Notes on Dynamic Optimization in Continuous Time*

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In economics, most of decision making is assumed to be done through some optimization process. This note presents the basis to solve dynamic optimization under certainty in continuous time. It also presents simple derivations of the basic recipes. There are other frameworks for dynamic optimization, this is the case of stochastic optimization and/or discrete time problems that will not be discussed here.

The three methods shown here are the calculus of variations, the maximum principle, and dynamic programming. Under some conditions, all the methods yield the same solution and it is possible to interpret their connections.

Good exposition of dynamic optimization are in Dixit (1976) and Intriligator (1971). Somewhat more advanced is Kamien and Schwartz (1981), which is also the basis for these notes. For a rigorous mathematical approach you can see Fleming and Rishel (1975). Dynamic programming methods in discrete time are developed in Bertsekas (1976), Sargent (1987) and Stokey and Lucas (1989). Stochastic intertemporal optimization can be found in Fleming and Rishel (1975).

These notes are written to show the simplest derivation of the basic results, and they do not pretend to be a rigorous treatment of the topic. Rigor has been sacrificed in favor of intuition. In particular discussions as existence, uniqueness and sufficiency have been avoided in order to concentrate on the basic tools. Readers interested in more details are encouraged to look at the references for more details.

The general problem to be solved is the following:

[P.1]

$$\max J \equiv \int_0^T F(x(t), u(t), t)dt \tag{1}$$

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subject to:

$$\dot{x} = G(x(t), u(t), t) \tag{2}$$

$$\Psi(x(t), u(t), t) \geq 0 \tag{3}$$

$$x(t = 0) = x_0 \tag{4}$$

$$x(t = T) = x_T \tag{5}$$

$x(t)$ is the state variable. It defines the state of the system at time t, and its motion is given by equation (2). For example, in many problems it represents the capital stock, public debt, the stock of human capital, and in general stock variables. In rational expectations models these are the variables that cannot jump.

The variable $u(t)$ is the control variable. It is piecewise continuous: It is continuous in [0,T] except at a finite number of points t_1, t_2, $\ldots$, t_m which belong to the interior of [0,T] and $u(t)$ has right and left-hand finite limits at each t_i, for example: consumption, prices, and in many cases variables associated with $\dot{x}$. The state variable is determined by the choice of the control variable and the initial conditions. Given a value of $x(t)$, once we decide $u(t)$, we are determining, through (2), the evolution of $x(t)$, more precisely we determine $x(t + dt)$, since $u(t)$ and $x(t)$ determine the change in x.

The presence of control and state variables is what makes a dynamic problem essentially different from a static problem. We cannot solve the problem as a sequence of static problems. $u(t)$ can be decided at each instant, but this decision will affect the state of the system in the future, so it will not only affect current returns but also future returns (in terms of J).

Equation (3) is a standard constraint, that in what follows it will be omitted. T can be ∞, but we will present everything for finite T, highlighting the main changes when the optimization is until infinity. We also could consider a "free terminal conditions problem." In this case x_T and or T are optimally determined instead of given exogenously.

1 The calculus of variations

This special case can be applied when there is one state variable and we can write $u(t)$ as a function of $x(t)$, $\dot{x}(t)$ and t. That is, we can transform the $G(\cdot)$ function to:

$$u(t) = H(x(t), \dot{x}(t), t)$$

Without loss of generality we can consider $u(t) = \dot{x}(t)$, so $G(\cdot) = u(t)$. That is, the control variable is $\dot{x}(t)$.

Then **[P.1]** becomes:

[P.2]

$$\max J \equiv \int_0^T F(x(t), \dot{x}(t), t)dt \tag{6}$$

subject to:

$$x(t=0) = x_0 \tag{7}$$

$$x(t=T) = x_T \tag{8}$$

Now, we can solve the problem, i.e. we find some differential equation (or in general a system) that together with the endpoint conditions characterize the optimal function for x. Call this optimum $x(t)$, we solve the problem by considering an "admissible variation" of $x(t)$. An admissible variation is a class of functions (continuously differentiable) such that satisfies the endpoint conditions (see figure 1).

The function $\tilde{x}(t)$ is an admissible variation and then can be written as:

$$\tilde{x}(t) = x(t) + \epsilon h(t)$$

where $\epsilon \in \Re$ and $h(t)$ is a continuously differentiable function that satisfies the following restriction: $h(0) = h(T) = 0$. Then, $J(\epsilon)$ is:

$$J(\epsilon) \equiv \int_0^T F(x(t) + \epsilon h(t), \dot{x}(t) + \epsilon(\dot{h}), t)dt \tag{9}$$

by assumption $J(\epsilon)$ is maximized at $\epsilon = 0$, because $x(t)$ is the optimum path for $x(t)$. Therefore:

$$\frac{dJ(\epsilon = 0)}{d\epsilon} = 0 \tag{10}$$

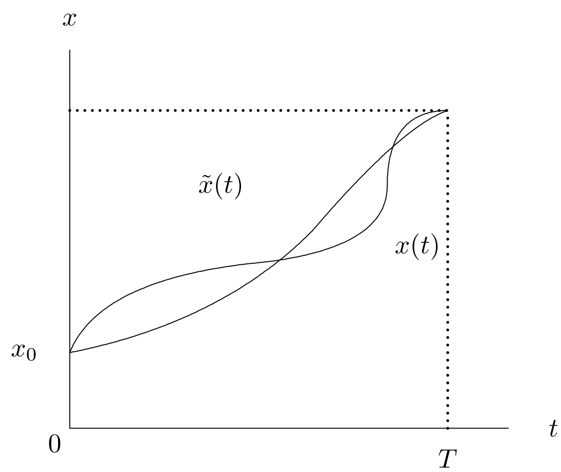

Figure 1: Admissible Functions

Differentiating (9) we have:[1]

$$\frac{dJ(\epsilon=0)}{d\epsilon} = \int_0^T \left[F_x(x,\dot{x},t)h(t) + F_{\dot{x}}(x,\dot{x},t)\dot{h}(t)\right] dt = 0 \tag{11}$$

The problem with equation (11) is that we have h and $\dot{h}$ in the integral, so we cannot go further unless we are able to transform this expression. To have only h in the integral we can solve the second term in the integrand using integration by parts:

$$\int u dv = uv - \int v du$$

Choosing $u = F_{\dot{x}}$ then $du = \dot{F}_{\dot{x}}dt$. On the other hand $dv = \dot{h}dt$, then $v = h$. Therefore:

$$\int_0^T F_{\dot{x}}\dot{h}dt = hF_{\dot{x}}\Big|_0^T - \int_0^T h\dot{F}_{\dot{x}}dt \tag{12}$$

Since $h(0) = h(T) = 0$, the first expression at the RHS of (12) is zero, and consequently (11) becomes:

$$\frac{dJ(\epsilon=0)}{d\epsilon} = \int_0^T \left[F_x - \dot{F}_{\dot{x}}\right] h(t)dt \tag{13}$$

[1]Note that the fact that $\epsilon = 0$ is embodied in the argument of the function F and its partial derivatives. Partial derivatives are denoted by subscripts.

At $\epsilon = 0$ J is maximized, hence the integral in (13) has to be zero for any function $h(t)$ that is zero at the endpoints. Therefore the expression in square brackets has to be zero, which yields the fundamental equation of calculus of variations, "the Euler equation":

$$F_x = \dot{F}_{\dot{x}} \tag{14}$$

recall that because "dots" are derivatives with respect to time, hence, the Euler equation can be written as:

$$\frac{\partial F}{\partial x} = \frac{d[\partial F/\partial \dot{x}]}{dt}$$

What the method of the calculus of variation intuitively does is to trade off some x today for having more x tomorrow (admissibility). At the optimum, the cost of doing this today is equal to the benefits achieved tomorrow. A good example of this reasoning is in Blanchard and Fischer (1989) pp. 41-43, when they discuss the optimality of the Ramsey rule. This kind of argument is easy to apply to discrete time problems, so we can obtain directly the first order conditions. For a good example you can see Mankiw, Rotemberg and Summers (1985).

2 Pontriagyn's Maximum Principle, Optimal Control, or "Hamiltonians"

We can handle also more general problems, like:

[P.3]

$$\max J \equiv \int_0^T F(x(t), u(t), t)dt \tag{15}$$

subject to:

$$\begin{aligned} \dot{x} &= G(x(t), u(t), t) && (16) \\ x(t=0) &= x_0 && (17) \\ x(t=T) &\geq x_T && (18) \end{aligned}$$

Note that the terminal state is free to take any value greater than x_T, which in many cases we may think is zero. In contrast with the problem in the previous section, this simplification will make easy the solution and at the same time allows us to discuss the role of the transversality conditions. In the calculus of variation, we can assume also a free terminal state, although we would need an additional condition. That condition is obtained using a variation around the optimal x_t.[2]

To solve this problem, let us write the Lagrangian:

$$\begin{aligned}\mathcal{L} &= \int_0^t \left[F(x(t),u(t),t) + \lambda(t)[G(x(t),u(t),t) - \dot{x}(t)]\right] dt \\ &\quad + \eta_0(x_0 - x(0)) + \eta_T(x_T - x(T)) \qquad (19)\end{aligned}$$

$\lambda(t)$ is called *costate variable* and later on we will interpret its meaning, that as you may guess it is related to some shadow price. η_0 and η_T are the lagrange multipliers associated to constraints (17) and (18).

We are interested in knowing the optimal path for $u(t)$ and $x(t)$. However, in the Lagrangian we have $\dot{x}(t)$. Then, in order to have $\mathcal{L}$ as a function of only $u(t)$ and $x(t)$, and not their time derivatives, we can use again integration by parts. Using $u = -\lambda$ and $dv = \dot{x}dt$, we have:

$$\int_0^T -\lambda(t)\dot{x}(t)dt = -\lambda(t)x(t)\Big|_0^T + \int_0^T x(t)\dot{\lambda}(t)dt$$

Hence the lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= \int_0^t \left[F(x(t),u(t),t) + \lambda(t)G(x(t),u(t),t) + \dot{\lambda}(t)x(t)\right] dt \\ &\quad + \lambda(0)x_0 + \lambda(T)x(T) + \eta_0(x_0 - x(0)) + \eta_T(x_T - x(T)) \qquad (20)\end{aligned}$$

Now we can differentiate the lagrangian with respect to $u(t)$ and $x(t)$ and equate to zero:

$$\frac{\partial \mathcal{L}}{\partial u} = F_u + \lambda G_u = 0 \qquad (21)$$

$$\frac{\partial \mathcal{L}}{\partial x} = F_x + \lambda G_x + \dot{\lambda} = 0 \qquad (22)$$

[2]For details see Kamien and Schwartz (1981), part I, section 9.

At the extreme values, we have the following necessary conditions:

$$\lambda(0) = -\eta_0 \qquad \text{and} \qquad \lambda(T)) = \eta_T$$

Finally, by Kuhn-Tucker's complementary slackness we have the following *transversality condition* (TVC):

$$\lambda(T)(x(T) - x_T) = 0$$

In the particular case that x_T is zero, this condition becomes:

$$\lambda(T)x(T) = 0$$

so for $x(T) > 0$, $\lambda(T) = 0$ and for $x(T) = 0$, $\lambda(T) \geq 0$. If x(T) were $x_T > 0$, the transversality condition would be $\lambda(T) = 0$.

When T goes to infinity, this transversality condition is:[3]

$$\lim_{T\to\infty} \lambda(T)x(T) = 0$$

Before providing the intuition for the transversality condition, we will first interpret $\lambda(t)$. At the optimum, $\mathcal{L} = J$, therefore $\lambda(0)$ and $\lambda(T)$ are:

$$\begin{aligned}\frac{\partial J}{\partial x_0} &= \lambda(0)\\ \frac{\partial J}{\partial x_T} &= -\lambda(T)\end{aligned}$$

Then, $\lambda(0)$ is the marginal value of having one more unit of x at the beginning. $\lambda(T)$ is the marginal loss of leaving one more unit of x at the end of the planning horizon. Thus, they are shadow values. In general, using Bellman's principle of optimality (see section 3), $\lambda(t)$ can be interpreted as the shadow value of $x(t)$.

The intuition for the TVC is that at T the value of what is left is zero. When one unit of x at T is valuable in terms of the objective function ($\lambda(T) > 0$), it is driven to its minimum feasible value, x_T. When some x in excess of x_T is left, it is because it has no value ($\lambda(T) = 0$). The importance of the

[3] The transversality condition in infinite horizon is not always necessary. For further discussion in infinite horizon, see Benveniste and Scheinkman (1982), Michel (1982) and references therein.

TVC is that rule out many trajectories that satisfies the system of differential equations given by (21), (22), and (16). In fact, it is important in finding a unique optimal path.

Equations (21), (22), the constraint (16) and the TVC describe the system for λ, x and u. So, you may wonder where is the hamiltonian. The hamiltonian is a function that makes easy to solve for the optimal conditions, and is defined as:

$$\mathcal{H} = F(x, u, t) + \lambda(t)G(x, u, t) \tag{23}$$

so it consists of the first two terms inside the integral of the lagrangian (19), that is: "$\mathcal{L} = \int \mathcal{H} +$ something else." Note that:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u} &= F_u + \lambda G_u \\ \frac{\partial \mathcal{H}}{\partial x} &= F_x + \lambda G_x \end{aligned}$$

comparing these two expressions with (21) and (22) we can see that the necessary conditions for optimality can be written as:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \tag{24}$$

$$\frac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda} \tag{25}$$

which are of course very easy to remember: the partial derivatives of the hamiltonian with respect to the control variables are zero, and the partial derivatives with respect to the state variable are minus the derivative of the costate variable with respect to time. Finally, you may note that the partial derivative of $\mathcal{H}$ with respect to the costate variable is equal to $G(\cdot)$, which is the rate of accumulation of x:

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{x} \tag{26}$$

so it recovers the constraint. Therefore the final system of differential equations that characterize the optimal solution is given by (24), (25) and (26), and the two endpoint conditions.

To analyze in more detail the solution in many applications it is possible from (24) to obtain $u(t)$ as a function of $\lambda(t)$ and $x(t)$. Then we can substitute this expression in (25) and (26). These two equations will conform a system of two differential equations for $x(t)$ and $\lambda(t)$. In addition, the endpoints conditions: $x(t=0)=x_0$, and $\lambda(T)(x(T)-x_T)=0$, will give us description of the system of differential equations.

With our three equations we can find a relation between $\dot{u}$ and λ, so instead of having a system for x and λ (capital and q in investment models), we can have one for u and x (consumption and capital in Ramsey). Of course, a phase diagram may help to understand the solution without solving analytically the system of differential equations.

The connection with calculus of variations is simple: The Euler equation is a simple representation of (24) and (25) without going through the costate variable. You may check that using the maximum principle to [**P.2**] we obtain the Euler equation.

The maximum principle can be derived using a variation on the optimal path. However, a variation on the control variable will not imply a straightforward variation on the state variable, because they are linked through the general function G. In contrast, in the calculus of variation $u=\dot{x}$x. Finally, the second order necessary conditions for a maximization problem is $\mathcal{H}_{uu}\leq 0$.

Most intertemporal problems in economics involve discounting, so it may be useful to define the current value of the costate variable, instead of its present value. Consider the following version of the function $F(\cdot)$ in [**P.3**]:

$$F(x(t),u(t),t)\equiv e^{-\rho t}f(x(t),u(t))$$

We can write the hamiltonian as:

$$\mathcal{H}=[f(x,u)+\lambda'(t)G(x,u,t)]e^{-\rho t} \tag{27}$$

$\lambda'(t)$ is called the current value of the costate variable and the expression in square brackets is called the current value hamiltonian ($\mathcal{H}'$):

$$\lambda(t)=\lambda'(t)e^{-\rho t} \qquad \text{and} \qquad \mathcal{H}(t)=\mathcal{H}'(t)e^{-\rho t}$$

then, the necessary conditions can be made in terms of the current values. Substituting the current values in the optimal conditions (24) and (25) we obtain:

$$\frac{\partial \mathcal{H}'}{\partial u} = 0 \tag{28}$$

$$\frac{\partial \mathcal{H}'}{\partial x} = \dot{\lambda}'+\rho\lambda' \tag{29}$$

and the TVC for the case of $x_T = 0$:

$$\lambda'(T)e^{-\rho T}x(T) = 0$$

When there is discounting, it is very convenient to write the hamiltonian as in (27), because $F(\cdot)$ is valued at time t. However it is enough to remember (24) and (25), writing the last condition as $\partial\mathcal{H}/\partial x = d(\lambda'(t)e^{\rho t})/dt$, and $e^{-\rho t}$ will cancel out at both sides of the equation.

3 Dynamic Programming

The method of dynamic programming is based on Bellman's Principle of optimality:

> "An optimal policy has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman, 1957).

Let us denote an optimal path of u for the period $[0, T]$ by $u^*(t)$ and denote by $x^*(t)$ the evolution for the state variable associated to this optimal policy and the initial state. Take a period $[t_1, T]$, such that $t_1 > 0$. If the initial state at t_1 is $x^*(t_1)$, Bellman's principle tell us that the optimal policy for this period will still be $u^*(t)$ for $[t_1, T]$. Therefore, no matter what has happened in the pass, given the "right" initial condition, the optimal policy is the same.

Bellman's principle is not always valid. This is the case of time inconsistency (e.g. Kydland and Prescott, 1977), where the optimal policy at 0 is no longer optimal starting at t_1 with $x^*(t_1)$. However, for our purposes, we will assume that the principle is valid in our problem.

To derive the solution to the dynamic programming problem define the optimal value function at time t_0 and with initial state x_0 :

$$V(t_0, x_0) = \max_u \int_{t_0}^{T} F(t, x, u)dt$$

subject to:

$$\dot{x} = G(x, t, u)$$

we can write:

$$V(t_0, x_0) = \max_u \int_{t_0}^{t_0+\Delta t} F(t, x, u)dt + \int_{t_0+\Delta t}^{T} F(t, x, u)dt \tag{30}$$

the first integral is to a first order Taylor approximation $F(x_0, t_0, u)\Delta t$. By Bellman's principle the second integral is $V(t_0 + \Delta t, x_0 + \Delta x)$, which to a first order Taylor approximation is $V(x_0, t_0) + V_t(x_0, t_0)\Delta t + V_x(x_0, t_0)\Delta x$. Using these two approximations in (30), dividing by Δt and taking the limit $\Delta t \to 0$, we have:

$$-V_t = \max[F(t, x, u) + V_x G(t, x, u)] \tag{31}$$

the fundamental partial differential equation (for $V(x, t)$) of dynamic programming, known as "Bellman equation." With known forms for $F(\cdot)$ and $G(\cdot)$ we can maximize the expression at the RHS of (31) and then substitute back the optimal value of u as function of x and t to have a partial differential equation for $V(x, t)$.

The relation between dynamic programming and optimal control is found identifying the hamiltonian as follows:

$$\mathcal{H} = F(x, t, u) + V_x G$$

assuming that $\lambda = V_x$. Maximizing (31) with respect to u we obtain:

$$F_u + V_x G_u = 0 \qquad \Leftrightarrow \qquad \mathcal{H}_u = 0$$

which shows the equivalence between the condition implicit in the Bellman equation and condition (24) from the maximum principle.

Because the Bellman equation holds for all x when u is set optimally, we can differentiate the Bellman equation with respect to x to obtain equation (25) of the maximum principle:

$$-V_{tx} = F_x + V_{xx}G + V_x G_x \tag{32}$$

known as Hamilton-Jacobi equation. Substituting in (32): $\dot{V}_x = V_{xt} + V_{xx}G$ we finally obtain an equivalent to (25):

$$\dot{V}_x = -(F_x + V_x G_x) \qquad \Leftrightarrow \qquad \dot{\mathcal{H}} = \dot{\lambda}$$

This last equation shows also that in fact λ is equal to V_x. Therefore, the interpretation of the costate variable in the hamiltonian is immediate: the marginal value of x at time t.

A particular and common problem in dynamic optimization in economics is the so called "infinite horizon autonomous problem", which corresponds to:

$$V(t_0, x_0) = \max_u \int_0^\infty f(x,u) e^{-\rho(t-t_0)} dt$$

subject to:

$$\dot{x} = g(x,u)$$

it is called autonomous because its only dependence on time is through discounting. Define:

$$W(x) = V(x,t)/e^{-\rho t} \qquad \forall (x,t)$$

so $W(x)$ is the optimal current value function. To write the Bellman equation in terms of $W(x)$ we have that:

$$\begin{aligned} V_t(x,t) &= -\rho W(x) e^{-\rho t} \\ V_x(x,t) &= W'(x) e^{-\rho t} \end{aligned}$$

and thus (31) becomes:

$$\rho W(x) = \max_u [f(x,u) + W'(x) g(x,u)] \tag{33}$$

This equation has a simple interpretation in terms of arbitrage or indifference condition at the optimum. Let us consider a discrete time approximation of (33):

$$\rho W(x) \Delta t \simeq \max_u [f(x,u) \Delta t + \Delta W(x)]$$

the LHS corresponds to the flow of interest accrued during Dt (annuity value of the optimal policy in (33)). The RHS is the best that an optimal deviation from that policy during Δt can do: to obtain a reward of f during the deviation period plus a capital gain which is due to the change in the optimal value caused by the choice of u. At the optimum both alternatives should yield the same value.

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