Introduction to DSGE Models

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Motivation

Ref: Chari and Kehoe (2006) "Modern Macroeconomics in Practice: How Theory is Shaping Policy"

Developments in academic macroeconomics over the last three decades:

- Lucas Critique of policy evaluation. Decision Rules of the private agents are not invariant to changes in policies. Technologies and preferences are deep parameters
- Time inconsistency problems of discretionary policy. A regime where policymakers set a state-contingent rule once for all is better than a discretionary regime in which policymakers sequentially choose policy optimally given the current situation
- Quantitative Dynamic General Equilibrium models. This was a response to Lucas Critique, because policy analysis should be based on DSGE in which preferences and technologies are reasonably argued to be invariant to policy. DSGE models have became increasingly sophisticated, including financial markets imperfections, sticky prices and other monetary non-neutralities, imperfect competition, incomplete markets, labor market frictions, and so on.

Lessons:

- Monetary policy should be conducted so as to keep nominal interest rates and inflation rates low
- Tax rates on labor and consumption should be roughly constant over time
- Capital income taxes should be roughly zero
- Returns on debt and taxes on assets should fluctuate to provide insurance against adverse shocks.

DSGE: Structure and main elements

- Agents: Households, Government (Foreign Agents)
- Preferences and technology
- Markets structure
- Nominal and real rigidities
- Policy functions
- Processes for exogenous variables

Simple Dynamic General Equilibrium

- Household Problem
  - Preferences (inelastic labor supply $l_t = 1$):

$$\left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\} \tag{1}$$

  - Budget constraint:

$$c_t + k_{t+1} = w_t l_t + z_t k_t + (1-\delta) k_t \tag{2}$$

  - Complete market: focus in a representative household
  - FOC: choose a sequence for $\{c_t, k_{t+1}\}$ to max (1) s.t. (2) and given k_0.

    Lagragian:

$$\mathcal{L} = \left\{ \sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \lambda_t \left(w_t l_t + z_t k_t + (1-\delta) k_t - c_t - k_{t+1} \right) \right] \right\}$$

$$1/c_t = \lambda_t \text{ [marginal utility of income]} \tag{3}$$

$$\lambda_t = \beta \lambda_{t+1} [z_{t+1} + (1-\delta)] \tag{4}$$

– Transversality condition: $\lim_{t\to\infty}[\beta^t \lambda_t k_{t+1}] = 0$

- Firms Problem
  - Production function: $y_t = A(k_t^d)^{\alpha}(l_t^d)^{1-\alpha}$
  - FOC: choose l_t^d and k_t^d in order to max $A(k_t^d)^{\alpha}(l_t^d)^{1-\alpha} - z_t k_t^d - w_t l_t^d$

$$z_t = \alpha A(l_t^d/k_t^d)^{1-\alpha} \tag{5}$$

$$w_t = (1-\alpha)A(k_t^d/l_t^d)^{\alpha} \tag{6}$$

- Equilibrium Conditions
  - Labor market: $l_t^d = l_t = 1$ [Inelastic labor supply]
  - Capital services: $k_t^d = k_t$
  - Final goods: $y_t = c_t + inv_t = c_t + k_{t+1} - (1-\delta)k_t$ [Closed economy; No Gov't]
- Summary of conditions
  - Resource constraint:

$$c_t + k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t \tag{7}$$

– Euler Equation:

$$1 = \beta \frac{c_t}{c_{t+1}} [\alpha A (k_{t+1})^{\alpha-1} + (1-\delta)] \tag{8}$$

– (7) and (8) is a set of dynamic non-linear equations: Solution ?

Some words in Dynamic Programming

- Consider the following problem

$$\max_{\{\mathbf{u}_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(\mathbf{u}_t, \mathbf{x}_t)$$
$$\text{s.t. } \mathbf{x}_{t+1} = g(\mathbf{u}_t, \mathbf{x}_t)$$

- $\mathbf{x}_t \in \mathbb{R}^n$ *state* variables; $\mathbf{u}_t \in \mathbb{R}^k$ *control* variables
- Under some qualifications[1], solution: $\mathbf{u}_t = h(\mathbf{x}_t)$ [time-invariant *policy function*]

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[1] r is concave function, $\{(\mathbf{x}'_{t+1}, \mathbf{x}'_t) : \mathbf{x}_{t+1} \leq g(\mathbf{x}_t, \mathbf{u}_t), \mathbf{u}_t \in \mathbb{R}^k\}$ is a convex and compact set

Back to the Simple DGE model

- Our simple model: $c_t = h(k_t)$. Find function $h()$ that solves (7) and (8)
- Assume $\delta = 1$ [closed form solution] $\Rightarrow$

  [see Ljungqvist and Sargent (2000), Ch. 2 pp. 33-34]

  $c_t = h(k_t) = (1 - \beta\alpha)Ak^{\alpha}$ [in logs $\log(c_t) = \log((1 - \beta\alpha)A) + \alpha \log(k_t)$]

- When $\delta \neq 1$: Not closed form solution $\Rightarrow$ resort to numerical approximations
- One popular approach: log-linearize the equations and solve for h in the family of log-linear functions: $\log(\mathbf{u}_t) = \mathbf{h}_0 + \mathbf{H}_1 \log(\mathbf{x}_t)$. Find $\mathbf{h}_0 \in \mathbb{R}^k$, $\mathbf{H}_1$ $n \times k$ matrix
- log-deviation of x_t from its steady state $\overline{x}$:

$$\widehat{x}_t = \log\left(\frac{x_t}{\overline{x}}\right)$$

  then $x_t - \overline{x} = \overline{x}(\exp(\widehat{x}_t) - 1) \approx \overline{x}\widehat{x}_t$

- Combining (7) and (8) we get:

$$Ak_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2} = \beta(Ak_t^{\alpha} + (1-\delta)k_t - k_{t+1})[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)] \quad (9)$$

- Steady state: $\overline{k}$ such that $1 = \beta[\alpha A \overline{k}^{\alpha-1} + (1-\delta)]$
- log-linearize (9) around the steady state:
  - Define $f(k) = Ak^{\alpha} + (1-\delta)k$

$$\begin{aligned} LHS = f(k_{t+1}) - k_{t+2} &\approx f(\overline{k}) - \overline{k} + f'(\overline{k})(k_{t+1} - \overline{k}) - (k_{t+2} - \overline{k}) \\ &\approx f(\overline{k}) - \overline{k} + f'(\overline{k})\overline{k}\widehat{k}_{t+1} - \overline{k}\widehat{k}_{t+2} \end{aligned}$$

  - Similarly:

$$\begin{aligned} RHS &= \beta(f(k_t) - k_{t+1})f'(k_{t+1}) \\ &\approx \beta(f(\overline{k}) - \overline{k})f'(\overline{k}) + \beta(f'(\overline{k})\overline{k}\widehat{k}_t - \overline{k}\widehat{k}_{t+1})f'(\overline{k}) \\ &\quad + \beta(f(\overline{k}) - \overline{k})f''(\overline{k})\overline{k}\widehat{k}_{t+1} \end{aligned}$$

  - Rearranging terms we get a 2nd order linear difference equation:

$$\widehat{k}_{t+2} = A_0\widehat{k}_t + A_1\widehat{k}_{t+1} \quad (10)$$

where $A_0 = -\beta f'(\overline{k})$, $A_1 = f'(\overline{k}) + \beta - \beta(f(\overline{k}) - \overline{k})f''(\overline{k})$

- Solution: $\widehat{k}_{t+1} = P\widehat{k}_t$. P is an unknown coefficient. Plug this solution in (10):

$$(A_0 + A_1 P - P^2)\widehat{k}_t = 0 \Rightarrow P : A_0 + A_1 P - P^2 = 0$$

- Two solutions for P: $P = \dfrac{A_1 \pm \sqrt{A_1^2 + 4A_0}}{2}$
- General solution: $\widehat{k}_t = \omega_1 P_1^t k_{1,0} + \omega_2 P_2^t k_{2,0}$
- Explosive solution if $P_i > 1$, then $\omega_i = 0$
- Saddle-path unstable: $0 < P_1 < 1 < P_2$
- Putting some values: $\beta = 0.99$ (quarterly); $\alpha = 0.34$; $\delta = 0.02$ (quarterly); $A = 1$ implies $P_1 = 0.96$, $P_2 = 1.04$
- Excel and Matlab code

Homework 1 *Consider the simple DGE described above. Use the same parameters values for* β*,* α*, and* δ*.*

1. *Compute the new steady state value for* k *if* A *changes from* 1 *to* 1.05 *(a* 5% *change). Define the return to capital as* $z_{t+1} + (1 - \delta)$*. Does the increase in*

productivity increase the return to capital at the steady state in this economy? Explain.

2. *For the new value of capital compute the new values for A_0 and A_1 in equation (10). Solve for the two roots for P. Check that only one root is less than 1.*
3. *Assume that this economy is in the steady state when $A = 1$. At $t = 0$, there is a permanent increase in A of 5% ($A = 1.05$). Define $\widehat{k}_0 = \log(\overline{k}(A = 1)/\overline{k}(A = 1.05))$ as the log-deviation of k_t at $t = 0$ with respect to its new steady state. Graph the evolution of k_t after this permanent increase in productivity using the log-linear approximation. What happen with wages (w_t) and the rental rate of capital (z_t) along this path? Explain the economic intuition. [Hint: Recall that in this economy $w_t = (1-\alpha)A(k_t)^{\alpha}$ and $z_t = \alpha A(k_t)^{\alpha-1}$]*

Including stochastic shocks to the model

- Example: A can be a random variable (A_t stochastic process)
- Preferences becomes:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\} \tag{11}$$

  From the perspective of $t = 0$, c_t $[t > 0]$ is a random variable

- In each period there is an event s_t that is realized ($s_t \in \mathbb{S}$). This event contains all relevant uncertainty involves. s_t follows a Markov process [pdf $Pr\left(s_t = s' | s_{t-1} = s\right)$]
- history of events from $t = 0$ to t: $(s_0, s_1, \ldots, s_t) \equiv s^t$. $Pr(s^t|s^{t-1})$.
- $x_t(s^t)$: value of variable at t when the history until is s^t.
- Again: Complete markets allows to focuss in a representative agents w/o loss of generality:

$$\begin{array}{l} \left\{ \sum_{t=0}^{\infty} \beta^t Pr(s^t|s_{-1}) \log(c_t(s^t)) \right\} \\ \text{s.t. } c_t(s^t) + k_{t+1}(s^t) = w_t(s^t) l_t(s^t) + z_t(s^t) k_t(s^{t-1}) + (1-\delta) k_t(s^{t-1}) \end{array} \tag{12}$$

- Lagragian:

$$\mathcal{L} = \left\{ \sum_{t=0}^{\infty} \beta^t Pr(s^t|s_{-1}) \left[\log(c_t(s^t)) + \lambda_t(s^t) \begin{pmatrix} w_t(s^t)l_t(s^t) + z_t(s^t)k_t(s^{t-1}) \\ +(1-\delta)k_t(s^{t-1}) - c_t(s^t) - k_{t+1}(s^t) \end{pmatrix} \right] \right\}$$

- FOC:

$$Pr(s^t|s_{-1})/c_t(s^t) = Pr(s^t|s_{-1})\lambda_t(s^t) \text{ [marginal utility of income]} \tag{13}$$

$$\begin{aligned} \beta^t Pr(s^t|s_{-1})\lambda_t(s^t) &= \sum_{s_{t+1}\in\mathbb{S}} \beta^{t+1} Pr((s^t, s_{t+1})|s_{-1})\lambda_{t+1}(s^t, s_{t+1})[z_{t+1}(s^t, s_{t+1}) + (1-\delta)] \\ &= \beta^{t+1} Pr(s^t|s_{-1}) \sum_{s_{t+1}\in\mathbb{S}} \left\{ Pr((s^t, s_{t+1})|s^t)\lambda_{t+1}(s^t, s_{t+1}) \begin{bmatrix} z_{t+1}(s^t, s_{t+1}) \\ +(1-\delta) \end{bmatrix} \right\} \\ &= \beta^{t+1} Pr(s^t|s_{-1}) E_{s^t} \left\{ \lambda_{t+1}(s^t, s_{t+1})[z_{t+1}(s^t, s_{t+1}) + (1-\delta)] \right\} \\ &= Pr(s^t|s_{-1}) E_t \left\{ \lambda_{t+1}[z_{t+1} + (1-\delta)] \right\} \end{aligned} \tag{14}$$

$$\Rightarrow \lambda_t = \beta E_t \left[\lambda_{t+1}(z_{t+1} + 1 - \delta) \right] \tag{15}$$

- Asset Pricing parenthesis:
  - *No arbitrage opportunities* $\Rightarrow$
    $\exists \Lambda_{t,t+1} : 1 = E_t[\Lambda_{t,t+1} R^e_{t+1}] \; \forall R^e_{t+1}$
  - In GE $\Rightarrow$ consumer based asset pricing: $\Lambda_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} = \beta \frac{u_{c,t+1}}{u_{c,t}}$

- Transversality condition: $\lim_{t\to\infty}[Pr(s^t|s_{-1})\beta^t\lambda_t(s^t)k_{t+1}(s^t)] = 0$,
- Firms problem is static, then

$$z_t(s^t) = \alpha A_t(s^t)(l_t^d(s^t)/k_t^d(s^t))^{1-\alpha} \tag{16}$$

$$w_t(s^t) = (1-\alpha)A_t(s^t)(k_t^d(s^t)/l_t^d(s^t))^{\alpha} \tag{17}$$

- Equilibrium conditions:

$$l_t^d(s^t) = 1 \tag{18}$$

$$k_t^d((s^{t-1}, s_t)) = k_t(s^{t-1}), \forall s_t \in \mathbb{S} \tag{19}$$

- Again we can combine these equations and get an expression similar to (9):

$$1 = \beta(A_t k_t^{\alpha} + (1-\delta)k_t - k_{t+1})E_t\left[\frac{\alpha A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta)}{A_{t+1}k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}}\right] \tag{20}$$

- Only source of uncertainty: A_t. Log-linear approx of stochastic process for A_t: $\widehat{a}_{t+1} = \rho_a \widehat{a}_t + \varepsilon_{a,t+1}$. ρ_a is parameter, $E[A_t] = 1$

- Log-linearizing (20):
  1. Steady state values for k, c
  2. log-linearizing resource constraint:
  $$c\widehat{c}_t + k\widehat{k}_{t+1} = Ak^{\alpha}(\widehat{a}_t + \alpha\widehat{k}_t) + (1-\delta)k\widehat{k}_t$$
  3. Log-linearizing Euler equation $1/c_t = \beta E_t\left[(1/c_{t+1})(\alpha A_{t+1}k_{t+1}^{\alpha-1} + 1 - \delta)\right]$:
  $$E_t[\widehat{c}_t - \widehat{c}_{t+1} + \beta\alpha Ak^{\alpha-1}(\widehat{a}_{t+1} + (\alpha-1)\widehat{k}_{t+1})] = 0$$

- Then, log-linearized version of (20):
$$E_t[F\widehat{k}_{t+2} + G\widehat{k}_{t+1} + H\widehat{k}_t + L\widehat{a}_{t+1} + M\widehat{a}_t] = 0 \tag{21}$$
F, G, H, L and M are scalars that are functions of parameters β, α, δ, ρ_a.

Homework 2 *Find expressions for F, G, H, L and M.*

- A_t is *exogenous state* variable, solution $\Rightarrow \widehat{k}_{t+1} = P\widehat{k}_t + Q\widehat{a}_t$.

- Plug the solution in (21):

$$(FP^2 + GP + H)\widehat{k}_t + (FPQ + AQ\rho_a + GQ + L\rho_a + M)\widehat{a}_t = 0 \tag{22}$$

Homework 3 *Find* P *and* Q *as functions of* F, G, H, L *and* M.

Some words in Stochastic Dynamic Programming

- Consider the following problem:

$$\max_{\{\mathbf{u}_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t r(\mathbf{x}_t, \mathbf{u}_t) \right\}$$
$$\text{s.t. } \mathbf{x}_{t+1} = g(\mathbf{x}_t, \mathbf{u}_t, \varepsilon_{t+1})$$

- At t $\mathbf{x}_t$ is known, but not $\mathbf{x}_{t+j}$, $j \geq 1$

- ε_t is a vector of iid random variables

- Again under some qualifications, solution: $\mathbf{u}_t = h(\mathbf{x}_t)$ [time-invariant *policy function*]

- *Recursive formulation: Bellman equation*

$$V(\mathbf{x}) = \max_{\mathbf{u}} \{r(\mathbf{x}, \mathbf{u}) + \beta E[V(g(\mathbf{x}, \mathbf{u}, \varepsilon)|\mathbf{x}]\} \tag{23}$$

- $h()$ satisfies:

$$V(\mathbf{x}) = r(\mathbf{x}, h(\mathbf{x})) + \beta E[V(g(\mathbf{x}, h(\mathbf{x}), \varepsilon)|\mathbf{x}] \tag{24}$$

Recap for log-linear approximations and solutions:

- Set of conditions for equilibrium
- Determine steady state
- Log-linear approximation of set of conditions around the steady state
- Solve set of linear difference equations