# Log-linearization of Equilibrium Conditions

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## Log-linearization

**Proposition 1 (Taylor expansion)** *A function $f(x)$ continuously differentiable can be approximated around $x = x_0$ as*

$$f(x) \approx f(x_0) + \frac{\partial f(x_0)}{\partial x}(x - x_0) + \frac{1}{2}\frac{\partial f^2(x_0)}{\partial x^2}(x - x_0)^2 + ...$$

**Proof.** See any math book. ■

- A variable $X_t$ can be written as

$$X_t = X_0 \exp\left[\ln\left(\frac{X_t}{X_0}\right)\right]$$

- Lets define $\widehat{x}_t = \ln \frac{X_t}{X_0}$ (log-deviation of $X_t$ with respect to $X_0$), where $\widehat{x}_0 = 0$
- We can write variable $X_t$ as

$$X_t = X_0 \exp(\widehat{x}_t)$$

- A first order Taylor expansion of $X_t$ around $X_0$ is obtained as follows,

$$X_t = X_0 \exp(\widehat{x}_t) \approx X_0 (1 + \widehat{x}_t)$$

  when $\widehat{x}_t$ is small

- Consider now the following polynomial:

$$f(X_t) = Y_t = (a + bX_t^c)^d$$

- This polynomial may be written as

$$Y_t = (a + bX_0^c \exp(c\widehat{x}_t))^d$$

where $X_0$ is a particular value for variable $X_t$. Taking a first order Taylor expansion to both sides of the polynomial we obtain

$$\begin{aligned} Y_0 (1 + \widehat{y}_t) &\approx f(X_0) + f'(X_0)(X_t - X_0) \\ &\approx (a + bX_0^c)^d + d\,(a + bX_t^c)^{d-1} \left(cbX_0^{c-1}\right)(X_t - X_0) \\ &\approx (a + bX_0^c)^d \left(1 + dc\frac{bX_0^c}{a + bX_0^c}\widehat{x}_t\right) \end{aligned}$$

- However, we know that $Y_0 = \left(a + bX_0^c\right)^d$. Therefore we can simplify the previous expression to obtain:

$$\widehat{y}_t = dc\frac{bX_0^c}{a + bX_0^c}\widehat{x}_t$$

**Example 2 (Budget constraint)** *Consider the budget constraint of a household*

$$\frac{B_t}{1+i_t} = B_{t-1} + Y_t - C_t$$

*The budget constraint may be written as*

$$\frac{B_0}{(1+i_0)}\left(1+\widehat{b}_t\right)\left(1-\widehat{i}_t\right) \approx B_0\left(1+\widehat{b}_{t-1}\right) + Y_0\left(1+\widehat{y}_t\right) - C_0\left(1+\widehat{c}_t\right)$$

$$\frac{B_0}{1+i_0} + \frac{B_0}{1+i_0}\widehat{b}_t - \frac{B_0}{1+i_0}\widehat{i}_t - \frac{B_0}{1+i_0}\widehat{b}_t\widehat{i}_t =$$
$$B_0 + B_0\widehat{b}_{t-1} + Y_0 + Y_0\widehat{y}_t - C_0 - C_0\widehat{c}_t$$

*We know that in steady-state* $\frac{B_0}{1+i_0} - B_0 - Y_0 + C_0 = 0$. *Assuming that* $\widehat{b}_t\widehat{i}_t \approx 0$ *we have that:*

$$\widehat{b}_t = (1+i_0)\,\widehat{b}_{t-1} + \widehat{i}_t + \frac{(1+i_0)\,Y_0}{B_0}\widehat{y}_t - \frac{(1+i_0)\,C_0}{B_0}\widehat{c}_t$$

**Remark 3** *Usually, variables and equations are approximated around their steady-state values. Thus, in the example* $B_0$ *would correspond to the steady-state level of* $B_t$.

**Remark 4** *Typically is assumed that the product* $\widehat{x}_t\widehat{y}_t$ *could be neglected as it is of second order*

**Remark 5** *For the case of the interest rate, variable* $\widehat{i_t}$ *is defined as:* $\ln\frac{1+i_t}{1+i_0}$

**Remark 6** *For the case of the inflation rate, variable* $\widehat{\pi}_t$ *is defined as:* $\ln\left(\frac{P_t/P_{t-1}}{1+\pi_0}\right)$

**Example 7 (Euler equation)** *The log-linearization of the Euler relies on some particular assumptions. Consider the first order condition for consumption*

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{P_t}{P_{t+1}}(1+i_t)\right]=1 \tag{1}$$

*This condition may be written as*

$$\beta E_t \exp\left[\ln C_t-\ln C_{t+1}+\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right)\right]=1 \tag{2}$$

*Assume that* $\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right)$ *and* $\ln C_{t+1}$ *are jointly normally distributed.*

**Remark 8** *If* $x$ *is normally distributed then* $E\left[\exp(x)\right]=\exp(E\left[x\right]+Var(x)/2)$

**Example 9** *Take logs in the previous expression and rewrite it as*

$$\begin{aligned}1 &= \ln\beta+\ln\left\{E_t\exp\left[\ln C_t-\ln C_{t+1}+\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right)\right]\right\}\\ &= \ln\beta+\ln\left\{\exp\left[E_t\ln C_t-E_t\ln C_{t+1}+E_t\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right)\right.\right.\\ &\quad \left.\left.+\frac{1}{2}Var\left(\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right)-\ln C_{t+1}\right)\right]\right\}\end{aligned}$$

*Lets define* $u_{t+1} = \frac{1}{2} Var\left(\ln\left(\frac{1+i_t}{1+\pi_{t+1}}\right) - \ln C_{t+1}\right)$.

*The linear expression for the Euler equation is given by,*

$$\widehat{c}_t = -\left(\widehat{i}_t - E_t\widehat{\pi}_{t+1}\right) + E_t\widehat{c}_{t+1} - u_{t+1}$$

*where* $\widehat{c}_t = \ln\frac{C_t}{C}$, $\widehat{i}_t = \ln\frac{1+i_t}{1+i}$, $\widehat{\pi}_{t+1} = \ln\frac{1+\pi_{t+1}}{1+\pi}$. *The term* $u_{t+1}$ *is usually neglected as it is of second order.*

*Other way to obtain this approximation is:*

$$\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{P_t}{P_{t+1}}(1+i_t)\right] \\
&= \beta E_t\left[\exp(\widehat{c}_t - \widehat{c}_{t+1})\frac{1+i_0}{1+\pi_0}\exp(\widehat{i}_t - \widehat{\pi}_{t+1})\right] \\
&= \beta\left(\frac{1+i_0}{1+\pi_0}\right)E_t\left[\exp(\widehat{c}_t - \widehat{c}_{t+1} + \widehat{i}_t - \widehat{\pi}_{t+1})\right] \\
&= E_t\left[\exp(\widehat{c}_t - \widehat{c}_{t+1} + \widehat{i}_t - \widehat{\pi}_{t+1})\right] \\
&\approx E_t\left[1 + \widehat{c}_t - \widehat{c}_{t+1} + \widehat{i}_t - \widehat{\pi}_{t+1}\right]
\end{aligned}$$

**Example 10 (NKPC)** *The problem of a firm is to maximize profits*

$$\max_{P_t(z)} \sum_{i=0}^{\infty} (\phi\beta)^i E_t \left\{ \Lambda_{t,t+i} \frac{P_t(z) - MC_{t+i}(z)}{P_{t+i}} Y_{t+i}(z) \right\}$$

*subject to* $Y_t(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} C_t$ *where* $MC_t(z)$ *are nominal marginal cost for firm* $z$. *The optimal resetting price is*

$$P_t^{new} = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{i=0}^{\infty} (\phi\beta)^i E_t \left\{ \Lambda_{t,t+i} MC_{t+i} P_{t+i}^{\epsilon} \frac{C_{t+i}}{P_{t+i}} \right\}}{\sum_{i=0}^{\infty} (\phi\beta)^i E_t \left\{ \Lambda_{t,t+i} P_{t+i}^{\epsilon} \frac{C_{t+i}}{P_{t+i}} \right\}} \tag{3}$$

*where subscript* $z$ *was dropped due to the symmetric adjustment of all firms setting price in* $t$.

*Linearizing each component of the expression:*

$$\begin{aligned}
P_t^{new} &\approx P^{new} \left(1 + \widehat{p}_t^{new}\right) \\
\Lambda_{t,t+i} MC_{t+i} P_{t+i}^{\epsilon} \frac{C_{t+i}}{P_{t+i}} &\approx \Lambda MC P^{\epsilon-1} C \begin{pmatrix} 1 + \widehat{\Lambda}_{t,t+i} + \widehat{MC}_{t+i} \\ +(\epsilon - 1)\widehat{p}_{t+i} + \widehat{c}_{t+i} \end{pmatrix} \\
\Lambda_{t,t+i} P_{t+i}^{\epsilon} \frac{C_{t+i}}{P_{t+i}} &\approx \Lambda P^{\epsilon-1} C \left(1 + \widehat{\Lambda}_{t,t+i} + (\epsilon - 1)\widehat{p}_{t+i} + \widehat{c}_{t+i}\right)
\end{aligned}$$

*Therefore:*

$$P_t^{new}\sum_{i=0}^{\infty}(\phi\beta)^i E_t\left\{\Lambda_{t,t+i}P_{t+i}^{\epsilon}\frac{C_{t+i}}{P_{t+i}}\right\}$$
$$\approx P^{new}\Lambda P^{\epsilon-1}C\left(1+\widehat{p}_t^{new}\right)E_t\left(\sum_{i=0}^{\infty}(\beta\phi)^i\left(1+\widehat{\Lambda}_{t,t+i}+(\epsilon-1)\widehat{p}_{t+i}+\widehat{c}_{t+i}\right)\right)$$
$$\approx P^{new}\Lambda P^{\epsilon-1}C\left(\frac{1}{1-\beta\phi}+\frac{\widehat{p}_t^{new}}{1-\beta\phi}+E_t\left[\sum_{i=0}^{\infty}(\beta\phi)^i\left(\widehat{\Lambda}_{t,t+i}+(\epsilon-1)\widehat{p}_{t+i}+\widehat{c}_{t+i}\right)\right]\right)$$

*and*

$$\frac{\epsilon}{\epsilon-1}\sum_{i=0}^{\infty}(\phi\beta)^i E_t\left\{\Lambda_{t,t+i}MC_{t+i}P_{t+i}^{\epsilon}\frac{C_{t+i}}{P_{t+i}}\right\}$$
$$\approx\frac{\epsilon}{\epsilon-1}\Lambda MCP^{\epsilon-1}CE_t\left(\sum_{i=0}^{\infty}(\beta\phi)^i\left(1+\widehat{\Lambda}_{t,t+i}+\widehat{MC}_{t+i}(\epsilon-1)\widehat{p}_{t+i}+\widehat{c}_t\right)\right)$$
$$\approx\frac{\epsilon}{\epsilon-1}\Lambda MCP^{\epsilon-1}C\left(\frac{1}{1-\beta\phi}+E_t\left[\sum_{i=0}^{\infty}(\beta\phi)^i\left(\begin{array}{c}\widehat{\Lambda}_{t,t+i}+\widehat{MC}_{t+i}\\+(\epsilon-1)\widehat{p}_{t+i}+\widehat{c}_{t+i}\end{array}\right)\right]\right)$$

*Using the fact that the steady-state inflation rate is zero* ($P^{new} = P$) *and* $P = MC(\epsilon - 1)/\epsilon$, *we can obtain:*

$$\begin{aligned} \widehat{p}_t^{new} &= (1-\phi\beta)\sum_{i=0}^{\infty}(\phi\beta)^i E_t\left\{\widehat{MC}_{t+i}\right\} \\ &= (1-\phi\beta)\left(\widehat{mc}_t + \widehat{p}_t\right) + \phi\beta E_t \widehat{p}_{t+1}^{new} \end{aligned}$$

*where* $\widehat{mc}_t = \widehat{MC}_t - \widehat{p}_t$ *is the log deviation of real marginal cost with respect to its steady state value.*

*The aggregate price level is:*

$$P_t = \left[(1-\phi)\left(P_t^{new}\right)^{\epsilon-1} + \phi\left(P_{t-1}\right)^{\epsilon-1}\right]^{\frac{1}{\epsilon-1}}$$

*Log-linearizing this expression of the price level we get:*

$$\widehat{p}_t = (1-\phi)\,\widehat{p}_t^{new} + \phi\widehat{p}_{t-1}$$

*Combining with the previous expression we get*

$$\begin{aligned} \widehat{p}_t - \widehat{p}_{t-1} &= (1-\phi)\,\widehat{p}_t^{new} + (\phi-1)\,\widehat{p}_{t-1} \\ &= (1-\phi)(1-\phi\beta)\left(\widehat{mc}_t + \widehat{p}_t\right) + (1-\phi)\,\phi\beta E_t\widehat{p}_{t+1}^{new} + (\phi-1)\,\widehat{p}_{t-1} \end{aligned}$$

*Now, we can expressed expected optimal price as* $E_t \widehat{p}_{t+1}^{new} = \frac{1}{1-\phi} E_t \pi_{t+1} + \widehat{p}_t$. *Then,*

$$\begin{aligned} \pi_t &= (1-\phi)(1-\phi\beta)(\widehat{mc}_t + \widehat{p}_t) + \phi\beta E_t \pi_{t+1} + (1-\phi)\phi\beta\widehat{p}_t + (\phi - 1)\widehat{p}_{t-1} \\ &= (1-\phi)(1-\phi\beta)\widehat{mc}_t + \phi\beta E_t \pi_{t+1} + (1-\phi)\pi_t \end{aligned}$$

*Finally, we obtain an expression for the NKPC*

$$\pi_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widehat{mc}_t + \beta E_t \pi_{t+1}$$