Answer

(a)

$$\text{Minimize} \quad E_0 \sum_{t=0}^{\infty} (1+r)^{-t}(a - b\tau_t)^2$$

subject to $t_t = \tau_t \cdot y_t$, $s_t = t_t - g_t$ and $b_t = (1+r)(b_{t-1} - s_{t-1})$.

Find the Euler equation.

Combining the constraints we get

$$b_t = (1+r)(b_{t-1} + g_{t-1} - \tau_{t-1} \cdot y_{t-1}).$$

Solving it forward yields

$$\sum_{j=0}^{\infty} \frac{\tau_{t+j} \cdot y_{t+j}}{(1+r)^j} = b_t + \sum_{j=0}^{\infty} \frac{g_{t+j}}{(1+r)^j}$$

The Lagrangian is

$$\mathrm{L} \quad = E_t \sum_{j=0}^{\infty} \left\{ \left(\frac{1}{1+r} \right)^j (a - b\tau_{t+j})^2 + \lambda \left[b_t + \sum_{j=0}^{\infty} \frac{g_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{\tau_{t+j} \cdot y_{t+j}}{(1+r)^j} \right] \right\}$$

The FOC with respect to τ_{t+j} is

$$\frac{\partial \mathrm{L}}{\partial \tau_{t+j}} = E_{t+j} \left\{ \left(\frac{1}{1+r} \right)^j (-2b)(a - b\tau_{t+j}) - \frac{\lambda y_{t+j}}{(1+r)^j} \right\} = 0$$

$$\Rightarrow \lambda = -2bE_{t+j} \frac{(a - b\tau_{t+j})}{y_{t+j}} = -2b \frac{(a - b\tau_{t+j})}{y_{t+j}}$$

Similarly,

$$\frac{\partial \mathrm{L}}{\partial \tau_{t+j+1}} \Rightarrow \lambda = -2bE_{t+j} \frac{(a - b\tau_{t+j+1})}{y_{t+j+1}}$$

Eliminating λ gives

$$\frac{(a - b\tau_{t+j})}{y_{t+j}} = E_{t+j}\frac{(a - b\tau_{t+j+1})}{y_{t+j+1}}$$

$$\Rightarrow (a - b\tau_{t+j}) = E_{t+j}(a - b\tau_{t+j+1})\frac{y_{t+j}}{y_{t+j+1}}$$

Or, setting $j = 0$ we have

$$(a - b\tau_t) = E_t(a - b\tau_{t+1})\frac{y_t}{y_{t+1}} \qquad (*)$$

If in addition there is no growth in output (as in part (b)) then the answer simplifies to:

$$\tau_t = E_t\tau_{t+1}$$

(b) Then:

$$\tau_t = y^{-1}(1+r)^{-1}r(b_t + g_t + E_t g_{t+1}/(1+r) + ...)$$

So that

$$\tau_t - \tau_{t-1} = \epsilon_t = \left(\frac{r}{(1+r)y}\right)\sum_{j=0}^{\infty}(1+r)^{-j}(E_t g_{t+j} - E_{t-1}g_{t+j})$$

$$= v_t(r/y)/(1+r-\rho)$$

Ojo que aquí v_t es el shock de impuestos en t.