \quad \text{en} \quad \Omega \times (T,\infty) \overset{u \geq 0}{\Rightarrow} u \equiv 0 \quad \text{en} \quad \Omega \times (T,\infty).$$

w satisface:

$$\begin{aligned} w_t &= v_t - u_t \\ &= -v^{1/2} - \Delta u + u^{1/2} \\ &= \Delta w - (v^{1/2} - u^{1/2}) \end{aligned}$$

Supongamos que $\exists (\bar{x}, \bar{t})$ tq $w(\bar{x},\bar{t}) < 0$ y consideremos $\Omega \times (0, \bar{t})$

Por compacidad y continuidad $\exists (x^*, t^*) \in \overline{\Omega_{\bar{t}}}$ donde w alcanza su mínimo, luego $w(x^*,t^*) < 0$.

Como antes, $\frac{\partial}{\partial t} w(x^*,t^*) \overset{\text{Hopf.}}{\leq} 0$ y tambien $\Delta w(x^*,t^*) \geq 0$

Pero,

$$\begin{aligned} 0 \geq w_t(x^*,t^*) &= \Delta w(x^*,t^*) - (v^{1/2}(x^*,t^*) - u^{1/2}(x^*,t^*)) \\ &\geq -(v^{1/2} - u^{1/2}) > 0 \end{aligned}$$

$$w(x^*,t^*) < 0 \iff v(x^*,t^*) < u(x^*,t^*) \Rightarrow v^{1/2}_{(*)} < u^{1/2}_{(*)}$$