PART IV

APPLICATIONS

We have grouped in this last part of the book exercises and examples for applications. These have various objectives and different degrees of difficulties. Leaving aside (except for special cases) the cases that are too academic, we will concern ourselves with applications of concrete nature, with an emphasis on the numerical aspect of the results. A few of these applications should be used as validation tests for numerical models.

18

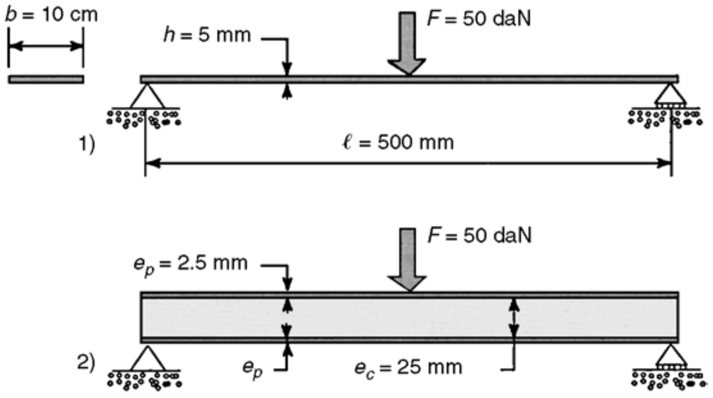
APPLICATIONS

18.1 LEVEL 1

18.1.1 Simply Supported Sandwich Beam

Problem Statement:

1. The following figure represents a beam made of duralumin that is supported at two points. It is subjected to a transverse load of $F = 50$ daN. Calculate the deflection—denoted as Δ—of the beam under the action of the force F.



2. We separate the beam of duralumin into two parts with equal thickness $e_p = 2.5$ mm, by imaginarily cutting the beam at its midplane. Each half is bonded to a parallel pipe made of polyurethane foam, making the skins of a sandwich beam having essentially the same mass as the initial beam (in neglecting the mass of the foam and the glue). The beam is resting on the same supports and is subjected to the same load F. Calculate the deflection caused by F, denoted by Δ'. Compare with the value of Δ found in Part 1. (Take the shear modulus of the foam to be: $G_c = 20$ MPa.)

Solution:

1. We will use the classical formula that gives the deflection at the center of the beam on two supports:

$$\Delta = \frac{F}{48EI} \quad \text{with} \quad I = \frac{bh^3}{12}$$

For duralumin (see Section 1.6): $E = 75{,}000$ MPa. One finds

$$\boxed{\Delta = 16.7 \text{ mm}}$$

2. Denoting by W the elastic energy due to flexure, one has[1]

$$W = \int_{\text{beam}} \frac{1}{2}\frac{M^2}{\langle EI\rangle}dx + \int_{\text{beam}} \frac{1}{2}\frac{k}{\langle GS\rangle}T^2 dx$$

with[2]:

$$\frac{k}{\langle GS\rangle} \# \frac{1}{G_c(e_c + 2e_p)\times b}$$

Using Castigliano theorem, one has $\Delta' = \dfrac{\partial W}{\partial F}$
then:

$$\Delta' = \int_{\text{beam}} \frac{M}{\langle EI\rangle}\frac{dM}{dF}dx + \int_{\text{beam}} \frac{k}{\langle GS\rangle}T\frac{dT}{dF}dx$$

$$0 \le x \le \ell/2: \quad M = Fx/2; \quad T = -F/2$$

$$\ell/2 \le x \le \ell: \quad M = \frac{F}{2}(\ell - x); \quad T = F/2$$

$$\Delta' = \frac{1}{\langle EI\rangle}\left\{\int_0^{\ell/2}\frac{Fx}{2}\times\frac{x}{2}dx + \int_{\ell/2}^{\ell}\frac{F}{2}(\ell - x)\frac{(\ell - x)}{2}dx\cdots\right\}$$

$$\cdots + \frac{k}{\langle GS\rangle}\left\{\int_0^{\ell/2} -\frac{F}{2}\times -\frac{dx}{2} + \int_{\ell/2}^{\ell}\frac{F}{2}\times\frac{dx}{2}\right\}$$

$$\boxed{\Delta' = \frac{F\ell^3}{48\langle EI\rangle} + \frac{F\ell}{4}\frac{k}{\langle GS\rangle}}$$

Approximate calculation:

$$\langle EI\rangle \# E_p \times e_p \times b \times \frac{(e_c + e_p)^2}{2} + E_c \times \frac{e_c^3 b}{12}$$

then:

$$\langle EI\rangle = 7090 \text{ MKS} + \underset{\text{negligible}}{7.8 \text{ MKS}} \quad \text{with} \quad E_c = 60 \text{ MPa } (cf.\ 1.6)$$

[1] To establish this relation, see Chapter 15, Equation 15.17.
[2] See calculation of this coefficient in 18.2.1, and more precise calculation in 18.3.5.

one obtains for Δ':

$$\Delta' = \underset{\substack{\text{bending}\\\text{moment}}}{0.18\ \text{mm}} + \underset{\text{shear}}{1.04\ \text{mm}}$$

$$\boxed{\Delta' = 1.22\ \text{mm}}$$

Comparing with the deflection Δ found in Part 1 above:

$$\boxed{\frac{\Delta}{\Delta'} = \frac{14}{1}}$$

Remarks:

- The sandwich configuration has allowed us to divide the deflection by 14 without significant augmentation of the mass: with adhesive film thickness 0.2 mm and a specific mass of 40 kg/m^3 for the foam, one obtains a total mass of the sandwich:

$$m = 700\ \text{g (duralumin)} + 50\ \text{g (foam)} + 48\ \text{g (adhesive)}$$

This corresponds to an increase of 14% with respect to the case of the full beam in Question 1.

- The deflection due to the shear energy term is close to 6 times more important than that due to the bending moment only. In the case of the full beam in question 1, this term is negligible. In effect one has:

$k = 1.2$ for a homogeneous beam of rectangular section, then:

$$\frac{k}{GS} = 8.27 \times 10^{-8}$$

(with $G = 29{,}000$ MPa, Section 1.6). The contribution to the deflection Δ of the shear force is then:

$$\int \frac{k}{GS} T \frac{dT}{dF} dx = 0.02\ \text{mm} \ll \Delta$$

18.1.2 Poisson Coefficient of a Unidirectional Layer

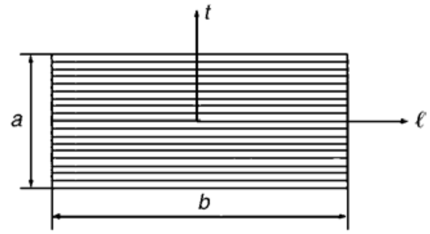
Problem Statement:

Consider a unidirectional layer with thickness e as shown schematically in the figure below. The moduli of elasticity are denoted as E_ℓ (longitudinal direction) and E_t (transverse direction).

Show that two distinct Poisson coefficients $\nu_{\ell t}$ and $\nu_{t\ell}$ are necessary to characterize the elastic behavior of this unidirectional layer. Numerical application: a layer of glass/epoxy. V_f = 60% fiber volume fraction.

Solution:

Let the plate be subjected to two steps of loading as follows:



1. A uniform stress σ_ℓ along the ℓ direction: the changes in lengths of the sides can be written as:

$$\frac{\Delta b_1}{b} = \frac{\sigma_\ell}{E_\ell}; \quad \frac{\Delta a_1}{a} = -\frac{\nu_{\ell t}}{E_\ell}\sigma_\ell$$

2. A uniform stress σ_t along the t direction: for a relatively important elongation of the resin, one can only observe a weak shortening of the fibers along ℓ. Using then another notation for the Poisson coefficient, the change in length can be written as:

$$\frac{\Delta b_2}{b} = -\frac{\nu_{t\ell}}{E_t}\sigma_t; \quad \frac{\Delta a_2}{a} = \frac{\sigma_t}{E_t}$$

Now calculating the accumulated elastic energy under the two loadings above:

- When σ_ℓ is applied first, and then σ_t is applied,

$$W = \frac{1}{2}\sigma_\ell \times a \times e \times \Delta b_1 + \frac{1}{2}\sigma_t \times b \times e \times \Delta a_2 + \sigma_\ell \times a \times e \times \Delta b_2$$

- When σ_t is applied first, and then σ_ℓ is applied,

$$W' = \frac{1}{2}\sigma_t \times b \times e \times \Delta a_2 + \frac{1}{2}\sigma_\ell \times a \times e \times \Delta b_1 + \sigma_t \times b \times e \times \Delta a_1$$

The final energy is the same:

$$W = W'$$

then:

$$\sigma_\ell \times a \times e \times \Delta b_2 = \sigma_t \times b \times e \times \Delta a_1$$

with the values obtained above for Δb_2 and Δa_1:

$$\sigma_\ell \times a \times e \times -\frac{\nu_{t\ell}}{E_t}\sigma_t \times b = \sigma_t \times b \times e \times -\frac{\nu_{\ell t}}{E_\ell}\sigma_\ell \times a$$

$$\boxed{\frac{\nu_{t\ell}}{E_t} = \frac{\nu_{\ell t}}{E_\ell}}$$

Numerical application: $\nu_{\ell t} = 0.3$, $E_\ell = 45{,}000$ MPa, $E_t = 12{,}000$ MPa (see Section 3.3.3):

$$\nu_{t\ell} = 0.3 \times \frac{12{,}000}{45{,}000}$$

$$\boxed{\nu_{t\ell} = 0.08}$$

Remark: The same reasoning can be applied to all balanced laminates having midplane symmetry, by placing them in the symmetrical axes.[3] However, depending on the composition of the laminate, the Poisson coefficients in the two perpendicular directions vary in more important ranges:

- in absolute value and
- one with respect to the other.

One can find in Section 5.4.2 in Table 5.14 the domain of evolution of the global Poisson coefficient ν_{xy} of the glass/epoxy laminate, from which one can deduce the Poisson coefficient ν_{yx} using a formula analogous to the one above, as:

$$\nu_{yx}/E_y = \nu_{xy}/E_x$$

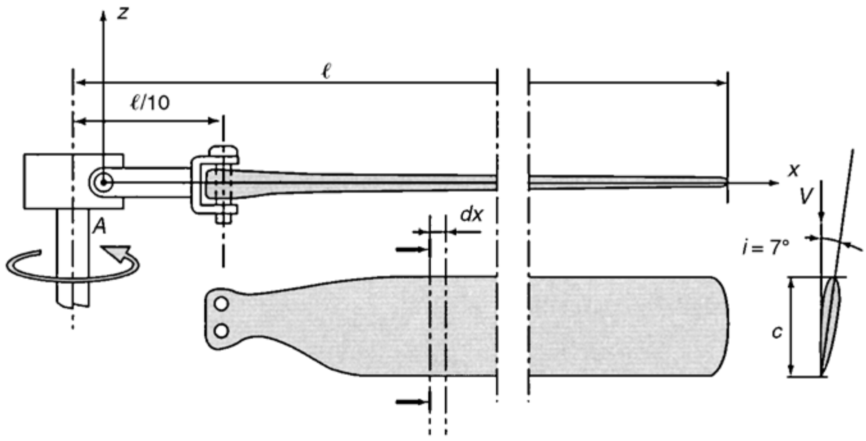
18.1.3 Helicopter Blade

This study has the objective of bringing out some important particularities related to the operating mode of the helicopter blade, notably the behavior due to normal load.

Problem Statement:

Consider a helicopter blade mounted on the rotor mast as shown schematically in the following figure.

[3] Or the orthotropic axes: see Chapter 12, Equation 12.9.



The characteristics of the rotor are as follows:

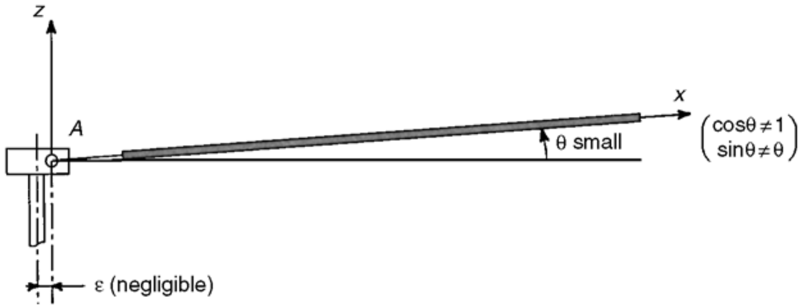
- Rotor with three blades; rotational speed: 500 revolutions per minute.
- The mass per unit length of a blade at first approximation is assumed to have a constant value of 3.5 kg/m.
- $\ell = 5$ m; $c = 0.3$ m.
- The elementary lift of a segment dx of the blade (see figure above) is written as:

$$dF_z = \frac{1}{2}\rho(c\,dx)C_z V^2$$

in which V is the relative velocity of air with respect to the profile of the blade. In addition, C_z (7°) = 0.35 (lift coefficient).

$\rho = 1.3$ kg/m^3 (specific mass of air in normal conditions).

We will not concern ourselves with the drag and its consequences. One examines the helicopter as immobile with respect to the ground (stationary flight in immobile air). In neglecting the weight of the blade compared with the load application and in assuming infinite rigidity, the relative equilibrium configuration in uniform rotation is as follows:



1. Justify the presence of the angle called "flapping angle" θ and calculate it.
2. Calculate the weight of the helicopter.
3. Calculate the normal force in the cross section of the blade and at the foot of the blade (attachment area).

The spar of the blade[4] is made of unidirectional glass/epoxy with 60% fiber volume fraction "R" glass ($\sigma_{\ell \text{ rupture}}$ # 1700 MPa). The safety factor is 6. Calculate:

4. The longitudinal modulus of elasticity E_ℓ of the unidirectional.
5. The cross section area for any x value of the spar, and its area at the foot of the blade.
6. The total mass of the spar of the blade.
7. The elongation of the blade assuming that only the spar of the blade is subject to loads.
8. The dimensions of the two axes to clamp the blade onto the rotor mast. Represent the attachment of the blade in a sketch.

Solution:

1. The blade is subjected to two loads, in relative equilibrium:
   - Distributed loads due to inertia, or centrifugal action, radial (that means in the horizontal plane in the figure, with supports that cut the rotor axis.
   - Distributed loads due to lift, perpendicular to the direction of the blade (Ax in the figure).

From this there is an intermediate equilibrium position characterized by the angle θ.

Joint A does not transmit any couple. The moment of forces acting on the blade about the y axis is nil, then:

$$\int_{\ell/10}^{\ell} dF_z \times x = \int_{\ell/10}^{\ell} dF_c \times x \sin\theta \; \# \; \theta \times \int_{\ell/10}^{\ell} dF_c \times x$$

with:

$$dF_z = \frac{1}{2}\rho c \; dx \; C_z V^2 = \frac{1}{2}\rho c \; dx \; C_z (x\cos\theta \times \omega)^2 \; \# \; \frac{1}{2}\rho c \; dx \; C_z x^2 \omega^2$$

$$dF_c = dm \; \omega^2 x \cos\theta \; \# \; m \; dx \; \omega^2 x \text{ (centrifugal load)}$$

then after the calculation:

$$\frac{1}{2}\rho c C_z \omega^2 \frac{(\ell^4 - \ell^4/10^4)}{4} = \theta \; m\omega^2 \frac{(\ell^3 - \ell^3/10^3)}{3}$$

$$\boxed{\theta \; \# \; \frac{3}{8}\frac{\rho c C_z}{m} \times \ell}$$

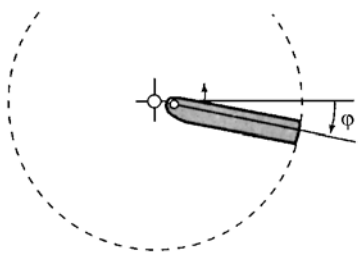
[4] See Section 7.2.3.

or numerically:

$$\boxed{\theta = 0.073 \text{ rad} = 4°11'}$$

Remarks:

- One verifies that $\sin\theta = 0.073 \;\#\; \theta$ and $\cos\theta = 0.997 \;\#\; 1$.
- When the helicopter is not immobile, but has a horizontal velocity, for example v_0, the relative velocity of air with respect to the blade varies between $v_0 + \omega x$ for the blade that is forward, and $-v_0 + \omega x$ for a blade that is backward. If the incidence i does not vary, the lift varies in a cyclical manner, and there is vertical "flapping motion" of the blade. This is why a mechanism for cyclic variation of the incidence is necessary.
- We have not taken into account the drag to simplify the calculations. This can be considered similarly to the case of the lift. It then gives rise to an equilibrium position with a second small angle, called φ, with respect to the radial direction from top view, as in the following figure. This is why a supplementary joint, or a drag joint, is necessary.



2. Weight of the helicopter: The lift and the weight balance themselves out. The lift of the blade is then:

$$F_z = \int_{\ell/10}^{\ell} dF_z \cos\theta \;\#\; \int_{\ell/10}^{\ell} dF_z = \frac{1}{2}\rho c C_z \omega^2 \frac{(\ell^3 - \ell^3/10^3)}{3}$$

then for the 3 rotor blades:

$$Mg = 3F_z$$

$$\boxed{Mg \;\#\; \frac{1}{2}\rho c C_z \omega^2 \ell^3}$$

numerically:

$$\boxed{Mg = 2340 \text{ daN}}$$

3. Normal load: It is denoted as $N(x)$:

$$N(x) = \int_x^\ell dF_c \cos\theta \# \int_x^\ell dF_c = \int_x^\ell m\omega^2 x \, dx$$

$$\boxed{N(x) = \frac{m\omega^2}{2}(\ell^2 - x^2)}$$

at the foot of the blade ($x = l/10$):

$$\boxed{N(\ell/10) \# 12{,}000 \text{ daN}}$$

4. Longitudinal modulus of elasticity:
Using the relation of Section 3.3.1:

$$E_\ell = E_f V_f + E_m V_m$$

with (Section 1.6): $E_f = 86{,}000$ MPa; $E_m = 4{,}000$ MPa.

$$\boxed{E_\ell = 53{,}200 \text{ MPa}}$$

5. Section of the spar of the blade made of glass/epoxy:
The longitudinal rupture tensile stress of the unidirectional is

$$\sigma_{\ell \text{ rupture}} \# 1700 \text{ MPa}$$

With a factor of safety of 6, the admissible stress at a section $S(x)$ becomes

$$\sigma = \frac{N(x)}{S(x)} = \frac{1700}{6} = 283 \text{ MPa}$$

then:

$$S(x) = \frac{N(x)}{\sigma}$$

$$\boxed{S(x) = \frac{m\omega^2}{2\sigma}(\ell^2 - x^2)}$$

at the foot of the blade:

$$\boxed{S(\ell/10) = 4.24 \text{ cm}^2}$$

6. Mass of the spar (longeron) of the blade:

$$m_{\text{spar}} = \int_{\ell/10}^{\ell} \rho_{\text{unidirect.}} S(x)\,dx$$

$$\boxed{m_{\text{spar}} = \rho_{\text{unidirect.}} \times \frac{m\omega^2}{\sigma} \times \frac{1.7}{6}\ell^3}$$

Specific mass of the unidirectional layer (see Section 3.2.3):

$$\rho_{\text{unidirect.}} = V_f\rho_f + V_m\rho_m = 1980\ \text{kg/m}^3$$

Then:

$$\boxed{m_{\text{spar}} = 2.38\ \text{kg}}$$

7. Elongation of the spar of the blade: The longitudinal constitutive relation is written as (see Section 3.1):

$$\varepsilon_x = \frac{\sigma_x}{E_x} = \frac{N(x)}{E_\ell \times S(x)} = \frac{\sigma}{E_\ell}$$

Elongation of a segment dx : $\varepsilon_x(x)\,dx$.

For the whole spar of blade:

$$\Delta\ell = \int_{\ell/10}^{\ell} \varepsilon_x\,dx$$

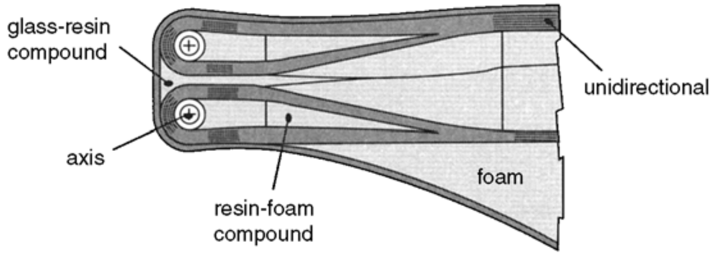
$$\boxed{\Delta\ell = 0.9\frac{\ell\sigma}{E_\ell}}$$

then:

$$\boxed{\Delta\ell = 2.4\ \text{cm}}$$

One has to reinforce the spar of the blade to diminish the elongation to resist the centrifugal force.

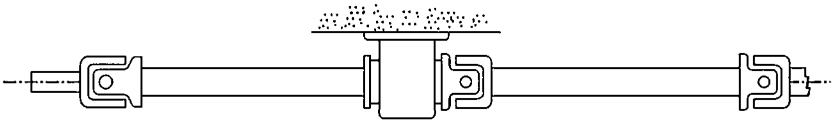
8. Clamped axes: For 2 axes in 30 NCD16 steel (rupture shear strength τ_{rupt} = 500 MPa; bearing strength σ_{bearing} = 1600 MPa); 4 sheared sections; factor of safety = 6:
   - diameter: $N(\ell/10)/\pi\phi^2 \leq \tau_{\text{rupt}}/6 \rightarrow \phi \geq 21.4$ mm
   - length: $N(\ell/10)/2h\phi \leq \sigma_{\text{bearing}}/6 \rightarrow h \geq 10.5$ mm



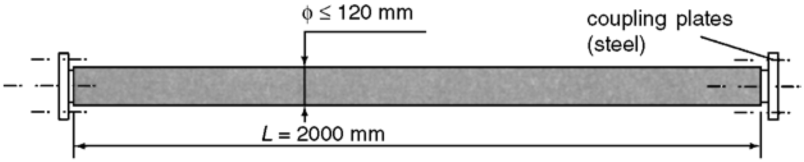
18.1.4 Transmission Shaft for Trucks

Problem Statement:

One proposes to replace the classical transmission shaft made of universal cardan joint and intermediary thrust bearing as shown below:



with a solution consisting of a long shaft made of carbon/epoxy, with the following dimensions:



The characteristics of the transmission shaft are as follows:

- Maximum torsional couple: $M_t = 300$ m daN
- Maximum rotation speed: $N = 4000$ revolutions/minute
- The first resonant flexural frequency of a beam on two supports is given by:

$$f_1 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^3}}$$

where m is the mass of the beam, and I is its flexure moment of inertia. It corresponds to a "critical speed" for a beam in rotation, which should not be reached during the operation.

- The carbon/epoxy unidirectional has $V_f = 60\%$ fiber volume fraction. The thickness of a cured ply is 0.125 mm.
  1. Give the characteristics of a suitable shaft of carbon/epoxy composite. One will make use of the tables in Section 5.4.2 and will use a factor of safety of 6.
  2. Study the adhesive fitting of the coupling plates to the shaft.
  3. Carry out an assessment on the saving in weight with respect to the "shaft in steel" solution (not including the coupling plates).

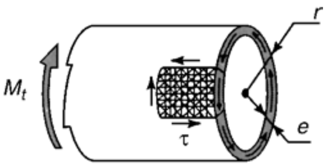
Solution:

1. Characteristics of the shaft: The shaft is assumed to be thin and hollow (thickness e is small compared with the average radius r as in the following figure).

The shear stress τ is as follows:

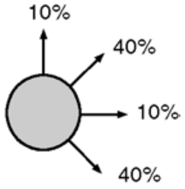
$$\tau = \frac{M_t}{2\pi r^2 e}$$

Taking into account the nature of the loading on the laminate making up the tube (pure shear), the composition of the tube requires



- Important percentages of unidirectionals in the direction of ±45° (see Section 5.2.2).
- Minimum percentages in the order of 10% in other directions (see Section 5.2.3.6).

This leads, for example, to the following distribution:



In Section 5.4, one finds Table 5.3, which gives the maximum shear stress that can be applied to a laminate subject to pure shear, as a function of the proportions of the plies at 0°, 90°, +45°, -45°. One reads for the proportions above:

$$\tau_{\text{max}} = 327 \text{ MPa}$$

from which the admissible stress taking into account a safety factor of 6:

$$\tau_{\text{admis.}} = 327/6 \text{ MPa}$$

One then has

$$\frac{M_t}{2\pi r^2 e} \le \tau_{\text{admis.}}$$

or numerically:

$$r^2 e \ge 8\ 760 \text{ mm}^3$$

For a minimum specified radius r = 60 mm, one obtains

$$e \geq 2.43 \text{ mm}$$

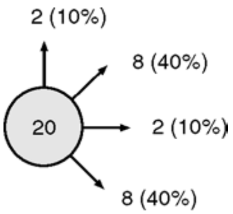
The corresponding number of plies of carbon/epoxy is

$$\frac{2.43}{0.125} \# 20 \text{ plies}$$

giving a thickness of:

$$e = 2.5 \text{ mm}$$

One can verify that a number of 20 plies allows one to satisfy the following:
(a) The required proportions



(b) The midplane symmetry, with the sequence:

$$\lfloor 90°/0°/\pm 45^{°}_{4} \rfloor_s$$

Critical speed of such a shaft:

$$f_1 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^2}}$$

- The longitudinal modulus E of the laminate (in the direction of the shaft) is (see Table 5.4 [longitudinal modulus] in Section 5.4.2)

$$E = 31{,}979 \text{ MPa}$$

- The specific density of the laminate is (see Section 3.2.3)

$$\rho_{\text{lam}} = V_f \rho_f + V_m \rho_m$$

with (Section 1.6): ρ_f = 1750 kg/m^3; ρ_m = 1,200 kg/m3. Then: ρ_{lam} = 1,530 kg/m^3 (see also Table 3.4 in Section 3.3.3).

- The second moment of inertia in flexure is $I = \pi r^3 e$ from which the first frequency is: $f_1 = 76$ Hz

It corresponds to a critical speed of 4,562 rev/minute, superior to the maximum speed of rotation of the shaft.[5]

2. Bonded fittings:

We will use the relation of Paragraph 6.2.3 (Figure 6.26) for simplification. This implies identical thicknesses for the tube making up the shaft and the coupling plate made of steel.[6] The maximum shear stress has an order of magnitude of:

$$\tau_{\max} = \frac{a}{\operatorname{th} a} \times \tau_{\text{average}} = \frac{a}{\operatorname{th} a} \times \frac{M_t}{2\pi r^2 \ell}$$

where ℓ is the bond length, and

$$a = \ell \sqrt{\frac{G_c}{2Gee_c}}$$

with G_c as the shear modulus of araldite, then G_c = 1,700 MPa (Section 1.6).

$$G_{\text{laminate}} = 28430 \text{ MPa (Section 5.4.2; Table 5.5)}$$

$$e_c = \text{bond thickness (Section 6.2.3): } e_c = \# \ 0.2 \text{ mm.}$$

- Thickness at bond location:

If one conserves the thickness found for the tube, as $e = 2.5$ mm, one obtains

$$a = l \times 244.5$$

The resistance condition can then be written as:

$$\tau_{\max} \leq \tau_{\text{rupture}} \text{ (15 MPa for araldite; see Section 6.2.3).}$$

Then:

$$\frac{a}{\operatorname{th} a} \times \frac{M_t}{2\pi r^2 \ell} \leq \tau_{\text{rupture}}$$

$$\frac{244.5}{\operatorname{th} a} \times \frac{M_t}{2\pi r^2} \leq \tau_{\text{rupture}}$$

numerically: $\operatorname{th} a \geq 2.16 \rightarrow$ impossible ($\operatorname{th} x \in]-1, +1[$).

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[5] One also has to verify the absence of buckling due to torsion of the shaft, see annex 2 for this subject.

[6] For different thicknesses for the tube made of carbon/epoxy and for the coupling plate part, one can use the more general relation established in application 18.3.1. This also allows different shear moduli for each material.

It is then necessary to augment the thickness of the tube at the bond location. One starts from the relation:

$$\frac{a}{\text{th } a} \times \frac{M_t}{2\pi r^2 \ell} \leq \tau_{\text{rupture}}$$

placed in the form:

$$\frac{\sqrt{\dfrac{G_C}{2Gee_C}}}{(1-\varepsilon)} \times \frac{M_t}{2\pi r^2} \leq \tau_{\text{rupture}} \text{ with } \varepsilon \ll 1$$

then:

$$\sqrt{\frac{G_C}{2Gee_C}} \leq \tau_{\text{rupture}} \times \frac{2\pi r^2}{M_t} \times (1-\varepsilon)$$

One finds numerically:

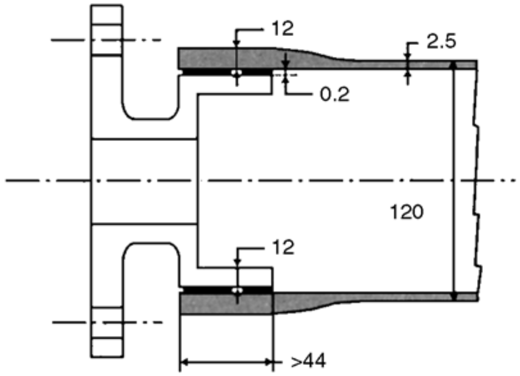
$$e > 11.7 \text{ mm;}$$

we retain

$$e = 12 \text{ mm (one then has th } a = 1 - \varepsilon = 0.987)$$

- Bond length

In accordance with Section 6.2.3, the resistance condition is written as:



schematic of fitting of tube

$$\tau_{\text{average}} = \frac{M_t}{2\pi r^2 \ell} \leq 0.2 \times \tau_{\text{rupture}}$$

then:

$$\boxed{\ell \geq 44 \text{ mm}}$$

3. Mass assessment:
   - Mass of the shaft in carbon/epoxy

$$m_{\text{laminate}} = \rho \times 2\pi re \times L$$

   with numerical values already cited:

$$m_{\text{laminate}} = 2.8 \text{ kg.}$$

   - If one takes a tubular shaft made of steel ($\tau_{\text{rupture}} = 300$ MPa) with a factor of safety that is 2 times less, say 3, and a minimum thickness of 2.5 mm, the resistance condition:

$$\frac{M_t}{2\pi r^2 e} \leq \frac{300}{3} \text{ MPa}$$

leads to a radius of the tube of

$$r \geq 43 \text{ mm.}$$

From this we find a mass of: ($\rho_{\text{steel}} = 7{,}800$ kg/m^3):

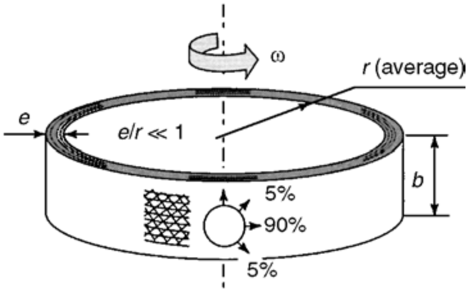
$$m_{\text{steel}} = 10.5 \text{ kg}$$

The saving in mass of the composite solution over the steel solution is 73%. The real saving is higher because it takes into account the disappearance of the intermediate bearing and of one part of the universal joint.

18.1.5 Flywheel in Carbon/Epoxy

Problem Statement:

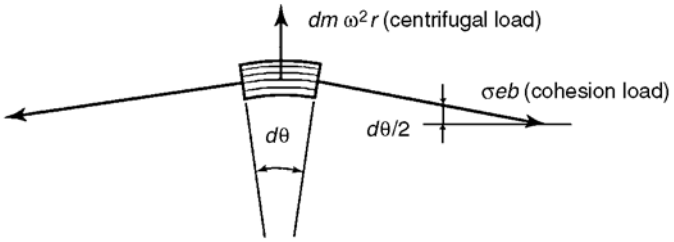
We show schematically in the figure below an inertia wheel made of carbon/epoxy with 60% fiber volume fraction, with the indicated proportions for the orientation of the fibers.



1. Calculate the maximum kinetic energy that one can obtain with such a flywheel with a mass of 1 kg.
2. Compare the maximum kinetic energy that one can obtain with a steel flywheel with a mass of 1 kg. One will take: $\sigma_{\text{rupture steel}} = 1{,}000$ MPa.

Solution:

1. The equilibrium of an element of the wheel (shown below) reveals inertia forces and cohesive forces.



One deduces from there the equilibrium equation along the radial direction:

$$dm \times \omega^2 r = 2\sigma eb \frac{d\theta}{2}$$

Denoted by ρ the specific mass:

$$\rho r \, d\theta \, eb\omega^2 r = \sigma ebd\theta$$

$$\rho(r\omega)^2 = \sigma$$

Denoted by $V = r\omega$ the circumferential speed, one obtains the maximum for the rupture strength of carbon/epoxy, as:

$$\boxed{V_{\max} = \sqrt{\frac{\sigma_{\text{rupture}}}{\rho}}}$$

Numerical application: For the composition of the carbon/epoxy laminate indicated above, one reads in Section 5.4.2, Table 5.1:

$$\sigma_{\text{rupture}} = 1{,}059 \text{ MPa}$$

and with $\rho = 1{,}530$ kg/m^3 (Table 3.4 of Section 3.3.3, or the calculation in Section 3.2.3):

$$V_{\max} = 832 \text{ m/s}$$

from this the maximum kinetic energy obtained with 1kg of composite[7]:

$$W_{\text{Kinetic}} = \frac{1}{2} \times 1 \text{ kg} \times V_{\max}^2$$

then:

$$\boxed{W_{\text{Kinetic}} = 346 \text{ kjoules}}$$

2. The maximum circumferential speed that one can obtain with a steel flywheel can be written as:

$$V_{\text{max. steel}} = \sqrt{\frac{\sigma_{\text{rupture steel}}}{\rho_{\text{steel}}}}$$

Therefore, the ratio of kinetic energies of composite/steel is

$$\frac{W_{\text{Kinetic carbon}}}{W_{\text{Kinetic steel}}} = \frac{V_{\text{max carbon}}^2}{V_{\text{max steel}}^2} = \frac{\sigma_{\text{rupt. carbon}} \times \rho_{\text{steel}}}{\sigma_{\text{rupt. steel}} \times \rho_{\text{carbon}}}$$

with $\rho_{\text{steel}} = 7800 \text{ kg/m}^3$ and $\sigma_{\text{rupt. steel}} = 1000$ MPa, one obtains

$$\boxed{\frac{W_{\text{Kinetic carbon}}}{W_{\text{Kinetic steel}}} = 5.4}$$

With respect to the same mass, it appears then possible to accumulate 5 times more kinetic energy with a flywheel in carbon/epoxy composite.

18.1.6 Wing Tip Made of Carbon/Epoxy

Problem Statement:

Wing tip refers to a part of airplane wing as shown in Figure 18.1. It is made of a sandwich structure with carbon/epoxy skins (Figure 18.2) fixed to the rest of the wing by titanium borders as shown. Under the action of the aerodynamic forces (Figure 18.3), the wing tip is subjected to bending moments, torsional moments, and shear forces as shown in Figure 18.4(a).

One can assume that the core of the sandwich structure transmits only shear forces, and the skins support the flexural moments. This is represented in Figure 18.4(b); the skins resist in their respective planes the in-plane stress resultants: N_x, N_y, and T_{xy}. Figure 18.5 shows the values of these stress resultants

[7] Recall the expression for the rotational kinetic energy of a mass m placed at a radius r and rotating at a speed of ω: $W_{\text{Kinetic}} = \frac{1}{2}I\omega^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}mV_{\text{circonfer}}^2$.

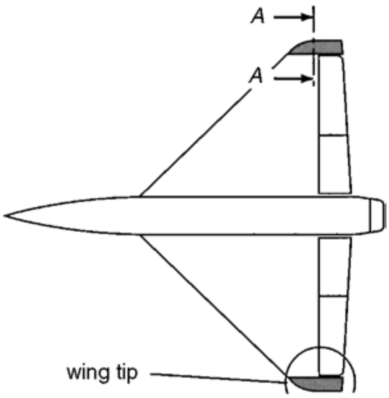


Figure 18.1

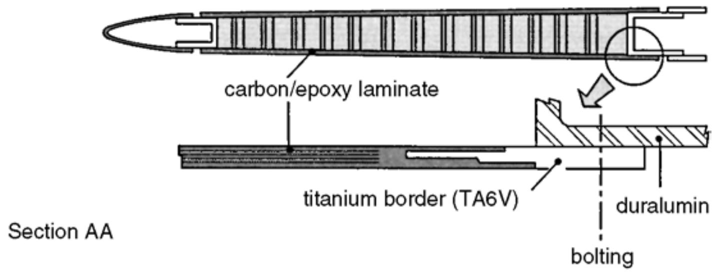


Figure 18.2

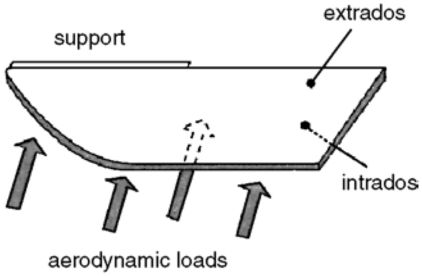


Figure 18.3

at a few points of the upper skin (or extrados).

1. According to Figures 18.4(a) and 18.4(b), deduce the elements of the stress resultants N_x, N_y, and T_{xy} from the knowledge of the moment resultants M_x, M_y, and M_{xy}.
2. Using a factor of safety of 2, define the carbon/epoxy skin that is suitable at the surrounding of the support made of titanium alloy (proportions, thickness, number of plies). One will use unidirectional plies with V_f = 60% fiber volume fraction.

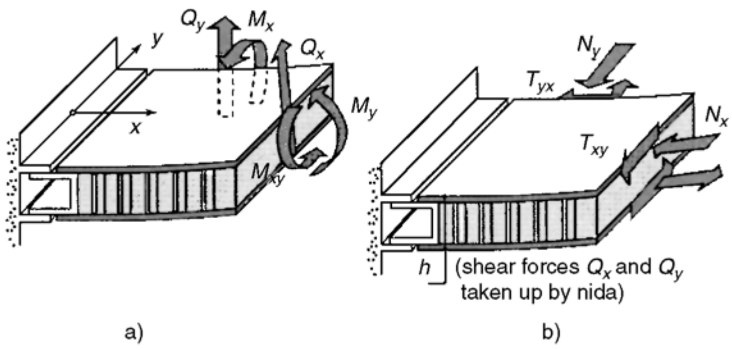


Figure 18.4

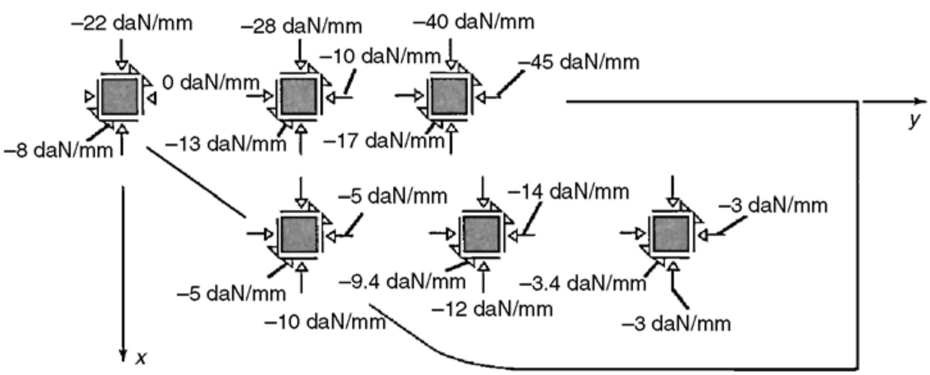


Figure 18.5

3. The skin is bonded on the edge of the titanium (Figure 18.2). Provide the dimensions of the bonded surface by using an average shear stress in the adhesive (araldite: $\tau_{\text{rupture}} = 30$ MPa).
4. The border of the titanium is bolted to the rest of the wing (Figure 18.2). Determine the dimensional characteristics of the joint: "pitch" of the bolts, thickness, foot, with the following data:
   - Bolts: 30 NCD 16 steel: $\varnothing = 6.35$ mm, adjusted, negligible tensile loading, $\sigma_{\text{rupture}} = 1{,}100$ MPa; $\tau_{\text{rupture}} = 660$ MPa; $\sigma_{\text{bearing}} = 1600$ MPa
   - TA6V titanium alloy: $\sigma_{\text{rupture}} = 900$ MPa; $\tau_{\text{rupture}} = 450$ MPa; $\sigma_{\text{bearing}} = 1100$ MPa
   - Duralumin: $\sigma_{\text{rupture}} = 420$ MPa; $\sigma_{\text{bearing}} = 550$ MPa

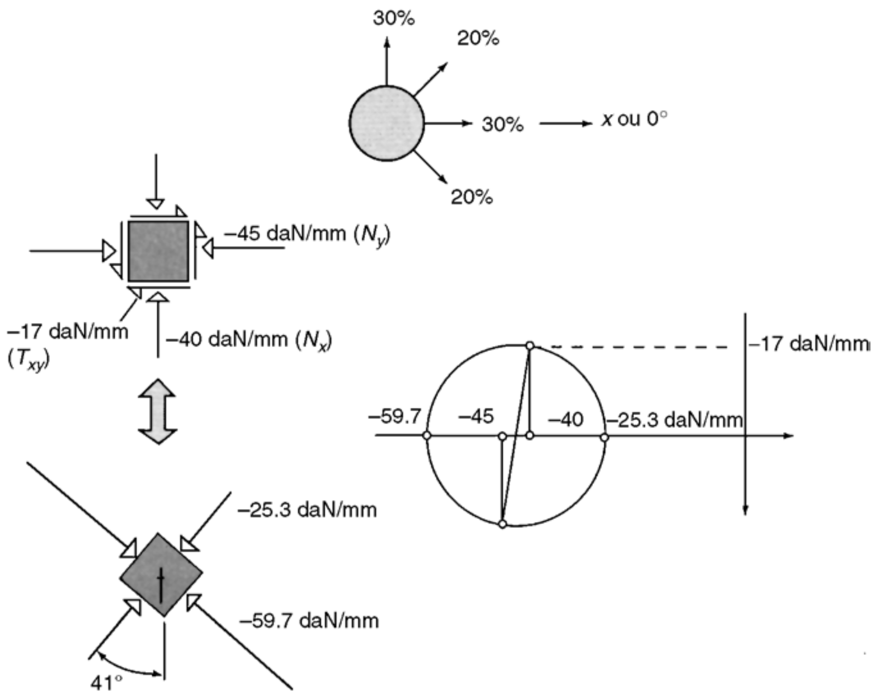
Solution:

1. The moment resultants M_x, M_y, M_{xy} (and M_{yx}, not shown in Figure 18.4a) are taken up by the laminated skins. One then has in the upper skin (Figure 18.4b), h being the mean distance separating the two skins:

$$N_x = \frac{M_y}{h};\; N_y = \frac{-M_x}{h};\; T_{xy} = -\frac{M_{xy}}{h}$$

Remark: The moment resultants, that means the moments per unit width of the skin – 1mm in practice – have units of daN × mm/mm. The stress resultants N_x, N_y, T_{xy} have units of daN/mm.

2. Looking at the most loaded region of the skin, we can represent the principal directions and stresses by constructing Mohr's circle (shown in the following figure). Then we note that there must be a nonnegligible proportion of the fibers at ±45°. However, the laminate has to be able to resist compressions along the axes x and y. The estimation of the proportions can be done following the method presented in Section 5.4.3. One then obtains the following composition[8]:



Let σ_ℓ, σ_t, $\tau_{\ell t}$ be the stresses along the principal axes l, t of one of the plies for the state of loading above, the thickness e of the laminate (unknown *a priori*) such that one finds the limit of the Hill-Tsai criterion of failure.[9] One then has

$$\frac{\sigma_\ell^2}{\sigma_{\ell\ \text{rupture}}^2} + \frac{\sigma_t^2}{\sigma_{t\ \text{rupture}}^2} - \frac{\sigma_\ell \sigma_t}{\sigma_{\ell\ \text{rupture}}^2} + \frac{\tau_{\ell t}^2}{\tau_{\ell t\ \text{rupture}}^2} = 1$$

[8] The calculation to estimate these proportions is shown in detail in the example of Section 5.4.3, where one has used the same values of the resultants with a factor of safety of 2, as: N_x = –800 N/mm; N_y = –900 N/mm; T_{xy} = –340 N/mm.

[9] See Section 5.3.2 and also Chapter 14.

If one multiplies the two sides by the square of the thickness e:

$$\boxed{\frac{(\sigma_\ell e)^2}{\sigma^2_{\ell\ \mathrm{rupture}}}+\frac{(\sigma_t e)^2}{\sigma^2_{t\ \mathrm{rupture}}}-\frac{(\sigma_\ell e)(\sigma_t e)}{\sigma^2_{\ell\ \mathrm{rupture}}}+\frac{(\tau_{\ell t} e)^2}{\tau^2_{\ell t\ \mathrm{rupture}}} = e^2} \qquad [1]$$

one will obtain the values $(\sigma_\ell\, e)$, $(\sigma_t\, e)$, $(\tau_{\ell t}\, e)$, by multiplying the global stresses σ_x, σ_y, τ_{xy} with the thickness e, as $(\sigma_x\, e)$, $(\sigma_y\, e)$, $(\tau_{xy}\, e)$, which are just the stress resultants defined previously:

$$N_x = (\sigma_x e);\quad N_y = (\sigma_y e);\quad T_{xy} = (\tau_{xy} e)$$

Units: the rupture resistances are given in MPa (or N/mm^2) in Appendix 1. As a consequence:

$$\begin{aligned} N_x &= -400\ \text{MPa}\times\text{mm} \\ N_y &= -450\ \text{MPa}\times\text{mm} \\ T_{xy} &= -170\ \text{MPa}\times\text{mm} \end{aligned}$$

with a factor of safety of 2, one then has

$$\begin{aligned} N'_x &= -800\ \text{MPa}\times\text{mm} \\ N'_y &= -900\ \text{MPa}\times\text{mm} \\ T'_{xy} &= -340\ \text{MPa}\times\text{mm} \end{aligned}$$

We use the Plates in annex 1 which show the stresses σ_ℓ, σ_t, $\tau_{\ell t}$ in each ply for an applied stress resultant of unit value (1 MPa, for example):

(a) Plies at 0°:

- Loading $N'_x = -800$ MPa × mm only:

For the proportions defined in the previous question, one reads on Plate 1:

$$\left.\begin{aligned} \sigma_\ell &= 2.4 \\ \sigma_t &= 0.0 \\ \tau_{\ell t} &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &(\sigma_\ell e) = 2.4\times -800 = -1920\ \text{MPa}\times\text{mm} \\ &(\sigma_t e) = 0 \\ &(\tau_{\ell t} e) = 0 \end{aligned}\right.$$

- Loading $N'_y = -900$ MPa × mm only:

One reads from Plate 5:

$$\left.\begin{aligned} \sigma_\ell &= -0.54 \\ \sigma_t &= 0.12 \\ \tau_{\ell t} &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &(\sigma_\ell e) = -0.54\times -900 = 486\ \text{MPa}\times\text{mm} \\ &(\sigma_t e) = 0.12\times -900 = -108\ \text{MPa}\times\text{mm} \\ &(\tau_{\ell t} e) = 0 \end{aligned}\right.$$

■ Loading T'_{xy} l = –340 MPa × mm only:

One reads from Plate 9:

$$\left.\begin{array}{r}\sigma_\ell = 0\\ \sigma_t = 0\\ \tau_{\ell t} = 0.26\end{array}\right\} \rightarrow \left\{\begin{array}{l}(\sigma_\ell e) = 0\\ (\sigma_t e) = 0\\ (\tau_{\ell t} e) = 0.26 \times -340 \ = -89 \ \text{MPa} \times \text{mm}\end{array}\right.$$

The superposition of the three loadings will then give the plies at 0°a total state of stress of

$$\begin{aligned}(\sigma_\ell e) &= -1920 + 486 = -1434 \ \text{MPa} \times \text{mm}\\ (\sigma_t e) &= -108 \ \text{MPa} \times \text{mm}\\ (\tau_{\ell t} e) &= -89 \ \text{MPa} \times \text{mm}\end{aligned}$$

Then we can write the Hill-Tsai criterion in the modified form written above (relation denoted as [1]) in which one notes the denominator with values of the rupture strengths indicated at the beginning of annex 1:

$$e^2 = \frac{1434^2}{1130^2} + \frac{108^2}{141^2} - \frac{1434 \times 108}{1130^2} + \frac{89^2}{63^2} = 4.07$$

$$\underset{(0°)}{e} = 2.02 \ \text{mm}$$

One resumes the previous calculation as follows:

plies at 0°	$(\sigma_\ell e)$	$(\sigma_t e)$	$(\tau_{\ell t} e)$	
N'_x	–1920	0	0	
N'_y	486	–108	0	
T'_{xy}	0	0	–89	e = 2.02 mm
total (MPa × mm)	–1434	–108	–89	

(b) Plies at 90°: One repeats the same calculation procedure by using the Plates 2, 6, and 10. This leads to the following analogous table, with a thickness e calculated as previously (this is the minimum thickness of the laminate below which there will be rupture of the 90° plies).

plies at 90°	$(\sigma_\ell e)$	$(\sigma_t e)$	$(\tau_{\ell t} e)$	
N'_x	432	–96	0	
N'_y	–2160	0	0	
T'_{xy}	0	0	89	e = 2.16 mm
total (MPa × mm)	–1728	–96	89	

(c) Plies at +45°: Using Plates 3, 7, and 11 one obtains:

plies at 45°	$(\sigma_\ell e)$	$(\sigma_t e)$	$(\tau_{\ell t} e)$	
N'_x	–752	–48	72	
N'_y	–846	–54	–81	
T'_{xy}	–1384	55	0	e = 2.64 mm
total (MPa × mm)	–2982	–47	–9	

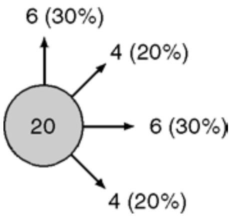
(d) Plies at –45°: Using Plates 4, 8, 12 one obtains:

plies at –45°	$(\sigma_\ell e)$	$(\sigma_t e)$	$(\tau_{\ell t} e)$	
N'_x	–752	–48	–72	
N'_y	–846	–54	81	
T'_{xy}	1384	–55	0	e = 1.13 mm
total (MPa × mm)	–214	–157	9	

Then the theoretical thickness to keep here is the largest out of the four thicknesses found, as:

$$e = 2.64 \text{ mm (rupture of the plies at +45°).}$$

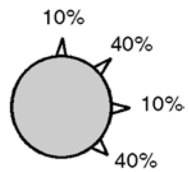
The thickness of each ply is 0.13 mm. It takes 2.64/0.13 = 20 plies minimum from which is obtained the following composition allowing for midplane symmetry:



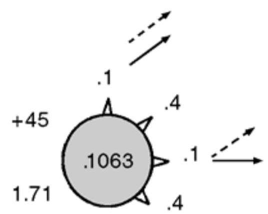
Remark: Optimal composition of the laminate: For the complex loading considered here, one can directly obtain the composition leading to the minimum thickness by using the tables in Section 5.4.4. One then uses the reduced stress resultants, deduced from the resultants taken into account above, to obtain

$$\begin{aligned}
\bar{N}_x &= -800/(|800| + |900| + |340|) = -39\% \\
\bar{N}_y &= -900/(|800| + |900| + |340|) = -44\% \\
\bar{T}_{xy} &= -17\%
\end{aligned}$$

Table 5.19 of Section 5.4.4 allows one to obtain an optimal composition close to

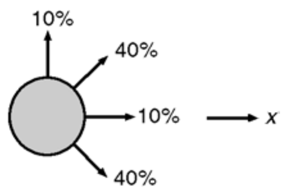


If one uses the previous exact stress resultants, the calculation by computer of the optimal composition leads to the following result, which can be interpreted as described in Section 5.4.4.



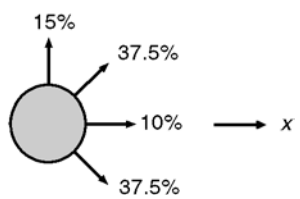
Then one has for the minimum thickness of the laminate:

$$\textit{thickness: } e = 0.1063 \times \frac{(|800| + |900| + |340|)}{100} = 2.17 \text{ mm}$$

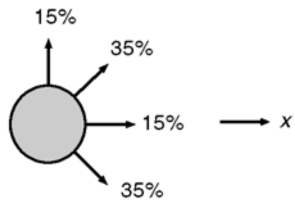


and for the two immediate neighboring laminates:

$$\textit{thickness: } e = 0.1068 \times \frac{(|800| + |900| + |340|)}{100} = 2.18 \text{ mm}$$



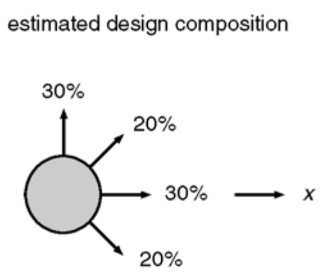
$$\textit{thickness: } e = 0.1096 \times \frac{(|800| + |900| + |340|)}{100} = 2.24 \text{ mm}$$



One notes a sensible difference between the initial composition estimated by the designer and the optimal composition. This difference in composition leads to a relative difference in thickness:

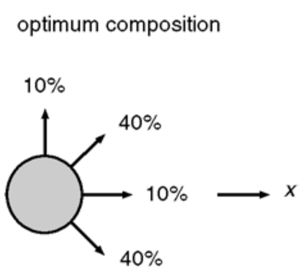
$$\frac{2.64 - 2.17}{2.17} = 21\%$$

which indicates a moderate sensibility concerning the effect of thickness and, thus, the mass. One foresees there a supplementary advantage: the possibility to reinforce the rigidity in given directions without penalizing very heavily the thickness. We can note this if we compare the moduli of elasticity obtained starting from the estimated design composition (Section 5.4.3) with the optimal composition, we obtain (Section 5.4.2, Tables 5.4 and 5.5) very different values noted below:



$$E_x = 55{,}333 \ MPa$$

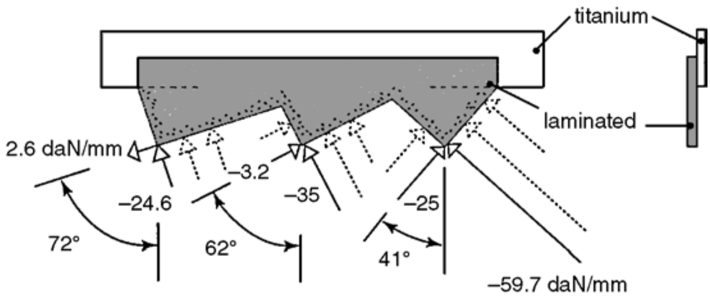
$$G_{xy} = 16{,}315 \ MPa$$



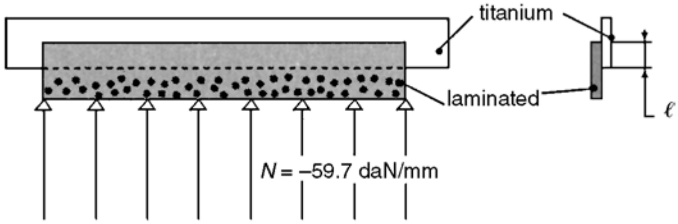
$$E_x = 31{,}979 \ MPa$$

$$G_{xy} = 28{,}430 \ MPa$$

3. Bonding of the laminate: We represent here after the principal loadings deduced from the values of the stress resultants in Figure 18.5, in the immediate neighborhood of the border of titanium:



One can for example, overestimate these loadings by substituting them with a fictitious distribution based on the most important component among them. Taking –59.7 daN/mm, one obtains then the simplified schematic below:



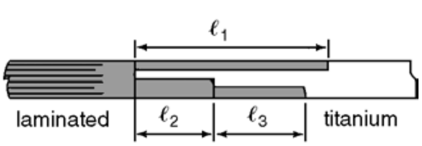
One must evaluate the width of bonding noted as ℓ. For a millimeter of the border, this corresponds to a bonding surface of $\ell \times 1$ mm. For an average rupture criterion of shear of the adhesive, one can write (see Section 6.2.3):

$$\frac{N}{\ell \times 1} \leq 0.2 \times \tau_{\text{adhesive rupture}}$$

then with $\tau_{\text{rupture}} = 30$ MPa:

$$\boxed{\ell \geq \frac{597}{0.2 \times 30} \,\#\, 100 \text{ mm}}$$

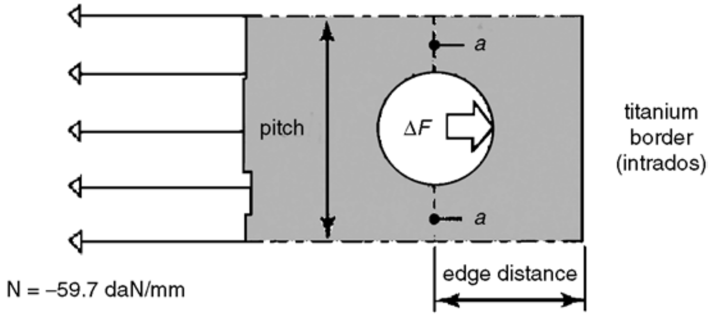
From this one obtains the following configuration such that $\ell_1 + \ell_2 + \ell_3 = 100$ mm.



4. Bolting on the rest of the wing:
   - "Pitch of bolts": The tensile of bolting is assumed to be weak, then bolting strength is calculated based on shear. The bolt load transmitted by a bolt is denoted as ΔF, and one has (*cf*. following figure):

$$\Delta F = N \times \text{pitch} \leq \frac{\pi \varnothing^2}{4} \times \tau_{\text{bolt rupture}}$$

where ∅ is the diameter of the bolt. One finds a pitch equal to 35 mm.



This value is a bit high. In practice, one takes pitch ≤ 5 ∅, for example, here:

$$\text{Pitch} = 30 \text{ mm.}$$

- Thickness of the border: the bearing condition is written as:

$$\frac{N \times \text{pitch}}{\varnothing e_{\text{titanium}}} \leq \sigma_{\text{bearing}}$$

then:

$$\boxed{e_{\text{titanium}} \geq 2.55 \text{ mm}}$$

- Verification of the resistance of the border in the two zones denoted *a* in the previous figure: the stress resultant in this zone, noted as N', is such that:

$$N \times \text{pitch} = N' \times (\text{pitch} - \varnothing)$$

then:

$$N' = N\frac{\text{pitch}}{\text{pitch} - \varnothing} = 75.4 \text{ daN/mm}$$

The rupture stress being:

$$\sigma_{\text{rupture}} = 900 \text{ MPa}$$

and the minimum thickness 2.55 mm, one must verify

$$\frac{N'(\text{daN/mm})}{e(\text{mm})} \leq \sigma_{\text{rupture}}(\text{daN/mm})$$

One effectively has

$$\frac{75.4}{2.55} \leq 90$$

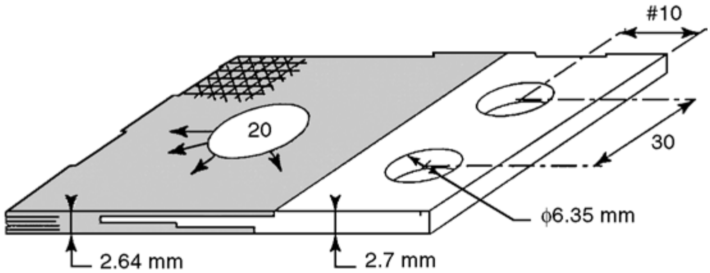
■ Verification of the edge distance (see previous figure): One has to respect the following condition:

$$\frac{\Delta F}{2 \times \text{edge distance} \times e} \leq \tau_{\text{titanium rupture}}$$

then:

$$\text{edge distance} \geq 7.8 \text{ mm}$$

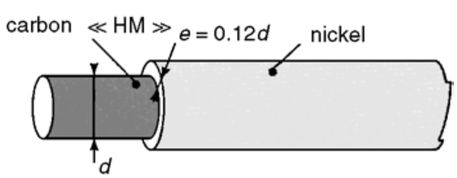
from which the configuration (partial) of the joint can be shown as in the following figure:



18.1.7 Carbon Fiber Coated with Nickel

Problem Statement:

With the objective of enhancing the electrical and thermal conductivity of a laminated panel in carbon/epoxy, one uses a thin coat of nickel with a thickness e for the external coating of the carbon fibers by electrolytic plating process (see following figure).



1. Calculate the longitudinal modulus of elasticity of a coated fiber.
2. Calculate the linear coefficient of thermal expansion in the direction of the coated fiber.

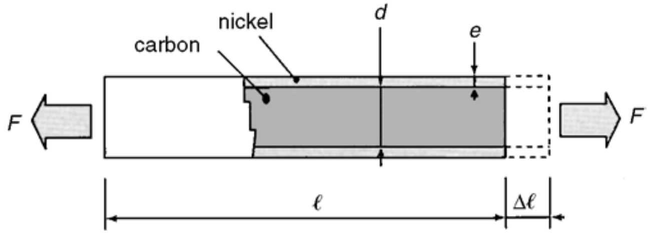
Solution:

1. Hooke's law applied to a fiber with length ℓ subject to a load F (following figure) can be written as:

$$F = E_f s \frac{\Delta \ell}{\ell}$$

where E_f is the modulus of the coated fiber that one wishes to determine, and

$$s = \pi \,(d/2 + e)^2$$



The load F is divided into F_C on the carbon fiber and F_N on the nickel coating. The equality of the elongations of the two components allow one to write

$$F_c = E_c \pi \frac{d^2}{4} \frac{\Delta \ell}{\ell}; \quad F_N = E_N \pi \left[\left(\frac{d}{2} + e \right)^2 - \frac{d^2}{4} \right] \frac{\Delta \ell}{\ell}$$

where, taking into account that $F = F_C + F_N$,

$$E_f \pi \left(\frac{d}{2} + e \right)^2 = E_c \pi \frac{d^4}{4} + E_N \pi \left[\left(\frac{d}{2} + e \right)^2 - \frac{d^2}{4} \right]$$

$$\boxed{E_f = E_c \frac{1}{\left(1 + \frac{2e}{d}\right)^2} + E_N \left[1 - \frac{1}{\left(1 + \frac{2e}{d}\right)^2} \right]}$$

Numerical application:

$$E_C = 390{,}000 \text{ MPa}; \; E_N = 220{,}000 \text{ MPa}; \; d = 6.5 \; \mu\text{m (Section 1.6)}$$

$$E_f = 330{,}500 \text{ MPa}.$$

2. Thermal expansion of an unloaded rod with length $\ell = 1$ m and corresponding to a temperature variation ΔT can be written as:

$$\Delta \ell_1 = \alpha \, \Delta T \times 1$$

where α is the thermal expansion coefficient of the material making up the rod. In addition, when this rod is subjected to a longitudinal stress σ, Hooke's law indicates a second expansion:

$$\Delta \ell_2 = \frac{\sigma}{E} \times 1$$

Superposition of the two cases simultaneously applied can be written as:

$$\Delta \ell = \Delta \ell_1 + \Delta \ell_2$$

or:

$$\Delta\ell = \left(\frac{\sigma}{E} + \alpha\Delta T\right) \times 1$$

When the coated fiber is subjected to a variation in temperature ΔT, each of the constituents will elongate an identical amount $\Delta\ell$. The whole coated fiber is not subjected to any external forces. The difference in the coefficients of thermal expansion of carbon and of nickel, which should lead to different free thermal expansions, then leads to the equilibrium of loads inside the coated fiber.

Let α_f be the thermal coefficient of expansion of the coated fiber. One has

$$\Delta\ell = \alpha_f \, \Delta T \times 1$$

Then for the carbon and for the nickel:

$$\Delta\ell = \frac{\sigma_c}{E_c} + \alpha_c\Delta T = \frac{\sigma_N}{E_N} + \alpha_N\Delta T \qquad [2]$$

The forces being in equilibrium

$$\pi\left[\left(\frac{d}{2} + e\right)^2 - \frac{d^2}{4}\right]\sigma_N + \pi\frac{d^2}{4}\sigma_c = 0 \qquad [3]$$

Equations [2] and [3] lead to

$$\sigma_c = (\alpha_N - \alpha_c)\Delta T \Big/ \left[\frac{1}{E_c} + \frac{1}{E_N} \times \frac{1}{(1 + \frac{2e}{d})^2 - 1}\right]$$

and taking into account that

$$\alpha_f\Delta T = \Delta\ell = \frac{\sigma_c}{E_c} + \alpha_c\Delta T$$

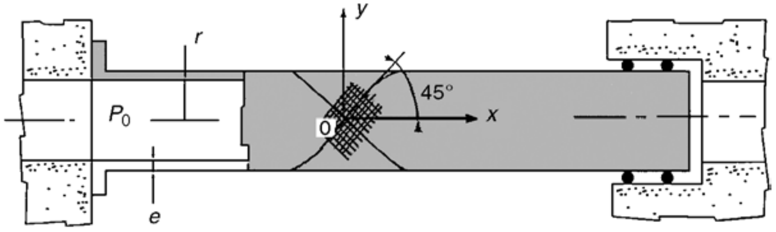
one obtains

$$\boxed{\alpha_f = \frac{\alpha_N + \alpha_c \frac{E_c}{E_N} \frac{1}{\left[\left(1 + \frac{2e}{d}\right)^2 - 1\right]}}{1 + \frac{E_c}{E_N} \frac{1}{\left[\left(1 + \frac{2e}{d}\right)^2 - 1\right]}}}$$

18.1.8 Tube Made of Glass/Epoxy under Pressure

Problem Statement:

Consider a thin tube made by filament winding of glass/epoxy with a winding angle of ±45°. The fiber volume fraction is $V_f = 0.6$. The tube is fixed at one end to a rigid undeformable mass and mounted to a sliding joint at the other end (see following figure).



The thickness e is considered to be small as compared with the average radius (e/r << 1). One applies on the inside of the tube a unit pressure of p_o = 1 MPa (or 10 bars). Use a safety factor of 8 to take into account the aging effect.

1. Calculate the stresses (σ_x, σ_y) along the axes x and y in the tangent plane at 0 of the tube.
2. What is the maximum stress allowable for the winding considered? From that deduce the minimum thickness of the tube for an average radius of r = 100 mm.
3. What are the moduli E_x, E_y, and G_{xy} of the laminate, and the Poisson coefficients ν_{xy} and ν_{yx}? Write the stress–strain behavior for the laminate in the coordinates $x - y$.
4. Calculate the strains ε_x and ε_y of the composite tube. From there deduce the strain in the direction that is perpendicular to the fiber direction at +45°, denoted as ε_t, which characterizes the strain in the resin.

This strain has to be less than 0.1% to avoid microfracture that can lead to the leakage of the fluid across the thickness of the tube (**weeping** phenomenon).

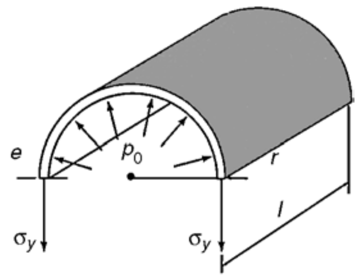
Solution:

1. The tube being free in the axial direction and neglecting the thickness e, $\sigma_x = 0$.

The equilibrium of a half-cylinder with unit length, represented in the figure below, allows one to write:

$$p_0 \times 2r \times l = (\sigma_y \times e \times l) \times 2$$

$$\boxed{\sigma_y = p_0 \times \frac{r}{e}}$$



2. Maximum admissible stress: One reads on Table 5.12, Section 5.4.2, for proportions of plies as 50% in the directions + and –45°:

$$\sigma_{y\ \max\ (\text{tension})} = 94\ \text{MPa}$$

then with $\sigma_{y\ \max} = p_o\ (r/e)$, the theoretical minimum thickness is

$$e_{\text{theoretical}} = \frac{p_o \times r}{\sigma_{y\ \max}} = \frac{1\ \text{MPa} \times 100\ \text{mm}}{94\ \text{MPa}} = 1.064\ \text{mm}$$

Taking into account the factor of safety of 8 for aging effect

$$e = 8.5\ \text{mm}$$

3. Moduli of the laminate: One reads on Table 5.14, Section 5.4.2:

$$E_x = 14{,}130\ \text{MPa} = E_y$$

$$\nu_{xy} = 0.57 = \nu_{yx}$$

and from Table 5.15:

$$G_{xy} = 12{,}760\ \text{MPa}.$$

Recalling the stress–strain relation for an anisotropic material described in Section 3.1, which is repeated here as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yx} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ \frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

one has

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{14130}\begin{bmatrix} 1 & -0.57 & 0 \\ -0.57 & 1 & 0 \\ 0 & 0 & 1.107 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

4. Strains: For $p_o = 1$ MPa and $e = 8.5$ mm, one has

$$\sigma_y = \frac{1 \text{ MPa} \times 100}{8.5} = 11.8 \text{ MPa}$$

then

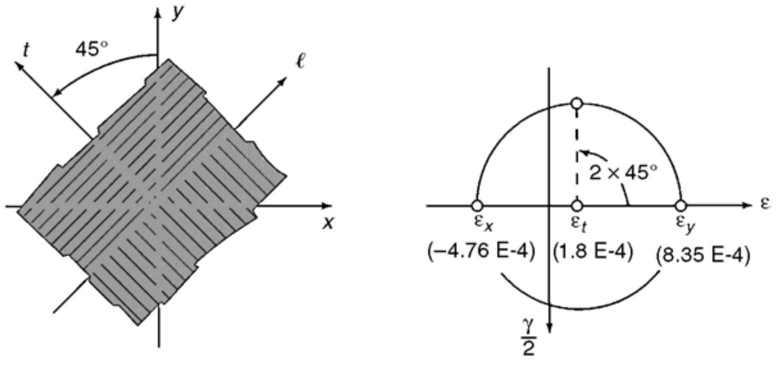
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{14\ 130}\begin{bmatrix} 1 & -0.57 & 0 \\ -0.57 & 1 & 0 \\ 0 & 0 & 1.107 \end{bmatrix}\begin{Bmatrix} 0 \\ 11.8 \\ 0 \end{Bmatrix}$$

from which

$$\varepsilon_x = -4.76 \times 10^{-4}$$

$$\varepsilon_y = 8.35 \times 10^{-4}$$

The Mohr's circle of strains, shown below, allows one to obtain the strain ε_t in the direction perpendicular to the fibers.



One obtains

$$\varepsilon_t = \frac{\varepsilon_x + \varepsilon_y}{2} = 1.8 \times 10^{-4}$$

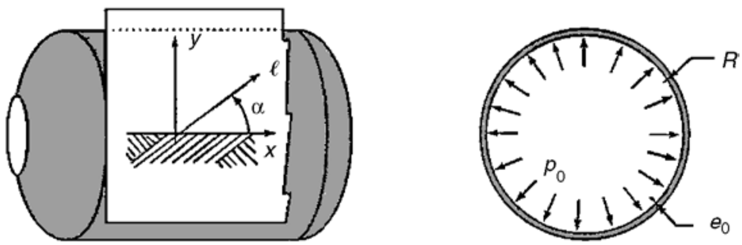
$$\varepsilon_t = 0.018\%$$

One verifies that the strain in the matrix is less than 0.1%, the maximum limit.

18.1.9 Filament Wound Vessel, Winding Angle

Problem Statement:

One considers a vessel having the form of a thin shell of revolution, wound of "R" glass/epoxy rovings. In the cylindrical portion (see figure) the thickness is e_o which is small compared with the average radius R. This vessel is loaded by an internal pressure of p_o.



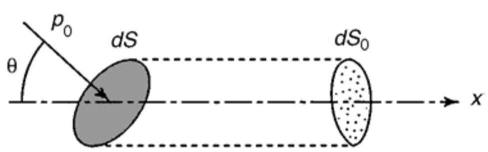
1. The resin epoxy is assumed to bear no load. Denoting by e the thickness of the reinforcement alone, calculate in the plane x,y (see figure) the stresses σ_{ox} and σ_{oy} in the wall, due to pressure p_o.
2. In the cylindrical part of the vessel, the winding consists of layers at alternating angles $\pm\alpha$ with the generator line (see figure). One wishes that the tension in each fiber along the direction ℓ could be of a uniform value σ_ℓ. (This uniform tension in all the fibers gives the situation of **isotensoid**.)
   (a) Evaluate the stresses σ_x and σ_y in the fibers as functions of σ_ℓ.
   (b) Deduce from the above the value of the helical angle α and the tension σ_ℓ in the fibers as functions of the pressure p_o.
   (c) What will be the thickness e_o for a reservoir of 80 cm in diameter that can support a pressure of 200 bars with 80% fiber volume fraction?

Solution:

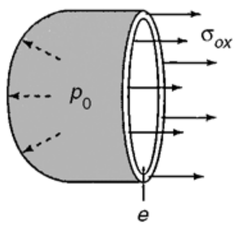
1. Preliminary remark: The elementary load due to a pressure p_o acting on a surface dS has a projection on the x axis as (see figure):

$$p_o \; dS \cos\theta = p_o \; dS_o$$

where dS_o is the x axis projection of dS in a plane perpendicular to this axis.



- Equilibrium along the axial direction: The equilibrium represented in the following figure leads to

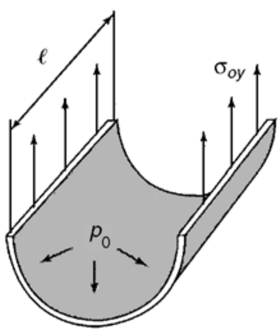


$$p_0 \times \pi R^2 = 2\pi R e \sigma_{ox}$$

then:

$$\boxed{\sigma_{ox} = \frac{p_0 R}{2e}}$$

- Equilibrium along the circumferential direction: The equilibrium represented in the following figure leads to

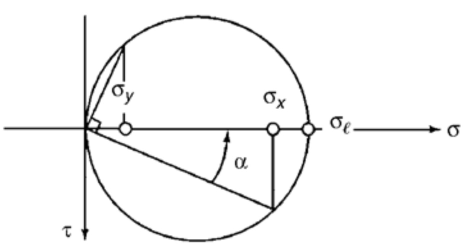


$$p_0 \times 2R \times \ell = \sigma_{oy} \times e \times \ell \times 2$$

then:

$$\boxed{\sigma_{oy} = \frac{p_0 R}{e}}$$

2. (a) Stresses σ_x and σ_y in the fibers: one can represent as follows the Mohr's circle of stresses starting from the pure normal stress σ_ℓ on a face normal to axis ℓ (see following figure). From there, the construction leading to the stress σ_x (see figure below) is geometrically as[10]:



$$\sigma_x = \sigma_\ell \cos^2\alpha$$

and for σ_y:

$$\sigma_y = \sigma_\ell \cos^2\left(\frac{\pi}{2} - \alpha\right) = \sigma_\ell \sin^2\alpha$$

Value of the helical angle α: Identification of these stresses with the values σ_{ox} and σ_{oy} found above leads to

$$\sigma_\ell \cos^2\alpha = \frac{p_0 R}{2e}; \quad \sigma_\ell \sin^2\alpha = \frac{p_0 R}{e}$$

[10] One obtains this result immediately by using the Equation 11.4.

from which:

$$tg^2\alpha = 2$$

then:

$$\boxed{\sin\alpha = \sqrt{\frac{2}{3}};\ \alpha = 54.7°}$$

Tension in the fibers is then:

$$\sigma_\ell = \frac{3}{2}p_0\frac{R}{e}$$

(c) Thickness e_0: One has for "R" glass[11]: σ_ℓ rupture = 3200 MPa.
The thickness e of the reinforcement is such as:

$$e = \frac{3}{2}\ \frac{p_0R}{\underset{\text{rupture}}{\sigma_\ell}} = 3.75 \text{ mm}$$

and the thickness of the glass/epoxy composite, V_f being the fiber volume fraction, is

$$e_0 = e/V_f = 4.7 \text{ mm}$$

18.1.10 Filament Wound Reservoir, Taking the Heads into Account

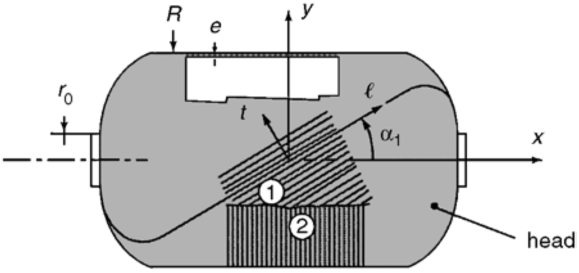
Problem Statement:

A reservoir having the form of a thin shell of revolution is wound with fibers and resin. It is subjected to an internal pressure p_o. The circular heads at the two ends of the reservoir have radius of r_o. We study the cylindrical part of the reservoir, with average radius R.

One part of the winding consists of filaments in helical windings making angles of $\pm\alpha_1$ with the generator (see figure). The other part consists of similar filaments in circumferential windings ($\alpha_2 = \pi/2$).

The resin is assumed to carry no load. The tension in the filaments of the helical layers is denoted by $\sigma_{\ell 1}$ and the tension in the filaments of the circumferential layers by $\sigma_{\ell 2}$.

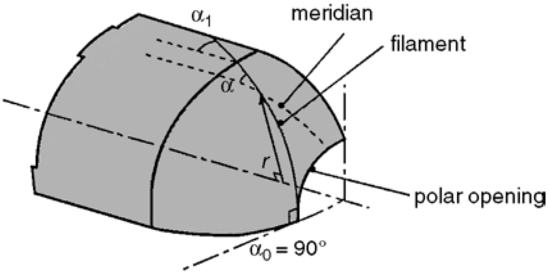
[11] See Section 1.6.



1. What has to be the value of α_1 so that the filaments can elongate on the heads along the lines of shortest distance?
2. Calculate the thickness e_1 of fibers of the helical layers and thickness e_2 of fibers of the circumferential layer as a function of p_o, R, α_1, $\sigma_{\ell 1}$, $\sigma_{\ell 2}$.
3. What is the minimum total thickness of fibers e_m that the envelope can have? What then are the ratios e_1/e_m and e_2/e_m? What is the real corresponding thickness of the envelope if the percentage of fiber volume, denoted as V_f, is identical for the two types of layers?

Note: It can be shown—and one admits—that on a surface of revolution the lines of shortest distance, called the **geodesic** lines, follow the relation (see following figure for the notations):

$$r \sin \alpha = \text{constant}$$



Solution:

1. The filaments wound helically (angle $\pm\alpha_1$) in the cylindrical part follow over the heads along the geodesic lines such that $r \sin \alpha$ = constant. The circle making up the head is a geodesic for which $r = r_o$. Then α = constant. Thus:

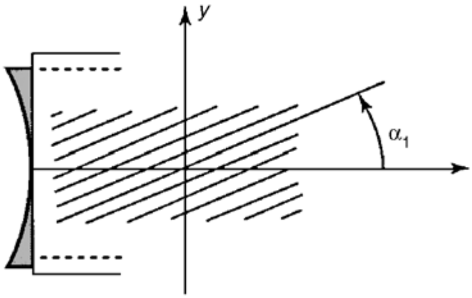
$$\alpha_o = \frac{\pi}{2}$$

One then has for the filaments joining the cylindrical part to the head:

$$r_o \sin \frac{\pi}{2} = R \sin \alpha_1$$

$$\sin \alpha_1 = \frac{r_o}{R}$$

2. Thicknesses of the layers: For an internal pressure p_o, the state of stresses in the cylindrical part of the thin envelope is defined in the tangent plane x, y (following figure) by[12]



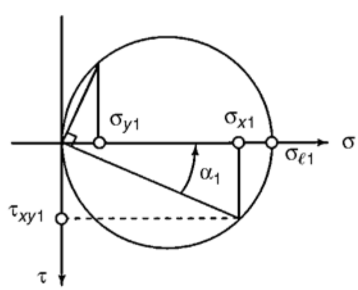
$$\begin{cases} \sigma_{ox} = \dfrac{p_o R}{2e} \\ \sigma_{oy} = \dfrac{p_o R}{e} \\ \tau_{oxy} = 0 \end{cases}$$

The resin being assumed to bear no load, e represents the thickness of the reinforcement alone. One can follow by direct calculation.[13] The state of stress in the helical layers reduce to

$$\sigma_\ell \ (\sigma_{t1} = \tau_{\ell t1} = 0).$$

One obtains for the state of stresses in plane x, y starting from the Mohr's circle (see following figure).[14]

$$\sigma_{x1} = \cos^2\alpha_1 \times \sigma_{\ell 1}; \quad \sigma_{y1} = \sin^2\alpha_1 \times \sigma_{\ell 1};$$
$$\tau_{xy1} = \cos\alpha_1 \sin\alpha_1 \sigma_{\ell 1}$$



and for the circumferential layers ($\alpha_2 = \pi/2$)

$$\sigma_{x2} = 0; \quad \sigma_{y2} = \sigma_{\ell 2}; \quad \tau_{xy2} = 0$$

[12] See Section 18.1.9.

[13] One can also consider a balanced laminate with the ply angles of $+\alpha_1$, $-\alpha_1$, and $\pi/2$. The role of the matrix is neglected. The elastic coefficients of a ply (see Equation 11.11) reduce to only one nonzero E_ℓ. The calculation is done as shown in detail in Section 12.1.3. It is more laborious than the direct method shown above here.

[14] See also the Equation 11.4 inverted.

One then has the following equivalents, in calculating the resultant forces on sections of unit width and normal x and y respectively:

- Along x:

$$\sigma_{x1} \times e_1 \times 1 + \sigma_{x2} \times e_2 \times 1 = \sigma_{ox} \times e \times 1$$

then:

$$e_1 \times \cos^2\alpha_1 \times \sigma_{\ell 1} = e\sigma_{ox} = e \times p_o \frac{R}{2e}$$

from which:

$$\boxed{e_1 = \frac{p_o}{\sigma_{\ell 1}} \times \frac{R}{2\cos^2\alpha_1}}$$

- Along y:

$$\sigma_{y1} \times e_1 \times 1 + \sigma_{y2} \times e_2 \times 1 = \sigma_{oy} \times e \times 1$$

$$e_1 \times \sin^2\alpha_1 \times \sigma_{\ell 1} + e_2 \times \sigma_{\ell 2} = e \times \sigma_{oy} = e \times \frac{p_o R}{e}$$

$$\boxed{e_2 = \frac{p_0}{\sigma_{\ell 2}} R\left(1 - \frac{\mathrm{tg}^2\alpha_1}{2}\right)}$$

3. Minimum thickness of the envelope: With the previous results, the thickness of the reinforcement is written as:

$$e = e_1 + e_2 = p_o R\left\{\frac{1}{2\sigma_{\ell 1}\cos^2\alpha_1} + \frac{2 - \mathrm{tg}^2\alpha_1}{2\sigma_{\ell 2}}\right\}$$

The reinforcements for the helical layers and for the circumferential layers are of the same type. They can be subjected to identical maximum tension. Therefore, at fracture, one has

$$\sigma_{\ell 1} = \sigma_{\ell 2} = \sigma_{\ell\ \mathrm{rupture}}$$

Then:

$$e_{\min} = \frac{p_o R}{2\sigma_{\ell\ \mathrm{rupture}}}\left(\frac{1}{\cos^2\alpha_1} + 2 - \mathrm{tg}^2\alpha_1\right)$$

$$\boxed{e_{\min} = \frac{3}{2} \times \frac{p_o R}{\sigma_{\ell\ \mathrm{rupture}}}}$$

Then for the ratios of thicknesses:

$$\boxed{\frac{e_1}{e_{\min}} = \frac{1}{3\cos^2\alpha_1}; \quad \frac{e_2}{e_{\min}} = \frac{2-\mathrm{tg}^2\alpha_1}{3}}$$

Real thickness of the envelope taking into account the percentage of fiber volume V_f:

$$\frac{dv_{\text{reinforcement}}}{dv_{\text{real}}} = V_f = \frac{2\pi R e_{\min}\ dx}{2\pi R e_{\text{real}}\ dx}$$

$$\boxed{e_{\text{real}} = \frac{3}{2} \times \underset{\text{rupture}}{\frac{p_o R}{\sigma_\ell}} \times \frac{1}{V_f}}$$

18.1.11 Determination of the Volume Fraction of Fibers by Pyrolysis

Problem Statement:

One removes a sample from a carbon/epoxy laminate made up of identical layers of balanced fabric. The measured specific mass of the laminate is denoted as ρ. The specific mass of carbon is denoted as ρ_f, that of the matrix is denoted as ρ_m.

One burns completely the epoxy matrix in an oven. The mass of the residual fibers is compared with the initial mass of the sample. One then obtains the fiber mass denoted as M_f (see Section 3.2.1.).

1. Express as a function of ρ, ρ_f, ρ_m, M_f:
   (a) The fiber volume fraction V_f
   (b) The matrix volume fraction, V_m
   (c) The volume fraction of porosities or voids, V_p
2. Numerical application:

$$\rho = 1{,}500\ \text{kg/m}^3; \quad \rho_f = 1{,}750\ \text{kg/m}^3; \quad \rho_m = 1{,}200\ \text{kg/m}^3; \quad M_f = 0.7$$

Solution:

1. (a) By definition (Section 3.2.2) one has

$$V_f = \frac{v_{\text{fibers}}}{v_{\text{total}}} = \frac{m_{\text{fibers}}}{\rho_f} \times \frac{\rho}{m_{\text{total}}} = M_f \times \frac{\rho}{\rho_f}$$

$$\boxed{V_f = M_f \times \frac{\rho}{\rho_f}}$$

(b) In an analogous manner:

$$V_m = M_m \times \frac{\rho}{\rho_m}$$

and with $M_f + M_m = 1$:

$$\boxed{V_m = (1 - M_f) \times \frac{\rho}{\rho_m}}$$

(c) Noting (Section 3.2.2) that:

$$V_f + V_m + V_p = 1$$

one deduces

$$\boxed{V_p = 1 - \rho \times \left(\frac{M_f}{\rho_f} + \frac{(1 - M_f)}{\rho_m}\right)}$$

2. Numerical application:

$$V_f = 60\%; \quad V_m = 37.5\%; \quad V_p = 2.5\%$$

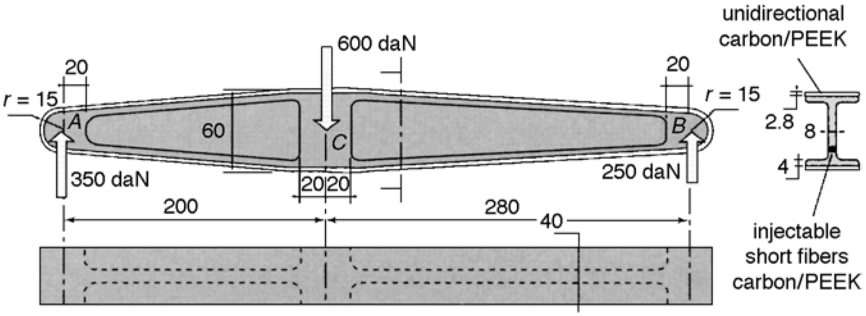
Remark: In practice, a small amount of carbon fibers is also pyrolyzed. About 0.125% of its mass is pyrolyzed per hour.

18.1.12 Lever Arm Made of Carbon/PEEK Unidirectional and Short Fibers

Problem Statement:

The following sketch shows a lever arm pinned at *A*, *B*, *C*. It is subjected to the loads indicated. The external skin is obtained from a plate of thermoformed unidirectional carbon/PEEK,[15] 2.8 mm in thickness, and placed in a mold into which one injects short fibers of carbon/PEEK at high temperature.

		DENSITY (kg/m^3)	σ_{rupture}	*MODULUS OF ELASTICITY* (mPa)
carbon/PEEK unidirectional	$V_f = 65$ %	1600	2100	$E_\ell = 125{,}000$ $G_{\ell t} = 4000$
short fibers carbon/PEEK	$V_f = 18$ %	1400	127	$E = 21{,}000$ $G = 8000$



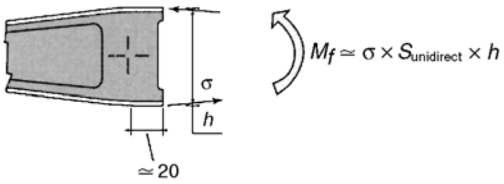
[15] Thermoformed polyether–ether–ketone resin, see Section 1.6.

1. Verify the resistance of this piece by a simplified calculation.
2. Estimate the order of magnitude of the displacements due to loads at A and B.
3. Make an assessment of the mass of the piece.

Solution:

1. Verification of the resistance of the piece:

- Unidirectional: Assume (simplified calculation) that the applied moment is resisted essentially by the unidirectional skins.[16] Then, in the section where the moment is maximum, one has (see following figure):



with

$$S_{\text{unidirectional}} = 2.8 \times 40 \text{ mm}^2; \quad h = 60 - 2 - 2.8 \# 55 \text{ mm};$$

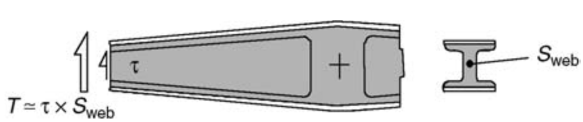
$$M_f = 650 \times 10^3 \text{ N} \times \text{mm}.$$

$$\boxed{\sigma = 106 \text{ MPa}}$$

Factor of safety: $\sigma_{\text{rupture}}/\sigma - 1 = 1880\%$.

Remark: In the injected layer, under the unidirectional skin, the order of magnitude of the normal stress is six times smaller.[16]

- Injected core: We assume that the shear stress due to the shear force is taken up essentially by the web, with an order of magnitude (see following figure):



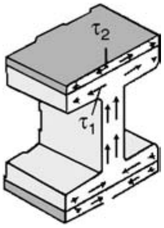
with

$$S_{\text{web}} = (33 - 5.6 - 8) \times 8 \text{ mm};^2 \; T = 3500 \text{ N}$$

$$\tau \# 23 \text{ MPa}$$

[16] This is because the longitudinal modulus of elasticity E_ℓ of the unidirectional is six times higher than that of the injected resin. For more "exact" calculation of the stresses, see Equation 15.16.

Remark: In fact, the distribution of the shear stresses is expanded in the flanges (injected part and unidirectional part in the figure below). The bonding being assumed to be perfect, the distortion is the same in the injected part and in the unidirectional part, as:



$$\gamma = \frac{\tau_2}{G_{\ell t}} = \frac{\tau}{G}$$

$$\tau_2 = 23 \times \frac{G_{\ell t}}{G} = 12 \text{ MPa}$$

2. Displacements under load: Keeping the central part C fixed in translation and in rotation, the deformation energy of each arm (right or left) is written as:

$$W = \frac{1}{2}\int_{\text{arm}} \sigma\varepsilon \, dV + \frac{1}{2}\int_{\text{arm}} \tau\gamma \, dV$$

then (with the previous approximations):

$$W = \frac{1}{2}\int_{\text{unidirect}} \frac{\sigma^2}{E} dS \times dx + \frac{1}{2}\int_{\text{web}} \frac{\tau^2}{G} \, dS \times dx$$

$$W = \frac{1}{2}\int \frac{M_f^2}{E_{\text{unidirect.}} \, (S_{\text{unidirect.}} \times h)^2} \times 2S_{\text{unidirect.}} dx \ldots$$

$$\cdots + \frac{1}{2}\int \frac{T^2}{G \times S_{\text{web}}^2} \times S_{\text{web}} \times dx$$

with $M_f = F(l - x)$; $T = F$; $h = h_{\text{average}}$; $S_{\text{web}} = S_{\text{average web}}$ at midlength of the arm in view of an estimation:

$$W = \frac{1}{2}\int \frac{F^2\ell^3/3}{E_{\text{unidirect.}}\left(S_{\text{unidirect.}} \times \frac{h_{\text{average}}^2}{2}\right)} + \frac{1}{2}\,\frac{F^2\ell}{GS_{\text{average web}}}$$

One obtains for the displacement at the point of application of the load F (Castigliano theorem):

$$\Delta = \frac{\partial W}{\partial F} = \left\{ \frac{\ell^3/3}{\underset{\text{unidirect.}}{E} \times \underset{\text{unidirect.}}{S} \times \frac{h^2_{\text{average}}}{2}} + \frac{\ell}{G \times \underset{\text{web ave.}}{S}} \right\} \times F$$

- Right arm: ℓ = 280 mm; F = 2,500 N; h_{average} = 45 mm – 2.8 mm

$$\boxed{\Delta_B = 1.8 \text{ mm}}$$

- Left arm: ℓ = 200 mm; F = 3,500 N; h_{average} = 45 mm –2.8 mm; $S_{\text{web average}}$ = 31.4 × 8 (mm^2)

$$\boxed{\Delta_A = 1.1 \text{ mm}}$$

3. Mass assessment: Unidirectional: 189 g; short fibers: 525 g; total mass before drilling

$$\boxed{m = 714 \text{ g}}$$

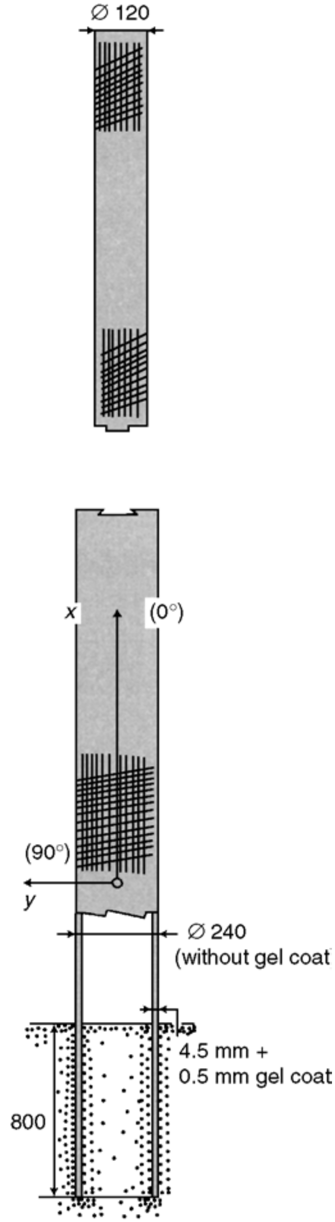
Remarks:

- Taking into account the low levels of stress in the unidirectional, the piece may be lightened in decreasing—uniformly and progressively—its thickness (here 40 mm). A reduction from 40 to 30 mm leads to a reduction of mass of 18% and an increase in displacements from 22 to 26% at A and B.
- To obtain a comparable mass in light alloys, one has to use folded and welded sheet. The price of the piece is higher. The composite piece is obtained by one single operation of injection after preforming of the unidirectional reinforcements.

18.1.13 Telegraphic Mast in Glass/Resin

Problem Statement:

A telegraphic mast 8 m long (of which 80 cm is buried in the ground) of glass/epoxy with 60% fiber volume fraction has the characteristics shown below.



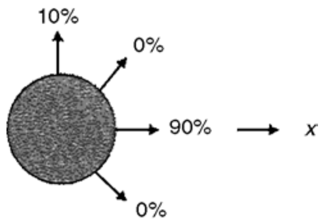
One has in the lower part of the mast:

- 27 layers at 0°—or along x direction
- three layers oriented in helix with an angle that will be taken practically equal to 90°

1. Give the elastic constants of the laminate in this zone.
2. What maximum horizontal load at the top is admissible for this lower zone?
3. Estimate the displacement of the top subject to this load.

Solution:

1. The composition of the laminate in the lower part is as:



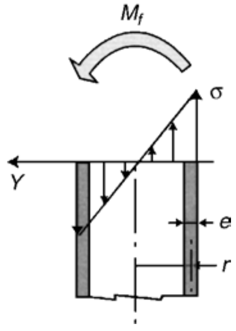
Tables 5.14 and 5.15 of Section 5.4.2 give for this composition:

$$\boxed{\begin{array}{l} E_x = 41{,}860 \text{ MPa}; \quad E_y = 15{,}360 \text{ MPa} \\ \nu_{xy} = 0.23; \quad \nu_{yx} = 0.09 \\ G_{xy} = 4500 \text{ MPa} \end{array}}$$

2. For maximum load at the top, three risks need to be taken into account:

- Risk of rupture due to classical flexure in this zone where the bending moment is maximum
- Risk of rupture due to shear load
- Risk of buckling by ovalization and then flattening of the tube

(a) Flexure moment: One has (see figure below)[17]:



$$\sigma = -\frac{M_f}{I} \times Y \quad \text{with} \quad I = \pi r^3 e$$

[17] See at the end of Section 5.4.5, Figure 5.31 the distribution of stresses in a composite beam. See also Equations 15.16 in Chapter 15.

The maximum is obtained when $Y = -r$:

$$\sigma_{\max} = \frac{M_f}{\pi r^2 e}$$

Note that for the laminate considered (Table 5.11, Section 5.4.2), the first ply fractures at a stress of

$$\sigma_{\text{tensile rupture}} = 128 \text{ MPa}$$

Then

$$M_f \leq 26 \times 10^6 \text{ N} \times \text{mm}$$

corresponding to a horizontal load at the top of the mast:

$$\underset{(M_f)}{F_{\max}} = \frac{26 \times 10^6}{7200} = 3600 \text{ N.}$$

(b) Shear load: On an average diameter of the tubular section (neutral plane), one can write

$$\tau = \alpha\, T/S$$

where T is the shear load, S is the area of the cross section, and α is the amplification factor ($\alpha > 1$).[18] Note that for the laminate considered (Table 5.13, Section 5.4.2), the first ply rupture occurs at $\tau_{\text{rupture}} = 63$ MPa, from which by taking $T = F_{\max(M_f)} = 3{,}600$ N,

$$\alpha < \frac{63 \times 3329}{3600} = 58$$

This condition is well certified (recall that for a thin tube of isotropic material, one has $\alpha = 2$).

(c) Ovalization of the mast: One has (Appendix 2, b)

$$M_{\text{critical}} = \frac{2\sqrt{2}}{9}\pi r e^2 \left[\frac{E_x E_y}{1 - \nu_{xy}\nu_{yx}}\right]^{1/2}$$

Here we have

$$M_{\text{critical}} = 6 \times 10^7 \text{ N mm}$$

[18] The exact value of α should be obtained from the complete study of the shear stresses in a composite beam (Equation 15.16).

which corresponds to a horizontal load at the top:

$$F_{\text{critical ovalisation}} = 8360 \text{ N}$$

One can then retain the maximum load as:

$$F_{\max} = 3600 \text{ N}$$

4. Deflection at the top: If the characteristics of the mast (dimension of the section, composition) were constant along the midline, in taking the average diameter of 180 mm, one would obtain for the previous maximum load the following deflection at the top:

$$\Delta = \frac{F_{\max} \times L^3}{3E_x I_z} \# 1 \text{ m}$$

To obtain a more precise value, it is necessary to discretize the mast into finite elements of shorter beams (four or five) with corresponding sections and moduli (helical angle increasing due to the decreasing diameter, the moduli E_x and E_y vary a little).

18.1.14 Unidirectional Ply of HR Carbon

Problem Statement:

Consider a unidirectional ply made of HR (high strength) carbon/epoxy. What is the fiber volume fraction one can predict to obtain a modulus of elasticity in the longitudinal direction that is comparable to duralumin (AU4G – 2024)?

Solution:

In the fiber direction, the modulus of elasticity E_ℓ is given by the relation (see Section 3.3.1):

$$E_\ell = E_f\, V_f + E_m\,(1 - V_f)$$

One reads in the tables in Section 1.6:

HR carbon: $E_f = 230{,}000$ MPa.

Epoxy resin: $E_m = 4500$ MPa

Duralumin: $E_{2024} = 75{,}000$ MPa.

The fiber volume fraction V_f has to be such that:

$$E_{2024} = E_f V_f + E_m (1 - V_f)$$

where:

$$V_f = \frac{E_{2024} - E_m}{E_f - E_m}$$

$$\boxed{V_f = 31\%}$$

18.1.15 Manipulator Arm of Space Shuttle

Problem Statement:

A manipulator arm is made of two identical tubular columns in carbon/epoxy ($V_f = 60\%$; thin cylindrical tubes of revolution) with pins as shown in Figure 18.6.

Among the different geometric configurations found when the arm is deployed, one can consider the geometries noted as (a), (b), and (c) in Figure 18.7. F represents the concentrated inertial force.

Note the following:

E_x = Longitudinal modulus of elasticity of the tube in x direction (Figure 18.6)
G_{xy} = Shear modulus in the tangent plane x, y (Figure 18.6)
I = Quadratic moment of flexure of a cross section (annular) of the tube with respect to its diameter

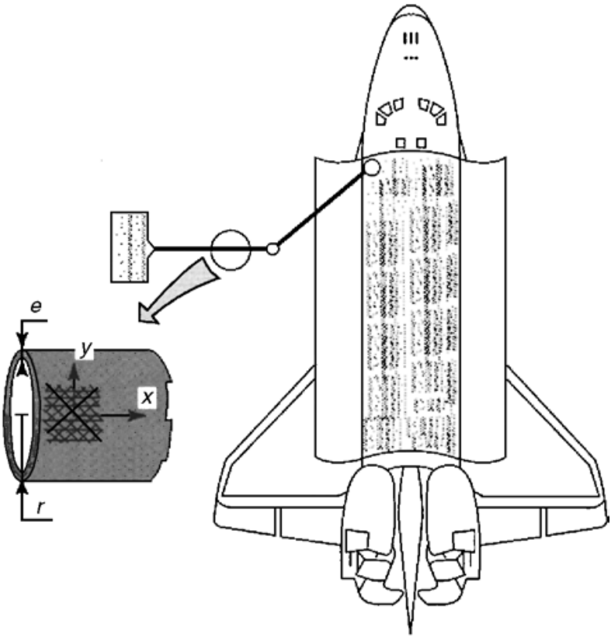


Figure 18.6

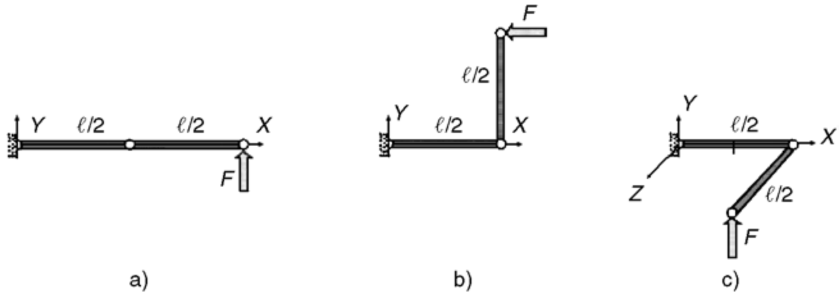


Figure 18.7

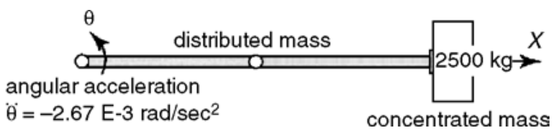


Figure 18.8

1. Calculate the deflection components along the directions X, Y, Z (Figure 18.7) at the point of application of the force F for each of the configurations (a), (b), (c) as function of F, ℓ, I, E_x, and G_{xy} (neglect the strains due to shear force and normal force). Comment on the relative values of these displacements.
2. What should be the ratio between E_x and G_{xy} to obtain identical deflections in the configurations (a) and (c)?
3. The tube is laminated starting from unidirectional layers. By means of the tables giving the moduli E_x and G_{xy} (Section 5.4.2), indicate by simple reading and without interpolation the composition of the laminate that verifies the ratio found in the previous question within a few percentages (choose G_{xy} as large as possible), as well as the values of the elastic characteristics.
4. Verify that this composition is preferable, for the mass assessment, to that of another tube of the same diameter but with different thickness, which has a modulus of elasticity E_x' as large as possible and with the same deflection as previously for configuration (c).
5. Keep the properties determined for the laminate in question 3. The arm has an average diameter of 0.3 m. Each of the two columns has a length of 7.5 m. One imposes a minimum stiffness for the arm $(F/\Delta)_{\text{minimum}} = 10^4$ N/m, where Δ is the deflection under the load F. Calculate the thickness of the tube, indicate the number of total unidirectional layers and the number of layers in each of the four orientations.
6. With the data given in Figure 18.8, verify that the distributed mass of the arm does not significantly influence the previous results during the adjustment in position of the apparatus.

Solution:

1. Starting from the relations of flexure and torsion of composite tubes (see Section 5.4.5, Figure 5.31):

$$E_x I \frac{d^2 v}{dX^2} = M_f; \quad G_{xy} I_o \frac{d\theta_X}{dX} = M_t$$

one obtains for the components of displacement at the end:

- Configuration (a):

$$\Delta_Y = \frac{F\ell^3}{3E_x I}$$

- Configuration (b):

$$\Delta_x = \frac{F(\ell/2)}{E_x I} \times \frac{\ell}{2} \times \frac{\ell}{2} - \frac{F(\ell/2)^3}{3E_x I} = -\frac{F\ell^3}{6E_x I}$$

$$\Delta_y = \frac{F(\ell/2)}{2E_x I} \times \left(\frac{\ell}{2}\right)^2 = \frac{F\ell^3}{16E_x I}$$

- Configuration (c):

$$\Delta_y = \frac{F(\ell/2)^3}{3E_x I} \times 2 + \frac{F(\ell/2)}{G_{xy} I_o} \times \frac{\ell}{2} \times \frac{\ell}{2} = \cdots$$

$$\cdots \frac{F\ell^3}{8E_x I}\left(\frac{2}{3} + \frac{E_x}{2G_{xy}}\right)$$

Remark: For configurations (a) and (b), one obtains a displacement that is as small as the modulus E_x is large. Then (see Section 5.4.2, Tables 5.4 and 5.5), G_{xy} is relatively small, which means that $E_x/G_{xy} \gg 1$. The displacement of configuration (c) is much larger than the others. This will create problems when operating the arm.

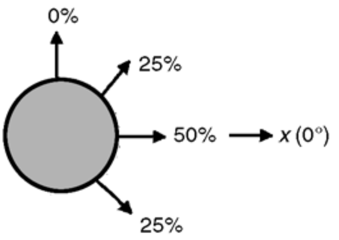
2. The deflections are identical for configurations (a) and (c) if

$$\frac{1}{3} = \frac{1}{8}\left(\frac{2}{3} + \frac{E_x}{2G_{xy}}\right)$$

then:

$$\boxed{\frac{E_x}{G_{xy}} = 4}$$

3. In looking for the modulus G_{xy} to be as high as possible, one reads on Tables 5.4 and 5.5 (Section 5.4.2.) a ratio $E_x/G_{xy} = 3.9$ ($\simeq 4$) for the composition:



$E_x = 75\,407$ MPa; $E_y = 20\,466$ MPa
$G_{xy} = 19343$ MPa; $\nu_{xy} = 0.71$
$\nu_{yx} = 0.19$

4. The maximum value of the longitudinal modulus of elasticity observed on Table 5.4 is

$$E'_x = 134{,}000 \text{ MPa}$$

This corresponds to a shear modulus (Table 5.5):

$$G'_{xy} = 4{,}200 \text{ MPa}$$

The same deflection as the previous one for the configuration (c) can be obtained by

$$\frac{F\ell^3}{8E'_x\,I'}\left(\frac{2}{3}+\frac{E'_x}{2G'_{xy}}\right) = \frac{F\ell^3}{3E_xI}$$

then:

$$\frac{I'}{I} = \frac{\pi r^3 e'}{\pi r^3 e} = \frac{3E_x}{8E'_x}\left(\frac{2}{3}+\frac{E'_x}{2G'_{xy}}\right) = 3.5$$
$$\frac{e'}{e} = 3.5$$

The tube with thickness e' and modulus E'_x will be more stiff for configuration (a) but will have a mass multiplied by 3.5 to keep the stiffness of configuration (c).

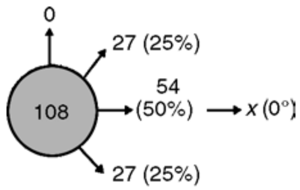
5. Configurations (a) and (c) are the more deformable. One then has to write

$$\frac{F}{\Delta_y} = \frac{3E_xI}{\ell^3} \geq \left(\frac{F}{\Delta}\right)_{\min}$$

with $\ell = 15$ m; $I = \pi r^3 e$; $r = 0.15$ m; $(F/\Delta)_{\min} = 10^4$ N/m; $E_x = 75{,}407$ MPa

$$e \geq 14 \text{ mm}$$

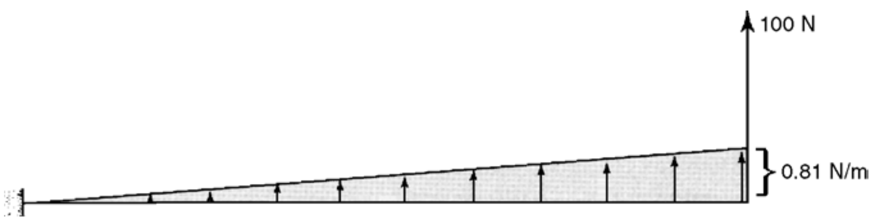
The ply thickness being 0.13 mm, one obtains 108 layers oriented as follows:



6. The specific mass of the laminate is indicated in Section 3.3.3, as $\rho = 1530$ kg/m.[3] The distributed mass of the arm is then:

$$\frac{m}{\ell} = 2\pi re \times \rho = 20.2 \text{ kg/m}$$

with the angular acceleration indicated in Figure 18.8, one obtains the following inertial load:



We then deduce from there:

- The deflection at the end due to the concentrated mass:

$$\Delta_{\text{concent.}} = \frac{100\ell^3}{3E_xI}$$

- The deflection at the end due to distributed load[19]:

$$\Delta_{\text{distributed}} = \frac{11}{120} \times \frac{0.81\ell^4}{E_xI}$$

from which we can obtain a total deflection:

$$\Delta_{\text{total}} = \frac{100\ell^3}{3E_xI}(1 + 0.033) \;\#\; \frac{100\ell^3}{3E_xI}$$

The rigidity (F/Δ_{total}) appears to be well related essentially to the concentrated inertial load at the extremity of the arm.

[19] Result obtained from the differential equation: $EI_x d^2v/dX^2 = -\frac{0.81}{6}\ell^2[2 - 3(X/\ell) + (X/\ell)^3]$

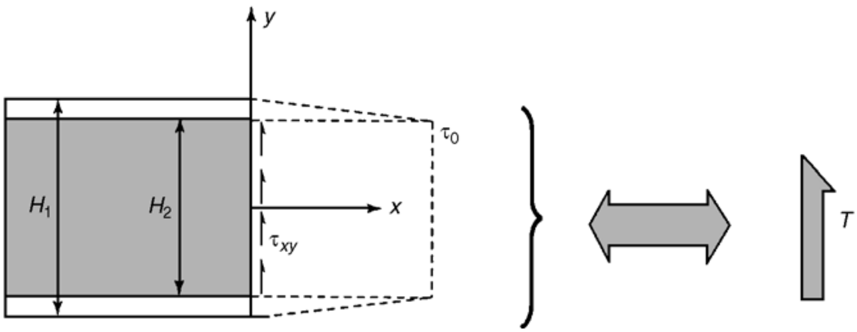
18.2 LEVEL 2

18.2.1 Sandwich Beam: Simplified Calculation of the Shear Coefficient

Problem Statement:

Represented below is the cross section of a sandwich beam. The thickness of the skins is small compared with that of the core. Under the action of a shear load T, the shear stresses in the section are assumed to vary in a piecewise-linear fashion[20] along the y direction. The constitutive materials, denoted as 1 and 2, are assumed to be isotropic, or transversely isotropic. The shear moduli are denoted as G_1 for material 1 (skin) and G_2 for material 2 (core). The beam has a width of unity.

1. Calculate the shear coefficient k for flexure in the plane x,y.



2. Give a simplified expression for the case—current in the applications—where $G_1 \gg G_2$ with the notations for the thicknesses:

$$e_p = \frac{H_1 - H_2}{2}; \quad e_c = H_2$$

Solution:

1. Let W be the strain energy due to shear stresses. One has (Equation 15.17):

$$\frac{dW}{dx} = \frac{1}{2}\frac{kT^2}{\langle GS \rangle} = \frac{1}{2}\int_{\text{section}} \frac{\tau_{xy}^2}{G_i}dy$$

In the upper skin, one has

$$\tau_{xy} = \frac{H_1 - 2y}{H_1 - H_2} \times \tau_o$$

[20] This representation of the shear stresses is only approximate. One will find in Application 18.3.5 the results concerning a more precise distribution of these stresses. In fact, the approximate representation of the shear proposed here is better approximated than the skins of the sandwich structure will have a small thickness as compared to that of the core.

On the other hand in the core: $\tau_{xy} = \tau_0$

Then with: $$T = \int_{\text{section}} \tau_{xy}(dy \times 1)$$

One deduces from there the maximum shear stress τ_o:

$$\tau_o = T \times \frac{2}{H_1 + H_2}$$

Strain energy:

$$\frac{dW}{dx} = \frac{1}{2}\int \frac{\tau_{xy}^2}{Gi} dy = \int_o^{H_2/2} \frac{\tau_o^2}{G_2} dy + \int_{H_2/2}^{H_1/2} \frac{\tau_o^2}{G_1} \frac{(H_1 - 2y)^2}{(H_1 - H_2)^2} dy$$

After calculation:

$$\frac{1}{2}\int \frac{\tau_{xy}^2}{G_i} dy = \frac{\tau_o^2}{2}\left(\frac{H_2}{G_2} + \frac{H_1 - H_2}{3G_1}\right) = \frac{2T^2}{(H_1 + H_2)^2}\left(\frac{H_2}{G_2} + \frac{H_1 - H_2}{3G_1}\right)$$

one then has

$$\frac{1}{2}\frac{kT^2}{\langle GS \rangle} = \frac{2T^2}{(H_1 + H_2)^2}\left(\frac{H_2}{G_2} + \frac{H_1 - H_2}{3G_1}\right)$$

Then:

$$k = \frac{4\langle GS \rangle}{(H_1 + H_2)^2}\left(\frac{H_2}{G_2} + \frac{H_1 - H_2}{3G_1}\right)$$

with (Equation 15.16): $\langle GS \rangle = G_1\ (H_1 - H_2) + G_2\ H_2$:

$$\boxed{k = \frac{4[G_1(H_1 - H_2) + G_2 H_2]}{(H_1 + H_2)^2}\left(\frac{H_2}{G_2} + \frac{H_1 - H_2}{3G_1}\right)}$$

2. Case where $G_2 \ll G_1$: One can rewrite

$$k = \frac{4\langle GS \rangle}{(e_c + 2e_p + e_c)^2} \times \frac{e_c}{G_c}\Bigg[1 + \underbrace{\frac{2}{3}\ \frac{e_p G_c}{e_c G_p}}_{\ll 1}\Bigg]$$

then:

$$k \# \frac{\langle GS \rangle}{e_c^2\left(1 + \frac{e_p}{e_c}\right)^2} \times \frac{e_c}{G_c}$$

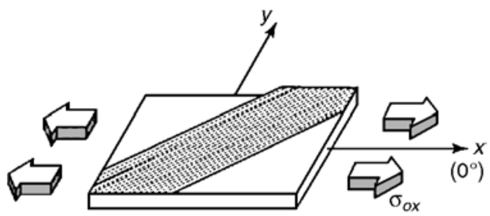
One obtains the following simplified form, valid if $e_p \ll e_c$ and $G_c \ll G_p$:

$$\boxed{\frac{k}{\langle GS \rangle} = \frac{1}{G_c(e_c + 2e_p)}}$$

18.2.2 Procedure for Calculation of a Laminate

Problem Statement:

Consider a balanced carbon/epoxy laminate with respect to the 0° direction (or x), having midplane symmetry. The plies have the orientations 0°, 90°, +45°, −45° with a certain proportions (recall that there are as many plies of +45° as there are −45°). This laminate is subjected to a unit uniaxial stress σ_{ox} = 1 MPa (see following figure).



Propose a procedure to establish a simple program allowing one to obtain the following:

1. The modulus of elasticity E_x of the laminate and the Poisson coefficient ν_{xy}[21]
2. The stresses in each ply and in the orthotropic axes of this ply[22]
3. The Hill–Tsai[23] expression for each ply
4. The largest stress $\sigma_{ox\ \max}$ admissible without failure of any ply

One gives in the following the characteristics of the unidirectional plies (identical) making up the laminate:

Carbon/epoxy ply with V_f = 60% fiber volume fraction

E_ℓ = 134,000 MPa[24]; E_t = 7000 MPa; G_{lt} = 4200 MPa; ν_{lt} = 0.25

Fracture strengths:
$\sigma_{\ell\ \text{tension}}$ = 1,270 MPa; $\sigma_{\ell\ \text{compression}}$ = 1130 MPa
$\sigma_{t\ \text{tension}}$ = 42 MPa; $\sigma_{t\ \text{compression}}$ = 141 MPa
$\tau_{\ell t}$ = 63 MPa

[21] See Equation 12.8.
[22] These are the stresses σ_ℓ, σ_t, $\tau_{\ell t}$ (see, for example, Equation 11.1).
[23] See Chapter 14.
[24] See Section 3.3.3, Table 3.4.

Solution:

Recall first the procedure for calculation (see also Section 12.1.3.):

1. Modulus E_x and Poisson coefficient ν_{xy}: The behavior of a laminate having midplane symmetry and working in its plane can be written as (Equation 12.7):

$$\begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} \qquad \text{[a]}$$

with:

$$\frac{1}{h}A_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^{k}\frac{e_k}{h}$$

where e_k is thickness of ply k, and h is the total thickness of the laminate. $[\bar{E}_{ij}]_k$ is the stiffness matrix for the ply k in the x,y axes (see Equation 11.8), as:

$$\underset{\text{ply } k}{\begin{Bmatrix} \sigma_{x} \\ \sigma_{y} \\ \tau_{xy} \end{Bmatrix}} = \underset{\text{ply } k}{\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}}\underset{\text{ply } k}{\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}} \qquad \text{[b]}$$

Note that $p^{0°}$ (%), $p^{90°}$ (%), $p^{45°}$ (%), $p^{-45°}$ (%) are the respective proportions of the plies in the directions 0°, 90°, +45°, –45°. The previous terms $(1/h)A_{ij}$ can be written as:

$$\frac{1}{h}A_{ij} = \bar{E}_{ij}^{0°}p^{0°} + \bar{E}_{ij}^{90°}p^{90°} + \bar{E}_{ij}^{45°}p^{45°} + \bar{E}_{ij}^{-45°}p^{-45°} \qquad \text{[c]}$$

Here the terms $\frac{1}{h}A_{13}$, $\frac{1}{h}A_{23}$, and their symmetrical counterparts are zero because the laminate is balanced (see Equation 11.8).

The relation denoted as [a] above is then inverted and can be written as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & 0 \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & 0 \\ 0 & 0 & \frac{1}{\bar{G}_{xy}} \end{bmatrix}\begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} \qquad \text{[d]}$$

where $\bar{E}_x$, $\bar{E}_y$, $\bar{G}_{xy}$, $\bar{\nu}_{xy}$, $\bar{\nu}_{yx}$ are the global moduli and Poisson coefficients of the laminate.

Here this laminate is subjected to a uniaxial stress $\sigma_{ox} = 1$ MPa, then:

$$\varepsilon_{ox} = \frac{\sigma_{ox}}{\bar{E}_x} = \frac{1\ \text{MPa}}{\bar{E}_x\,(\text{MPa})}; \quad \varepsilon_{oy} = -\frac{\bar{\nu}_{xy}}{\bar{E}_x}\sigma_{ox} = -\frac{\bar{\nu}_{xy}}{\bar{E}_x\,(\text{MPa})} \times 1\,\text{MPa}$$

One obtains as well the modulus and the Poisson coefficient required:

$$\boxed{\begin{aligned} \bar{E}_x\,(\text{MPa}) &= \frac{1}{\varepsilon_{ox}} \\ \bar{\nu}_{xy} &= -\varepsilon_{oy} \times \frac{\bar{E}_x(\text{MPa})}{1\quad\text{MPa}} \end{aligned}}$$

2. Stresses in the ply: The previous result gives us the global strains of the laminate, strains that each ply should follow as:

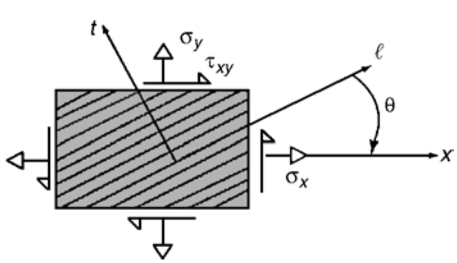
$$\varepsilon_{ox} = \frac{1}{\bar{E}_x}\sigma_{ox}; \quad \varepsilon_{oy} = \frac{\bar{\nu}_{xy}}{\bar{E}_x}\sigma_{ox}; \quad \gamma_{oxy} = 0$$

For a ply k, the relation mentioned above in [b] is then written as:

$$\underbrace{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}}_{\text{ply } k} = \underbrace{\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}}_{\text{ply } k} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ 0 \end{Bmatrix} \qquad [e]$$

This gives the stresses in ply k, expressed in the coordinates x, y. One can express them in the orthotropic axes of the ply (axes ℓ, t of the following figure, and Equation 11.4 recalled below):

$$\underbrace{\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix}}_{\text{ply } k} = \underbrace{\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ sc & -sc & (c^2 - s^2) \end{bmatrix}}_{\text{ply } k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \begin{aligned} c &= \cos\theta \\ s &= \sin\theta \end{aligned} \qquad [f]$$



3. Hill-Tsai expression: It is written as (Equation 14.7):

$$\alpha^2 = \frac{\sigma_\ell^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\sigma_t^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{\sigma_\ell \sigma_t}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\tau_{\ell t}^2}{\underset{\text{rupture}}{\tau_{\ell t}^2}}$$

One can then calculate the values $(\alpha^2)_k$ required for each ply k.

4. The largest stress $\sigma_{ox\,\max}$ without fracture:

The stresses σ_ℓ, σ_t, and $\tau_{\ell t}$ are calculated for a uniaxial stress: $\sigma_{ox} = 1$ MPa. Now apply the maximum stress found $\sigma_{ox\,\max}$ (MPa). The stresses σ_ℓ, σ_t, and $\tau_{\ell t}$ in the ply k are multiplied by the ratio:

$$\frac{\sigma_{ox\ \max}}{1\ \text{MPa}}$$

and the critical value of the Hill-Tsai expression is obtained as:

$$\frac{\sigma_{ox\ \max}^2}{(1\ \text{MPa})^2}\left\{\frac{\sigma_\ell^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\sigma_t^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{\sigma_\ell \sigma_t}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\tau_{\ell t}^2}{\underset{\text{rupture}}{\tau_{\ell t}^2}}\right\}_k = 1$$

With the values $(\alpha^2)_k$ found in the previous question for the Hill-Tsai expression between brackets, one obtains

$$\sigma_{ox\ \max}^2 \alpha_k^2 = (1\text{MPa})^2$$

Then:

$$\sigma_{ox\ \max} = \frac{1\ \text{MPa}}{\alpha_k}$$

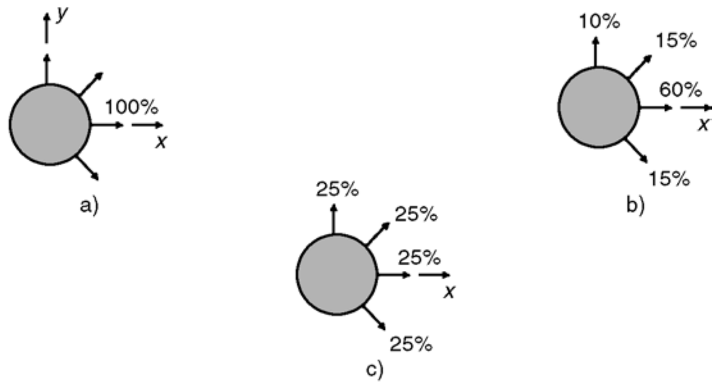
Examination of each ply will lead to a different value for $\sigma_{ox\,\max}$. One has to keep the minimum value as the critical stress that should initialize damage (failure of a ply) as:

$$\boxed{\sigma_{ox\ \max} = \min\frac{1}{\alpha_k}}$$

18.2.3 Kevlar/Epoxy Laminates: Evolution of Stiffness Depending on the Direction of the Load

Problem Statement:

Consider the balanced laminates of Kevlar/epoxy with $V_f = 60\%$ fiber volume fraction, working in their planes, with the following compositions:



1. Give the expression of the longitudinal modulus of elasticity for these laminates denoted as $E(\theta)$ for a direction i in the plane xy making an angle θ with the direction x.
2. Give for each of the laminates the expression for the "specific modulus" $E(\theta)/\rho$, ρ being the mass density. Use the tables in Section 5.4.2.
3. Represent in polar coordinates the variations of the specific modulus with θ for each of the laminates.
4. Compare with the specific moduli of conventional materials, steel, aluminum alloys Duralumin-2024, and titanium alloy TA6V.

Solution:

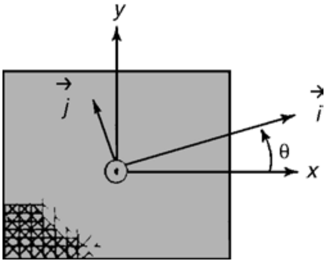
Each of the balanced laminates constitutes a thin plate of orthotropic material, with orthotropic axes x, y, z (see figures above and below). The constitutive relation corresponds with Equation 12.9.

For a balanced laminate, this law is reduced to the following expression:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{yx}}{E_y} & 0 \\ -\dfrac{\nu_{xy}}{E_x} & \dfrac{1}{E_y} & 0 \\ 0 & 0 & \dfrac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix}$$

1. E_x, E_y, G_{xy} are the moduli in the orthotropic axes x, y, which means the moduli of the laminate. In the axes i, j (see following figure) making an angle θ with the axes x, y, these coefficients are transformed according to the Equation 13.8. The modulus in the direction i is[25]

[25] Recalling (Section 9.3 and Application 18.1.2), the relation: $\nu_{xy}/E_x = \nu_{yx}/E_y$.



$$\boxed{E(\theta) = \frac{1}{\frac{\cos^4\theta}{E_x} + \frac{\sin^4\theta}{E_y} + \cos^2\theta\sin^2\theta\left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right)}}$$

2. Specific modulus: One finds in the tables of Section 5.4.2 the coefficients E_x, ν_{xy}, G_{xy} of the Kevlar/epoxy laminates (Tables 5.9 and 5.10). Table 5.9 also allows one to obtain the value E_y. For this it is sufficient to permute the 0° percentage and 90° percentage.

The specific mass ρ is shown in Table 3.4 in Section 3.3.3. It can also be calculated using the relation in Section 3.2.3.

One has ρ = 1350 kg/m^3. One then has for the expressions of the specific modulus:

- Laminate (a):

E_x = 85,000 MPa
E_y = 5600 MPa
G_{xy} = 2100 MPa
ν_{xy} = 0.34

$$\frac{E(\theta)}{\rho}(\mathrm{m/s})^2 = \frac{10^6/1350}{\frac{\cos^4\theta}{85{,}000} + \frac{\sin^4\theta}{5600} + \cos^2\theta\sin^2\theta\left(\frac{1}{2100} - 2\times\frac{0.34}{85{,}000}\right)}$$

- Laminate (b):

E_x = 56,600 MPa
E_y = 18,680 MPa
G_{xy} = 8030 MPa
ν_{xy} = 0.4

$$\frac{E(\theta)}{\rho}(\mathrm{m/s})^2 = \frac{10^6/1350}{\frac{\cos^4\theta}{56{,}600} + \frac{\sin^4\theta}{18{,}680} + \cos^2\theta\sin^2\theta\left(\frac{1}{8030} - 2\times\frac{0.4}{56{,}600}\right)}$$

- Laminate (c): The proportions of 25% along the directions 0° and 90° can be obtained from Table 5.9. In this view one has to evaluate by extrapolation, starting from the values corresponding to the percentages of 20% and 30%, as[26]:

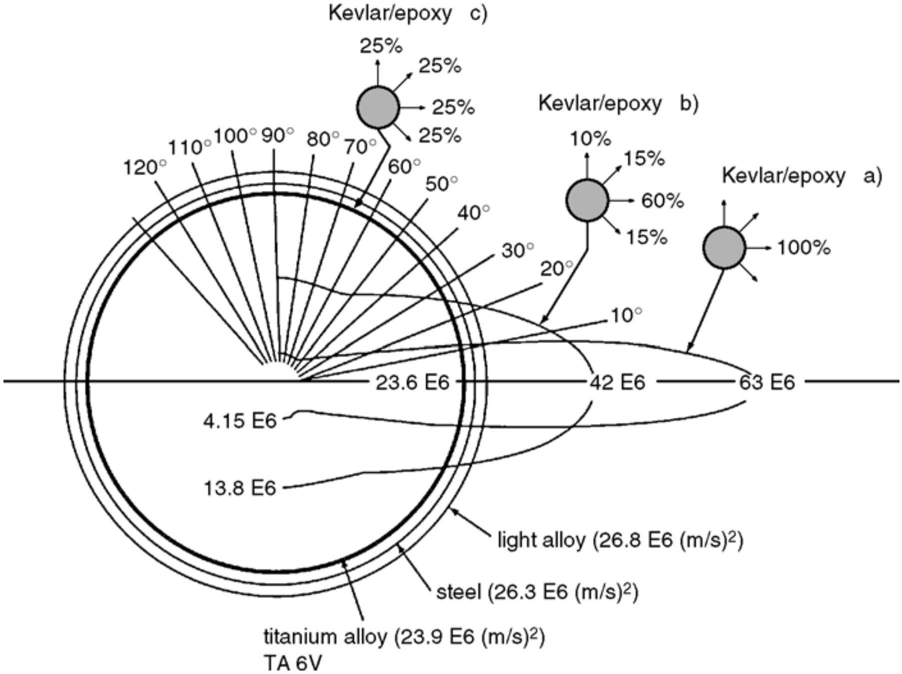
E_x = (1/2) (28,260 + 35,400) = 31,830 MPa

[26] See also Application 18.2.14.

$E_y = E_x$
$G_{xy} = 11{,}980$ MPa
$\nu_{xy} = 0.335$

$$\frac{E(\theta)}{\rho}(\text{m/s})^2 = \frac{10^6/1350}{\frac{\cos^4\theta + \sin^4\theta}{31{,}830} + \cos^2\theta\sin^2\theta\left(\frac{1}{11{,}980} - 2\times\frac{0.335}{31{,}830}\right)}$$

3. One obtains the evolutions in the following figure for the specific modulus. One verifies well that it is possible to "control" the anisotropy of the laminate by modifying the percentages of the plies at 0°, 90°, +45°, –45°.
4. For the other materials, one obtains immediately (Section 1.6):
   E/ρ (steel) = 26.3×10^6 (m/s)^2
   E/ρ (Duralumin-2024) = 26.8×10^6 (m/s)^2
   E/ρ (Titanium-TA6V) = 23.9×10^6 (m/s)^2

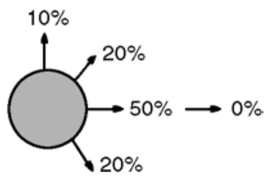


Remark: The notion of specific modulus is particularly important for aeronautical construction. When one compares on the above diagram the performances of Kevlar/epoxy with those of steel, Duralumin, and titanium, one sees clearly the advantage for the laminate inside angular borders for the directions of application of the loads.

18.2.4 Residual Thermal Stresses due to Curing of the Laminate

Problem Statement:

Consider a laminated panel in carbon/epoxy with V_f = 60% fiber volume fraction, with midplane symmetry, and a composition shown in the following figure:



It is cured in an autoclave at 180°C and demolded at 20°C.

1. Calculate the thermal deformations due to demolding.
2. Calculate the thermal residual stresses in the 90° plies.

Solution:

1. Thermal deformations:

The thermomechanical behavior of the laminate can be written as (see Equation 12.19):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\bar{E}_x} & -\dfrac{\bar{\nu}_{yx}}{\bar{E}_y} & \dfrac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\dfrac{\bar{\nu}_{xy}}{\bar{E}_x} & \dfrac{1}{\bar{E}_y} & \dfrac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \dfrac{\bar{\eta}_x}{\bar{E}_x} & \dfrac{\bar{\mu}_y}{\bar{E}_y} & \dfrac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix}$$

The panel is not subjected to any external mechanical loading. This law can then be written as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \Delta T \begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix}$$

The laminate being balanced, Equations 12.18, 12.17, and 11.10 lead to

$$\alpha_{oxy} = 0$$

Then Table 5.4 of Section 5.4.2 indicates for the laminate with the corresponding percentages of the composition above:

$$\alpha_{ox} = -0.072 \times 10^{-5}$$

One also deduces from Table 5.4, by permutation between 0° and 90°:

$$\alpha_{oy} = 0.44 \times 10^{-5}$$

Therefore, the thermal strains corresponding to $\Delta T = -160\,°C$ are

$$\varepsilon_{ox} = -160 \times (-0.0072 \times 10^{-5});\ \varepsilon_{oy} = -160 \times (0.44 \times 10^{-5})$$

or:

$$\varepsilon_{ox} = 115 \times 10^{-6}$$

$$\varepsilon_{oy} = -704 \times 10^{-6}$$

$$\gamma_{oxy} = 0$$

2. Thermal residual stresses in the 90° plies:
The Equation 11.10 allows one to write

$$\sigma_x = \bar{E}_{11}^{90°}\varepsilon_{ox} + \bar{E}_{12}^{90°}\varepsilon_{oy} - \Delta T\overline{\alpha E}_1^{90°}$$

where:

$$\overline{\alpha E}_1^{90°} = \bar{E}_t(\nu_{\ell t}\alpha_\ell + \alpha_t)$$

with (Equation 11.8):

$$\bar{E}_{11}^{90°} = \bar{E}_t \quad \text{and} \quad \bar{E}_{12}^{90°} = \nu_{t\ell}\bar{E}_\ell$$

The moduli of elasticity and coefficients of expansion are given in Section 3.3.3, Table 3.4.[27] Then:

$$\overline{\alpha E}_1^{90°} = 0.237$$

With the known values ε_{ox} and ε_{oy}, one has

$$\sigma_x = 7021 \times 115 \times 10^{-6} + 1717 \times (-704 \times 10^{-6}) - (-160)(0.237) = 37.5\ \text{MPa}$$

In an analogous manner:

$$\sigma_y = \bar{E}_{21}^{90°}\varepsilon_{ox} + \bar{E}_{22}^{90°}\varepsilon_{oy} - \Delta T\overline{\alpha E}_2^{90°}$$

[27] Recall also the property $\nu_{t\ell}/E_t = \nu_{\ell t}/E_\ell$ (see Sections 3.1 and 3.2 and Application 18.1.2).

with (Equation 11.8):

$$\bar{E}_{22}^{90^\circ} = \bar{E}_\ell \quad \text{and} \quad \overline{\alpha E_2}^{90^\circ} = \bar{E}_\ell(\alpha_\ell + \nu_{t\ell}\alpha_t)$$

One obtains

$$\sigma_y = -110.2 \text{ MPa}$$

and

$$\tau_{xy} = 0$$

One then has in the axes ℓ, t of the 90° plies (see Equation 11.4):

$$\sigma_\ell = -110.2 \text{ MPa}$$

$$\sigma_t = 37.5 \text{ MPa}$$

$$\tau_{\ell t} = 0$$

Remark: If one writes the Hill-Tsai expression (Section 5.3.2) for the 90° plies, one obtains with the rupture strengths of Section 3.3.3, Table 3.4:

$$\left(\frac{110.2}{1130}\right)^2 + \left(\frac{37.5}{42}\right)^2 - \left(\frac{(-110.2)(37.5)}{1130^2}\right) = 0.81$$

The factor of safety[28] is only:

$$\frac{1}{\sqrt{0.81}} - 1 = 11\%$$

This is due to high value of σ_t close to the rupture strength and explains the phenomenon of **microfracture** of the resin that happen during cooling. Subsequently, the microcracks favor the absorption of moisture by the resin and the fibers, which provoke expansions analogous to those induced by heating, with coefficients of expansion of hygrometric nature. Then, the residual stresses in the plies will be generally weaker.

18.2.5 Thermoelastic Behavior of a Tube Made of Filament-Wound Glass/Polyester

Problem Statement:

Obtain the thermoelastic behavior of a cylindrical tube made by filament winding E glass/polyester, with ±45° balanced composition, with a fiber volume fraction of V_f = 25%.

[28] See Section 14.2.3.

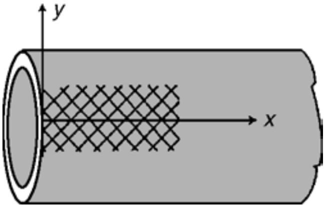
Solution:

In the x,y axes (following figure), the stress–strain law takes the form (see Equation 12.19):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix}$$

- Calculation of moduli:

First we have to evaluate the terms of the matrix $h^{-1}[A_{ij}]$ (see Equation 12.7). This calculation requires the knowledge of the stiffness coefficients for each ply $\bar{E}_{ij}$ (see Equation 11.18).



In this view, first calculate the elastic moduli of a ply in its principal axes (ℓ,t); one has (Equation 10.2 and those that follow and numerical values in Tables 1.3 and 1.4 in Section 1.6.)

$$E_\ell = 74{,}000 \times 0.25 + 4{,}000 \times 0.75 = 21{,}500 \text{ MPa}$$

$$\nu_{\ell t} = 0.25 \times 0.25 + 0.4 \times 0.75 = 0.36$$

$$E_t = 4000 \frac{1}{0.75 + \frac{4000}{74{,}000} \times 0.25} = 5240 \text{ MPa}$$

$$G_{\ell t} = 1400 \frac{1}{0.75 + \frac{1400}{30{,}000} \times 0.25} = 1840 \text{ MPa}$$

$$\nu_{t\ell} = (5240/21{,}500) \times 0.36 = 0.088$$

$$\bar{E}_\ell = 22{,}200 \text{ MPa}; \quad \bar{E}_t = 5410 \text{ MPa}$$

then (Equation 11.8):

$$\bar{E}_{11}^{+45°} = \bar{E}_{11}^{-45°} = \bar{E}_{22}^{+45°} = \bar{E}_{22}^{-45°} = 9720 \text{ MPa}$$

$$\bar{E}_{33}^{+45°} = \bar{E}_{33}^{-45°} = 5928 \text{ MPa}; \quad \bar{E}_{12}^{+45°} = \bar{E}_{12}^{-45°} = 6040 \text{ MPa}$$

$$\bar{E}_{13}^{+45°} = -\bar{E}_{13}^{-45°}; \quad \bar{E}_{23}^{+45°} = -\bar{E}_{23}^{-45°}$$

from which one can write (Equation 12.8)

$$\frac{1}{h}[A_{ij}] = \begin{bmatrix} 9720 & 6040 & 0 \\ 6040 & 9720 & 0 \\ 0 & 0 & 5928 \end{bmatrix} (\text{MPa})$$

The inversion of this matrix leads to (see Equation 12.9)

$$\begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & 0 \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & 0 \\ 0 & 0 & \frac{1}{\bar{G}_{xy}} \end{bmatrix} = \begin{bmatrix} 1.676 \times 10^{-4} & -1.041 \times 10^{-4} & 0 \\ -1.041 \times 10^{-4} & 1.676 \times 10^{-4} & 0 \\ 0 & 0 & \frac{1}{5928} \end{bmatrix} (\text{MPa}^{-1})$$

from which by identification:

$$\boxed{\begin{aligned} \bar{E}_x &= \bar{E}_y = 5966 \text{ MPa} \\ \bar{\nu}_{yx} &= \bar{\nu}_{xy} = 0.62 \\ \bar{G}_{xy} &= 5928 \text{ MPa} \end{aligned}}$$

- Calculation of coefficient of thermal expansion:

One has to calculate first $h^{-1}(\alpha Eh)_x$, $h^{-1}(\alpha Eh)_y$, and $h^{-1}(\alpha Eh)_{xy}$ from the Equation 12.18. This calculation requires knowledge of the terms $\overline{\alpha E_1}$, $\overline{\alpha E_2}$, and $\overline{\alpha E_3}$ of each ply (Equations 12.17 and 11.10 and numerical values in Tables 1.3 and 1.4 of Section 1.6). For that, one has to know the coefficients of expansion α_ℓ and α_t of a ply in its principal axes (ℓ, t). It can be written (Equations 10.7 and 10.8 and numerical values in Tables 1.3 and 1.4 of Section 1.6):

$$\alpha_\ell = 1.55 \times 10^{-5}; \quad \alpha_t = 7.86 \times 10^{-5}$$

$$\overline{\alpha E_1}^{+45°} = \overline{\alpha E_1}^{-45°} = \overline{\alpha E_2}^{+45°} = \overline{\alpha E_2}^{-45°} = 0.476 \text{ MPa/°C}$$

$$\overline{\alpha E_3}^{+45°} = -\overline{\alpha E_3}^{-45°}$$

From which (Equation 12.17):

$$\begin{Bmatrix} \frac{1}{h}(\alpha Eh)_x \\ \frac{1}{h}(\alpha Eh)_y \\ \frac{1}{h}(\alpha Eh)_{xy} \end{Bmatrix} = \begin{Bmatrix} 0.476 \\ 0.476 \\ 0 \end{Bmatrix} (\text{MPa/°C})$$

then (Equation 12.18):

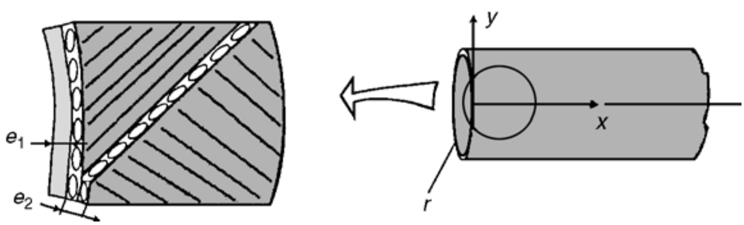
$$\begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix} = \begin{bmatrix} 1.676\times10^{-4} & -1.041\times10^{-4} & 0 \\ -1.041\times10^{-4} & 1.676\times10^{-4} & 0 \\ 0 & 0 & \frac{1}{5928} \end{bmatrix} \begin{Bmatrix} 0.476 \\ 0.476 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3.02\times10^{-5} \\ 3.02\times10^{-5} \\ 0 \end{Bmatrix}$$

In summary, the thermoelastic behavior of the filament-wound tube in glass/polyester can be written as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \underset{(\text{MPa}^{-1})}{\begin{bmatrix} \frac{1}{5966} & -\frac{0.62}{5966} & 0 \\ -\frac{0.62}{5966} & \frac{1}{5966} & 0 \\ 0 & 0 & \frac{1}{5928} \end{bmatrix}} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \underset{(°\text{C}^{-1})}{\begin{Bmatrix} 3.02\times10^{-5} \\ 3.02\times10^{-5} \\ 0 \end{Bmatrix}}$$

18.2.6 Polymeric Tube Under Thermal Load and Creep

Consider a cylindrical tube of revolution made of polyvinylidene fluoride (PVDF) reinforced externally by filament winding of glass/polyester at ±45° (see figure below).



The characteristics of the constituents are as follows:

- Polymer tube: thickness $e_1 = 10$ mm; isotropic material; modulus of elasticity $E_1 = 260$ MPa; Poisson coefficient ν_1; thermal expansion coefficient $\alpha_1 = 15 \times 10^{-5}$ (°C^{-1}).
- Glass/polyester reinforcement: thickness $e_2 = 3$ mm; modulus of elasticity E_2; Poisson coefficient ν_2; coefficient of thermal expansion $\alpha_2 = 0.7 \times 10^{-5}$ (°C^{-1}). These coefficients are valid for the behavior in the coordinate axes x, y (see figure). Fiber volume fraction $V_f = 60\%$.

Problem Statement:

The thicknesses e_1 and e_2 are small relative to the average radius of the tube, denoted as r.

1. Give the numerical values of E_2 and ν_2 (noting that the moduli of elasticity of epoxy resins and polyester resins are equivalent).
2. When taking into account the temperature variation, denoted as ΔT, the mechanical behavior of the polymer and of the reinforcement, respectively, can be written in the x,y axes as:

$$\begin{Bmatrix} \varepsilon_{1x} \\ \varepsilon_{1y} \\ \gamma_{1xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_1}{E_1} & 0 \\ -\frac{\nu_1}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_1} \end{bmatrix} \begin{Bmatrix} \sigma_{1x} \\ \sigma_{1y} \\ \tau_{1xy} \end{Bmatrix} + \alpha_1 \Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix};$$

$$\begin{Bmatrix} \varepsilon_{2x} \\ \varepsilon_{2y} \\ \gamma_{2xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_2} & -\frac{\nu_2}{E_2} & 0 \\ -\frac{\nu_2}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_2} \end{bmatrix} \begin{Bmatrix} \sigma_{2x} \\ \sigma_{2y} \\ \tau_{2xy} \end{Bmatrix} + \alpha_2 \Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

where one can recognize the strains and stresses in each of the materials. Starting with an assembly (polymer + reinforcement) not stressed nor strained at ambient temperature (20°C), which is heated up to 140°C.

(a) Write the equations for the external equilibrium of the assemblage.
(b) Write the equality of the strains. Deduce a system of equations that allows the calculation of stresses σ_{1x}, σ_{1y}, σ_{2x}, σ_{2y}.
(c) Numerical application: Calculate the stresses in each of the two components (polymer and glass/polyester reinforcement) as well as their strains.

3. Being subjected to high temperature, the internal tube in polymer obeys creep law. The stresses calculated previously do not remain constant in time. They evolve and stabilize at a certain final state. When this state is achieved, if one separates the internal polymer envelope (by imagination) from its reinforcement and cools it quickly from 140°C to 20°C, one will observe residual strains denoted as $\Delta\varepsilon_{1x} = \Delta\varepsilon_{1y} = \Delta\varepsilon$. Note that in the absence of creep, there are no residual strains.
   (a) Write the four relations allowing the calculation of the stresses in the assembly at 140°C **after** creep in the polymer, denoted as σ'_{1x}, σ'_{1y}, σ'_{2x}, σ'_{2y}.

(b) Numerical application: From experiments one finds: $\Delta\varepsilon = -0.6 \times \alpha_1 \Delta T$. Calculate the stresses after creep.

4. Considering the assembly at 140°C already crept, one cools the whole reinforced tube quickly, from 140°C to 20°C. Calculate the final stresses in the assembly, denoted as σ''_{1x}, σ''_{1y}, σ''_{2x}, σ''_{2y} at the end of the cooling. Remark.

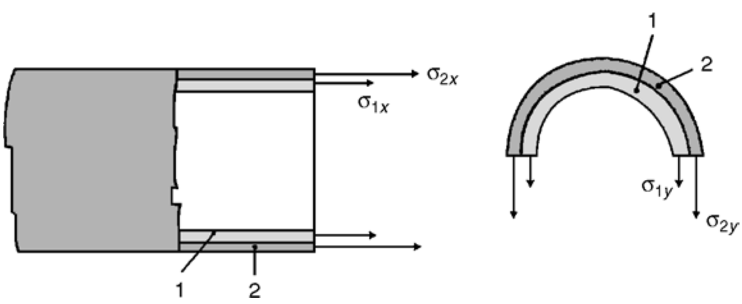
Solution:

1. We will use for the elastic characteristics of a unidirectional ply of glass/polyester at $V_f = 0.6$ those of a glass/epoxy ply from Table 3.4. For a laminate at ±45°, Table 5.14 (Section 5.4.2) shows:

$$E_2 = 14{,}130 \text{ MPa}$$

$$\nu_2 = 0.57$$

2. (a) Equilibrium of the assembly: Sections cut from the tube do not show any external resultant force, in spite of the existence of stresses of thermal origin (see following figure).



In addition, because the thicknesses are assumed to be small compared with the radius, the stresses will be taken to be uniform over the thicknesses. From there we have the relations:

$$2\pi r(\sigma_{1x} e_1 + \sigma_{2x} e_2) = 0; \quad 1 \times 2(\sigma_{1y} e_1 + \sigma_{2y} e_2) = 0$$

then:

$$\boxed{\begin{aligned} \sigma_{1x} e_1 + \sigma_{2x} e_2 &= 0 \\ \sigma_{1y} e_1 + \sigma_{2y} e_2 &= 0 \end{aligned}} \qquad [1], [2]$$

Due to the symmetry of revolution for the stress distribution, there are no shear stresses: $\tau_{1xy} = \tau_{2xy} = 0$.

(b) Equality of strains: This is assured by the assumed perfect bonding between the components 1 and 2 as:

$$\varepsilon_{1x} = \varepsilon_{2x}; \qquad \varepsilon_{1y} = \varepsilon_{2y}; \qquad \gamma_{1xy} = \gamma_{2xy}.$$

With the behavior as mentioned in the problem statement, the above equalities become:

$$\boxed{\begin{aligned} \frac{\sigma_{1x}}{E_1} - \frac{v_1}{E_1}\sigma_{1y} + \alpha_1\Delta T &= \frac{\sigma_{2x}}{E_2} - \frac{v_2}{E_2}\sigma_{2y} + \alpha_2\Delta T \\ -\frac{v_1}{E_1}\sigma_{1x} + \frac{\sigma_{1y}}{E_1} + \alpha_1\Delta T &= -\frac{v_2}{E_2}\sigma_{2x} + \frac{\sigma_{2y}}{E_2} + \alpha_2\Delta T \end{aligned}} \qquad [3], [4]$$

The above relations [1], [2], [3], [4] constitute a system of four equations for four unknowns σ_{1x}, σ_{1y}, σ_{2x}, σ_{2y}.

(c) In performing successively [3] – [4], [3] + [4], then substituting σ_{2x}, σ_{2y} obtained from [1] and [2], one obtains

$$\begin{cases} \sigma_{1x} - \sigma_{1y} = 0 \\ \sigma_{1x} + \sigma_{1y} = 2\Delta T \dfrac{(\alpha_2 - \alpha_1)}{\left(\dfrac{1-v_1}{E_1}\right) + \dfrac{e_1}{e_2}\left(\dfrac{1-v_2}{E_2}\right)} \end{cases}$$

from which:

$$\boxed{\sigma_{1x} = \sigma_{1y} = \Delta T \frac{(\alpha_2 - \alpha_1)}{\left(\frac{1-v_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1-v_2}{E_2}\right)}}$$

One deduces from there, with $\Delta T = 140 - 20 = 120°\text{C}$:

$$\sigma_{1x} = \sigma_{1y} = -6.14 \text{ MPa}$$

$$\sigma_{2x} = \sigma_{2y} = 20.4 \text{ MPa}$$

The internal envelope in polymer is in a state of biaxial compression. The external envelope in glass/polyester is in a state of biaxial tension. The mechanical behavior (in the Problem Statement) then indicates:

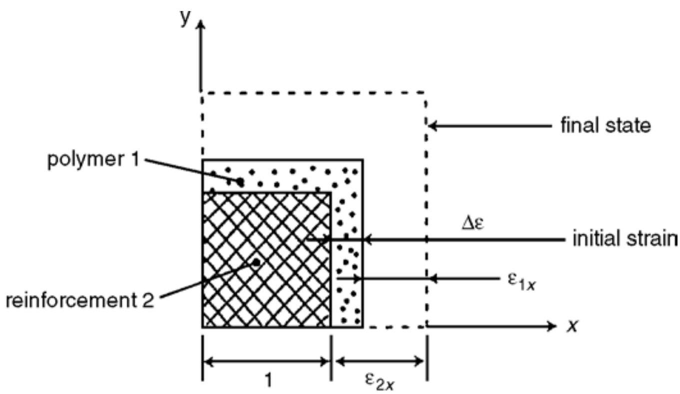
$$\varepsilon_{1x} = \varepsilon_{2x} = \varepsilon_{1y} = \varepsilon_{2y} = 1.47 \times 10^{-3}$$

3. Creep
   (a) The equilibrium relations are formally unchanged as:

$$\sigma'_{1x}\ e_1 + \sigma'_{2x}\ e_2 = 0 \tag{5}$$

$$\sigma'_{1y}\ e_1 + \sigma'_{2y}\ e_2 = 0 \tag{6}$$

The property of the perfect bond is now written in conformity with the following figure:



with the constitutive relations recalled in the Problem Statement, these equalities become

$$\begin{aligned}\frac{\sigma'_{1x}}{E_1} - \frac{\nu_1}{E_1}\sigma'_{1y} + \alpha_1\Delta T + \Delta\varepsilon &= \frac{\sigma'_{2x}}{E_2} - \frac{\nu_2}{E_2}\sigma'_{2y} + \alpha_2\Delta T \\ -\frac{\nu_1}{E_1}\sigma'_{1x} + \frac{\sigma'_{1y}}{E_1} + \alpha_1\Delta T + \Delta\varepsilon &= -\frac{\nu_2}{E_2}\sigma'_{2x} + \frac{\sigma'_{2y}}{E_2} + \alpha_2\Delta T\end{aligned} \qquad [7],\ [8]$$

   (b) Numerical application: In performing successively [7] – [8], [7] + [8], then substituting σ'_{2x} and σ'_{2y} obtained from [5] and [6], one obtains

$$\boxed{\sigma'_{1x} = \sigma'_{1y} = \Delta T\frac{(\alpha_2 - 0.4\alpha_1)}{\left(\frac{1-\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1-\nu_2}{E_2}\right)}}$$

then:

$$\boxed{\begin{aligned}\sigma'_{1x} &= \sigma'_{1y} = -2.28\ \text{MPa} \\ \sigma'_{2x} &= \sigma'_{2y} = 7.6\ \text{MPa}\end{aligned}}$$

4. Cooling: It is sufficient to suppress the increase in temperature ΔT in the previous equations [7] and [8]. The system of equations becomes

$$\begin{cases} \sigma''_{1x} e_1 + \sigma''_{2x} e_2 = 0 \\ \sigma''_{1y} e_1 + \sigma''_{2y} e_2 = 0 \\ \dfrac{\sigma''_{1x}}{E_1} - \dfrac{\nu_1}{E_1}\sigma''_{1y} + \Delta\varepsilon = \dfrac{\sigma''_{2x}}{E_2} - \dfrac{\nu_2}{E_2}\sigma''_{2y} \\ -\dfrac{\nu_1}{E_1}\sigma''_{1x} + \dfrac{\sigma''_{1y}}{E_1} + \Delta\varepsilon = -\dfrac{\nu_2}{E_2}\sigma''_{2x} + \dfrac{\sigma''_{2y}}{E_2} \end{cases}$$

In adopting a method of resolution analogous to that used in the previous problems, one obtains

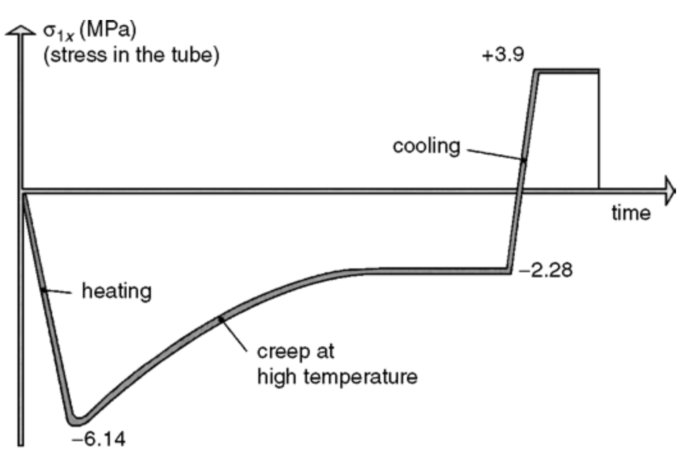
$$\sigma''_{1x} = \sigma''_{1y} = \Delta T \frac{0.6\alpha_1}{\left(\frac{1-\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1-\nu_2}{E_2}\right)}$$

then:

$$\boxed{\begin{aligned} \sigma''_{1x} &= \sigma''_{1y} = 3.9\ \text{MPa} \\ \sigma''_{2x} &= \sigma''_{2y} = -12.9\ \text{MPa} \end{aligned}}$$

It is worth noting that the polymer envelope is loaded now in biaxial tension. Subsequently, for one cycle of operation, the polymer envelope is successively compressed, released by creep, then extended, as shown in the following figure.

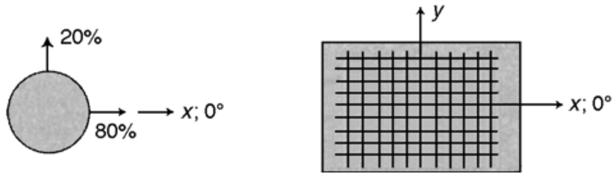
These cycles repeat themselves during the life of the tube, and this gives rise to fatigue. Therefore, an overdimension of the tube is necessary to lead to low admissible stresses in the polymer, to prevent the risk of buckling of the tube subject to compression at the location of bond defects or of failure in tension while cooling.



18.2.7 First Ply Failure of a Laminate—Ultimate Rupture

Problem Statement:

Consider a carbon/epoxy laminate with 60% fiber volume fraction and the following composition:



1. (a) Give the values of the moduli of elasticity and Poisson coefficients of this laminate.
   (b) What is the maximum tensile stress, denoted as $\sigma_{x\ \text{maximum}}$, that can be applied without deterioration?
2. When the value $\sigma_{x\ \text{maximum}}$ is reached, the 90° plies are deteriorated by microcracks of the epoxy resin ("first ply" failure). The characteristics of the 90° plies that are cracked are then decreased with respect to their values for intact plies. One admits the following damage factors:

$$\underset{\text{fractured}}{E'_\ell} \# \underset{\text{intact}}{E_\ell}; \qquad \underset{\text{fractured}}{E'_t} \# 0.1 \times \underset{\text{intact}}{E_t}$$

$$\underset{\text{fractured}}{G'_{\ell t}} \# 0.1 \times \underset{\text{intact}}{G_{\ell t}}; \qquad \underset{\text{fractured}}{\nu'_{\ell t}} \# 0.1\ \underset{\text{intact}}{\nu_{\ell t}}$$

   (a) Calculate the new terms of the matrix $\frac{1}{h}[A]$ of the elastic behavior.[29] Deduce from there the new elastic moduli of the deteriorated laminate. Remark.
   (b) Calculate the maximum stress σ_{xM} leading to complete rupture of this laminate (rupture of 0° plies, or "last ply rupture").
3. What will be the rupture strength denoted as σ'_{xM} that one can obtain by eliminating all the elastic characteristics of the deteriorated 90° plies?

Remark: How one can obtain rapidly the value σ'_{xM}?

4. The design of an aeronautical piece is carried out using this laminate with the following considerations:
   - When this piece is subjected to a stress along the x direction called "limit load," the piece stays in a reversible elastic domain and is not altered in its structure.

[29] See Equation 12.7.

- When this piece is subjected to a stress along the x direction called "extreme loading," one obtains total rupture.

Moreover, one has from common practice:

$$\text{Extreme loading} = 1.5 \times \text{limit loading}$$

Indicate the values of σ_x that should be kept here for extreme load and for limit load, respectively.

Solution:

1. (a) According to Tables 5.4 and 5.5 in Section 5.4.2, one notes for the indicated composition:

$$E_x = 108{,}860 \text{ MPa}; \quad E_y = 32{,}477 \text{ MPa}$$

$$\nu_{xy} = 0.054; \quad \nu_{yx} = 0.016$$

$$G_{xy} = 4200 \text{ MPa}$$

(b) Table 5.1, Section 5.4.2, indicates for the rupture limit of the first ply:

$$\sigma_x = 659 \text{ MPa}$$

2. (a) Terms of matrix $\frac{1}{h}[A]$ are written as (Equations 12.7 and 12.8):

$$\frac{1}{h}A_{ij} = \bar{E}_{ij}^{0°} \times p^{0°} + \bar{E}_{ij}^{90°} \times p^{90°}$$

Coefficients $\bar{E}_{ij}$ are given by Equation 11.8 which lead to[30]

$$\bar{E}_{11}^{0°} = 134\ 440 \text{ MPa}; \quad \bar{E}_{22}^{0°} = 7\ 023 \text{ MPa}; \quad \bar{E}_{12}^{0°} = 1748 \text{ MPa};$$
$$\bar{E}_{33}^{0°} = 4200 \text{ MPa}$$

The 90° plies are deteriorated. One then has[31]

$$\bar{E}_{11}^{90°} = \bar{E}'_t = 700 \text{ MPa}; \quad \bar{E}_{22}^{90°} = \bar{E}'_\ell = 134{,}000 \text{ MPa}$$
$$\bar{E}_{12}^{90°} = \nu'_{t\ell}\ \bar{E}'_\ell = 17.5 \text{ MPa}; \quad \bar{E}_{33}^{90°} = 420 \text{ MPa}$$

[30] See Section 3.3.3 for the characteristics of a unidirectional ply of carbon/epoxy.

[31] $\nu'_{t\ell} = \nu'_{\ell t} \times E'_t / E'_\ell$ (See Application 18.1.2.).

or after calculation:

$$\frac{1}{h}[A] = \begin{bmatrix} 107{,}692 & 1402 & 0 \\ 1402 & 32{,}418 & 0 \\ 0 & 0 & 3444 \end{bmatrix} (\text{MPa})$$

The new moduli of the deteriorated laminate are obtained by inversion of the above matrix. One has (Equation 12.9)

$$h[A]^{-1} = \begin{bmatrix} 1/E'_x & -\nu'_{yx}/E'_y & 0 \\ -\nu'_{xy}/E'_x & 1/E'_y & 0 \\ 0 & 0 & 1/G'_{xy} \end{bmatrix}$$

which leads to

$$\boxed{\begin{aligned} E'_x &= 107{,}630 \text{ MPa} \\ E'_y &= 32{,}400 \text{ MPa} \\ \nu'_{xy} &= 0.043; \quad \nu'_{yx} = 0.013 \\ G'_{xy} &= 3444 \text{ MPa} \end{aligned}}$$

Note that only the shear moduli G_{xy} has its value modified with respect to the value of the intact laminate.

(b) The 90° plies being deteriorated, total rupture of the laminate corresponds to rupture of the 0° plies. Let σ_{xM} be the corresponding ultimate rupture strength. The mechanical behavior of the deteriorated laminate is written, following what happens previously:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} 9.29 \times 10^{-6} & -4.02 \times 10^{-7} & 0 \\ -4.02 \times 10^{-7} & 3.086 \times 10^{-5} & 0 \\ 0 & 0 & 2.9 \times 10^{-4} \end{bmatrix} \begin{Bmatrix} \sigma_{xM} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 9.29 \times 10^{-6} \times \sigma_{xM} \\ -4.02 \times 10^{-7} \times \sigma_{xM} \\ 0 \end{bmatrix}$$

from which the state of stress in the 0° plies (Equation 11.8):

$$\begin{aligned} \sigma_x &= \bar{E}_{11}^{0°} \varepsilon_{ox} + \bar{E}_{12}^{0°} \varepsilon_{oy} = 1.248 \times \sigma_{xM} = \sigma_\ell \\ \sigma_y &= \bar{E}_{12}^{0°} \varepsilon_{ox} + \bar{E}_{22}^{0°} \varepsilon_{oy} = 0.0134 \times \sigma_{xM} = \sigma_t \\ \tau_{xy} &= 0 = \tau_{\ell t} \end{aligned}$$

Writing that for σ_{xM} the Hill–Tsai criterion is saturated (Section 5.3.2) and using the rupture strength values of Section 3.3.3:

$$\left(\frac{1.248\sigma_{xM}}{1270}\right)^2 + \left(\frac{0.0134\sigma_{xM}}{42}\right)^2 - \frac{1.248 \times 0.0134\sigma_{xM}^2}{1270^2} = 1$$

one obtains:

$$\boxed{\sigma_{xM} = 973 \text{ MPa}}$$

3. If one cancels all elastic characteristics of the deteriorated plies at 90°, the matrix $\frac{1}{h}[A]$ becomes

$$\frac{1}{h}[A] = 0.8\begin{bmatrix} \bar{E}_\ell & \nu_{t\ell}\bar{E}_\ell & 0 \\ \nu_{\ell t}\bar{E}_t & \bar{E}_t & 0 \\ 0 & 0 & G_{\ell t} \end{bmatrix} \text{then } h[A]^{-1} = \frac{1}{0.8}\begin{bmatrix} \frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0 \\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix}$$

under the loading of an ultimate stress denoted as σ'_{xM}, one will have for the strains:

$$\varepsilon_{ox} = \frac{1}{0.8}\frac{\sigma'_{xM}}{E_\ell}; \quad \varepsilon_{oy} = \frac{1}{0.8}\times -\frac{\nu_{\ell t}}{E_\ell}\sigma'_{xM}; \quad \gamma_{oxy} = 0$$

then in the 0° plies:

$$\sigma_x = \sigma_\ell = \bar{E}_{11}^{0°}\,\varepsilon_{ox} + \bar{E}_{12}^{0°}\,\varepsilon_{oy} = \frac{\sigma'_{xM}}{0.8}$$

$$\sigma_y = \sigma_t = \bar{E}_{12}^{0°}\,\varepsilon_{ox} + \bar{E}_{22}^{0°}\,\varepsilon_{oy} = 0$$

The saturated Hill–Tsai criterion then takes the form:

$$\left(\frac{\sigma'_{xM}}{0.8\times 1270}\right)^2 = 1$$

then:

$$\boxed{\sigma'_{xM} = 1016 \text{ MPa}}$$

One immediately obtains this value when noting a stress resultant N_x written as:

$$N_x = \sigma_x \times h = \sigma_x^{0°}\times 0.8h + \cancel{\sigma_x^{90°}}\times 0.2h$$

then

$$\sigma'_{xM} = \sigma_{xM}^{0°}\times 0.8 = 1270 \times 0.8 = 1016 \text{ MPa}$$

Note that the rupture stress of the last ply calculated in the previous problem (σ_{xM}) is less than σ'_{xM}. It then appears to be dangerous to reason as if the 0° plies were the only ones to resist by occupying 80% of the thickness of the laminate.

(c) If one considers that the limit load corresponds to the rupture of the first ply, denoted as $\sigma_{x\,\text{limit}} = 659$ MPa, then the extreme load will be

$$\sigma_{x\,\text{extreme}} = 1.5 \times 659 = 988 \text{ MPa}$$

This is an excessive value because it is higher than the rupture strength of the last ply $\sigma_{xM} = 973$ MPa. One then is led to keep

- For the extreme load: $\sigma_{x\,\text{extreme}} = \sigma_{xM} = 973$ MPa
- For the limit load: $\sigma_{x\,\text{limit}} = \sigma_{xM}/1.5 = 649$ MPa (value less than the fracture strength of the first ply)

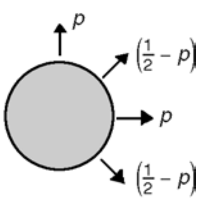
18.2.8 Optimum Laminate for Isotropic Stress State

Problem Statement:

Consider a laminate subjected to a state of plane uniform stresses:

$$\sigma_x = \sigma_y = \sigma_o; \ \tau_{xy} = 0 \text{ (state of isotropic stress)}$$

This laminate presents the following composition:



1. By means of a literal calculation show that the strain of the laminate is the same for any value of $p < 0.5$. Verify this property by means of Table 5.4 in Section 5.4.2 for p = 0%, 30%, 50%.
2. Show that the Hill–Tsai criterion has the same value in each ply, no matter what the proportion p. Comment.
3. Verify the previous property for a carbon/epoxy laminate by means of the tables in annex 1 for p = 0%, 30%, 50%.

Solution:

1. Determination of the apparent moduli of the carbon/epoxy laminate: We begin by calculating the terms of the matrix $\frac{1}{h}[A]$ (Equations 12.7 and 12.8):

$$\frac{1}{h}A_{11} = \bar{E}_{11}^{0°} \times p + \bar{E}_{11}^{90°} \times p + \bar{E}_{11}^{45°} \times \left(\frac{1}{2} - p\right) + \bar{E}_{11}^{-45°} \times \left(\frac{1}{2} - p\right)$$

then with the Equation 11.8:

$$\bar{E}_{11}^{0^\circ} = \bar{E}_\ell; \quad \bar{E}_{11}^{90^\circ} = E_t; \quad \bar{E}_{11}^{45^\circ} = \bar{E}_{11}^{-45^\circ} = \frac{\bar{E}_\ell + \bar{E}_t}{4} + \frac{1}{2}(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t})$$

$$\frac{1}{h}A_{11} = p(\bar{E}_\ell + \bar{E}_t) + 2\left(\frac{1}{2} - p\right)\left[\frac{\bar{E}_\ell + \bar{E}_t}{4} + \frac{1}{2}(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t})\right]$$

$$\frac{1}{h}A_{11} = p\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} - \nu_{t\ell}\bar{E}_\ell - 2G_{\ell t}\right] + \frac{1}{2}\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} + \nu_{t\ell}\bar{E}_\ell + 2G_{\ell t}\right]$$

$$\frac{1}{h}A_{22} = \frac{1}{h}A_{11}$$

$$\frac{1}{h}A_{12} = 2p\nu_{t\ell}\bar{E}_\ell + 2\left(\frac{1}{2} - p\right)\left[\frac{1}{4}(\bar{E}_\ell + \bar{E}_t - 4G_{\ell t}) + \frac{1}{2}\nu_{t\ell}\bar{E}_\ell\right]$$

$$\frac{1}{h}A_{12} = -p\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} - \nu_{t\ell}\bar{E}_\ell - 2G_{\ell t}\right] + \frac{1}{2}\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} + \nu_{t\ell}\bar{E}_\ell - 2G_{\ell t}\right]$$

$$\frac{1}{h}A_{13} = \frac{1}{h}A_{23} = 0$$

The constitutive law in Equation 12.7 here takes the form:

$$\begin{Bmatrix} \sigma_o \\ \sigma_o \\ 0 \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}$$

Its inverse is written as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} 1/\bar{E}_x & -\bar{\nu}_{yx}/\bar{E}_y & 0 \\ -\bar{\nu}_{xy}/\bar{E}_x & 1/\bar{E}_y & 0 \\ 0 & 0 & 1/\bar{G}_{xy} \end{bmatrix}\begin{Bmatrix} \sigma_0 \\ \sigma_0 \\ 0 \end{Bmatrix}$$

with:

$$\frac{1}{\bar{E}_x} = \frac{\frac{1}{h}A_{22}}{\frac{1}{h^2}(A_{11}A_{22} - A_{12}^2)} = \frac{1}{\bar{E}_y}; \quad \frac{\bar{\nu}_{yx}}{\bar{E}_y} = \frac{\frac{1}{h}A_{12}}{\frac{1}{h^2}(A_{11}A_{22} - A_{12}^2)}$$

and here:

$$\frac{1}{h^2}(A_{11}A_{22} - A_{12}^2) = 2\left[p\left(\frac{\bar{E}_\ell + \bar{E}_t}{2} - \nu_{t\ell}\bar{E}_\ell - 2G_{\ell t}\right) + G_{\ell t}\right]\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} + \nu_{t\ell}\bar{E}_\ell\right]$$

then we can obtain the strains:

$$\varepsilon_{ox} = \sigma_o\left(\frac{1}{\bar{E}_x} - \frac{\bar{\nu}_{yx}}{\bar{E}_y}\right) = \frac{\sigma_o}{\left(\frac{\bar{E}_\ell + \bar{E}_t}{2} + \nu_{t\ell}\bar{E}_{\ell,}\right)} = \varepsilon_{oy}; \quad \gamma_{oxy} = 0$$

In summary[32]:

$$\boxed{\varepsilon_{ox} = \varepsilon_{oy} = \varepsilon_o = \frac{\sigma_o}{\left[\frac{\bar{E}_\ell + \bar{E}_t}{2} + \nu_{t\ell}\bar{E}_\ell\right]}; \quad \gamma_{oxy} = 0}$$

The strain ε_o is independent of the proportion p and of the shear modulus $G_{\ell t}$. Each elastic characteristic that appears has the same weight: $\bar{E}_\ell$, $\bar{E}_t$, $\nu_{t\ell}\,\bar{E}_\ell = \nu_{\ell t}\,\bar{E}_t$

- Verification: Table 5.4, Section 5.4.2:

$$\boldsymbol{p} = \mathbf{0\%} : \bar{E}_x = \bar{E}_y = 15\,055 \text{ Mpa} ;\ \bar{\nu}_{xy} = 0.79 = \bar{\nu}_{yx}$$
$$\varepsilon_{ox} = \varepsilon_{oy} = \varepsilon_o = 1.39 \text{ E-5} \times \sigma_o(\text{MPa})$$
$$\boldsymbol{p} = \mathbf{30\%} : \bar{E}_x = \bar{E}_y = 55\,333 \text{ Mpa} ;\ \bar{\nu}_{xy} = 0.23 = \bar{\nu}_{yx}$$
$$\varepsilon_{ox} = \varepsilon_{oy} = \varepsilon_o = 1.39 \text{ E-5} \times \sigma_o(\text{MPa})$$
$$\boldsymbol{p} = \mathbf{50\%} : \bar{E}_x = \bar{E}_y = 70\,687 \text{ Mpa} ;\ \bar{\nu}_{xy} = 0.025 = \bar{\nu}_{yx}$$
$$\varepsilon_{ox} = \varepsilon_{oy} = \varepsilon_o = 1.38 \text{ E-5} \times \sigma_o(\text{MPa})$$

2. Hill–Tsai criterion:
   - 0° plies: Following the Equation 11.8:

$$\sigma_x^{0°} = \bar{E}_\ell\varepsilon_{ox} + \nu_{t\ell}\bar{E}_\ell\varepsilon_{oy} = \varepsilon_o\bar{E}_\ell(1 + \nu_{t\ell})$$
$$\sigma_y^{0°} = \nu_{t\ell}\bar{E}_\ell\varepsilon_{ox} + \bar{E}_t\,\varepsilon_{oy} = \varepsilon_o\bar{E}_t(1 + \nu_{\ell t})$$
$$\tau_{xy}^{0°} = 0$$

and following the Equation 11.4:

$$\sigma_\ell^{0°} = \sigma_x^o = \varepsilon_o\,\bar{E}_\ell(1 + \nu_{t\ell})$$
$$\sigma_t^{0°} = \sigma_y^o = \varepsilon_o\,\bar{E}_t(1 + \nu_{\ell t})$$
$$\tau_{\ell t}^{0°} = 0$$

[32] Recall (see Equation 11.8) that: $\bar{E}_\ell = E_\ell(1 - \nu_{\ell t}\nu_{t\ell})$; $\bar{E}_t = E_t(1 - \nu_{\ell t}\nu_{t\ell})$.

- 90° plies: Following Equations 11.8 and 11.4:

$$\begin{aligned}
\sigma_\ell^{90°} &= \sigma_y^{90°} = \varepsilon_o \bar{E}_\ell(1+\nu_{t\ell})\\
\sigma_t^{90°} &= \sigma_x^{90°} = \varepsilon_o \bar{E}_t(1+\nu_{\ell t})\\
\tau_{\ell t}^{90°} &= 0
\end{aligned}$$

- 45° plies: Following Equations 11.8 and 11.4[33]:

$$\begin{aligned}
\sigma_\ell^{45°} &= \frac{1}{2}(\sigma_x^{45°}+\sigma_y^{45°})+\tau_{xy}^{45°} = \varepsilon_o \bar{E}_\ell(1+\nu_{t\ell})\\
\sigma_t^{45°} &= \frac{1}{2}(\sigma_x^{45°}+\sigma_y^{45°})-\tau_{xy}^{45°} = \varepsilon_o \bar{E}_t(1+\nu_{\ell t})\\
\tau_{\ell t}^{45°} &= 0
\end{aligned}$$

- –45° plies: In an analogous manner:

$$\begin{aligned}
\sigma_\ell^{-45°} &= \varepsilon_o \bar{E}_\ell(1+\nu_{t\ell})\\
\sigma_t^{-45°} &= \varepsilon_o \bar{E}_t(1+\nu_{\ell t})\\
\tau_{\ell t}^{-45°} &= 0
\end{aligned}$$

- The Hill–Tsai criterion (see Section 5.3.2 or Equation 14.7) then has the same value in each of the plies, no matter what the proportion p and the value of the shear modulus $G_{\ell t}$. Rupture occurs simultaneously in all plies.
- One can also note that the minimum thickness h of the laminate that is capable of supporting the isotropic membrane load:

$$N_x = N_y; \quad T_{xy} = 0$$

will be independent of the proportion p (see Equation 12.10). One then can, for this particular case of loading, vary the modulus of elasticity $\bar{E}_x = \bar{E}_y$[34] **without varying the thickness.**

[33] Or still from Equation 11.7:

$$\varepsilon_\ell^{45} = \frac{1}{2}(\varepsilon_{ox}+\varepsilon_{oy}) = \varepsilon_o; \quad \varepsilon_t^{45} = \frac{1}{2}(\varepsilon_{ox}+\varepsilon_{oy}) = \varepsilon_o; \quad \gamma_{\ell t}^{45} = 0;$$

then following [11.6]:

$$\sigma_\ell^{45} = \varepsilon_o\bar{E}_\ell(1+\nu_{t\ell}); \quad \sigma_t^{45} = \varepsilon_o\bar{E}_t(1+\nu_{\ell t}); \quad \tau_{\ell t}^{45} = 0$$

[34] See Equation 12.9, or Tables 5.4, 5.9, 5.14 in Section 5.4.2.

- One automatically obtains such a laminate by using layers of balanced fabric at 0° and 45°. It is then convenient to calculate the thickness in considering the proper rupture resistances of the layer of fabric.[35]

3. Verification:

(cf. plates annex 1)

p = 0 %

plate 3 and 7

plies at +45°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	.94	.06	–.5
σ_y = 1 MPa	.94	.06	.5
total (MPa)	1.88	.12	.0
Hill-Tsai criterion: 1.02×10^{-5}			

plate 4 and 8

plies at –45°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	.94	.06	.5
σ_y = 1 MPa	.94	.06	–.5
total (MPa)	1.88	.12	0
Hill-Tsai criterion: 1.02×10^{-5}			

p = 30 %

plate 1 and 5

plies at 0°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	2.4	.0	.0
σ_y = 1 MPa	–.54	.12	.0
total (MPa)	1.86	.12	.0
Hill-Tsai criterion: 1.017×10^{-5}			

plate 2 and 6

plies at 90°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	–.54	.12	.0
σ_y = 1 MPa	2.4	.0	.0
total (MPa)	1.86	.12	.0
Hill-Tsai criterion: 1.017×10^{-5}			

plate 3 and 7

plies at +45°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	.94	.06	–.09
σ_y = 1 MPa	.94	.06	.09
total (MPa)	1.88	.12	.0
Hill-Tsai criterion: 1.02×10^{-5}			

plate 4 and 8

plies at –45°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	.94	.06	.09
σ_y = 1 MPa	.94	.06	–.09
total (MPa)	1.88	.12	0
Hill-Tsai criterion: 1.02×10^{-5}			

p = 50 %

plate 1 and 5

plies at 0°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	1.9	.02	.0
σ_y = 1 MPa	–.02	.1	.0
total (MPa)	1.88	.12	.0
Hill-Tsai criterion: 1.02×10^{-5}			

plate 2 and 6

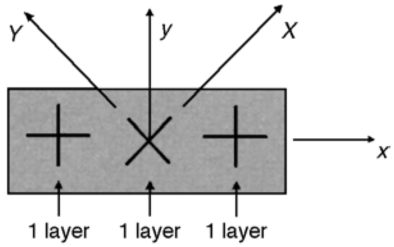
plies at 90°	σ_ℓ	σ_t	$\tau_{\ell t}$
σ_x = 1 MPa	–.02	.1	.0
σ_y = 1 MPa	1.9	.02	.0
total (MPa)	1.88	.12	.0
Hill-Tsai criterion: 1.02×10^{-5}			

18.2.9 Laminate Made of Identical Layers of Balanced Fabric

Problem Statement:

A carbon/epoxy laminate consists of a stacking of identical layers of balanced fabric with the composition illustrated below. The fiber volume fraction is $V_f = 60\%$.

[35] See Applications 18.2.9 and 18.2.10.



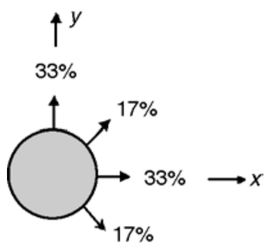
Give the elastic behavior law in the axes (x, y) and then in the axes (X, Y).

Solution:

- Axes x, y: The fabric being balanced, each layer can be replaced by two identical unidirectional plies crossed at 90°, with the resulting thicknesses (see Section 3.4.2):

$$e_{\text{warp}} = e_{\text{fill}} = e/2$$

The laminate is balanced and its composition is as follows (see figure):



One then notes (Table 5.4, Section 5.4.2):

$$E_x = 55{,}333 + \Delta E_x(\text{MPa})$$

One can evaluate ΔE_x starting from the expression:

$$dE_x = \frac{\partial E}{\partial p^{0°}} \times dp^{0°} + \frac{\partial E}{\partial p^{90°}} \times dp^{90°}$$

as:

$$\Delta E_x = (65{,}888 - 55{,}333) \times \frac{3}{10} + (53{,}545 - 55{,}333) \times \frac{3}{10} = 2630 \text{ MPa}$$

then:

$$E_x = 57{,}960 \text{ MPa} = E_y$$

Poisson coefficient: $\nu_{xy} = 0.23 + \Delta\nu_{xy}$.
From an analogous calculation:

$$\nu_{xy} = 0.20 = \nu_{yx}$$

Shear modulus: One notes (Table 5.5, Section 5.4.2)

$$G_{xy} = 16{,}315 + \Delta G_{xy} \text{ (MPa)}$$

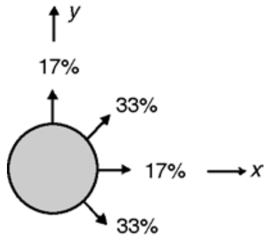
Then from an analogous calculation:

$$G_{xy} = 14{,}500 \text{ MPa.}$$

From which the elastic behavior relation in axes (x, y) can be written as (Equation 12.9):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{57{,}960} & -\frac{0.2}{57{,}960} & 0 \\ -\frac{0.2}{57{,}960} & \frac{1}{57{,}960} & 0 \\ 0 & 0 & \frac{1}{14{,}500} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} \text{(MPa)}$$

- Axes X, Y: The laminate is balanced and then has the composition:



In using the same tables as before, one obtains

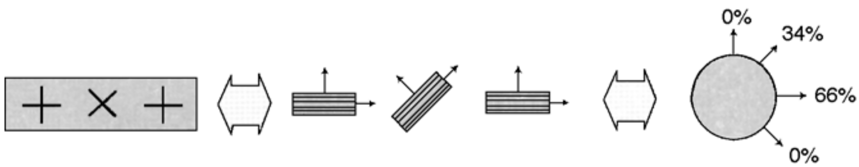
$$E_X = E_Y = 31{,}979 + \Delta E_X = 41{,}400 \text{ MPa}$$
$$\nu_{XY} = \nu_{YX} = 0.56 + \Delta\nu_{XY} = 0.43$$
$$G_{XY} = 28{,}430 + \Delta G_{XY} = 24{,}190 \text{ MPa}$$

from which the law for the behavior in the axes X, Y can be written as (Equation 12.9):

$$\begin{Bmatrix} \varepsilon_{oX} \\ \varepsilon_{oY} \\ \gamma_{oXY} \end{Bmatrix} = \begin{bmatrix} \frac{1}{41,400} & -\frac{0.43}{41,400} & 0 \\ -\frac{0.43}{41,400} & \frac{1}{41,400} & 0 \\ 0 & 0 & \frac{1}{24,190} \end{bmatrix} \begin{Bmatrix} \sigma_{oX} \\ \sigma_{oY} \\ \tau_{oXY} \end{Bmatrix} (\mathrm{MPa})$$

Remarks:

- One can note that a laminate constituted of layers of balanced fabric with four orientations 0°, 90°, +45°, –45° admits two systems of orthotropic axes x, y and X, Y.
- The elastic properties are suitably estimated when one uses Tables 5.4 and 5.5 in Section 5.4.2. **This is not the same** for the maximum admissible stresses indicated in Tables 5.1, 5.2, and 5.3 that are valid only for laminates made of unidirectional layers. In effect, the resistance to rupture for a layer of balanced fabric is clearly higher in tension than the first ply failure limit for an equivalent fabric, made up of layers at 0° (50%) and 90° (50%). For a calculation of first-ply failure or for the failure criterion of the laminate proposed in this application, it is convenient to consider a layer of fabric as an anisotropic ply with thickness e (see Section 3.4.2) with the values of rupture stresses $\sigma_{\ell\ \text{rupture}}$, $\sigma_{t\ \text{rupture}}$, and $\tau_{\ell t\ \text{rupture}}$ of the balanced fabric itself (see examples in Section 3.4.3).[36] One will then have the following equivalence[37]:



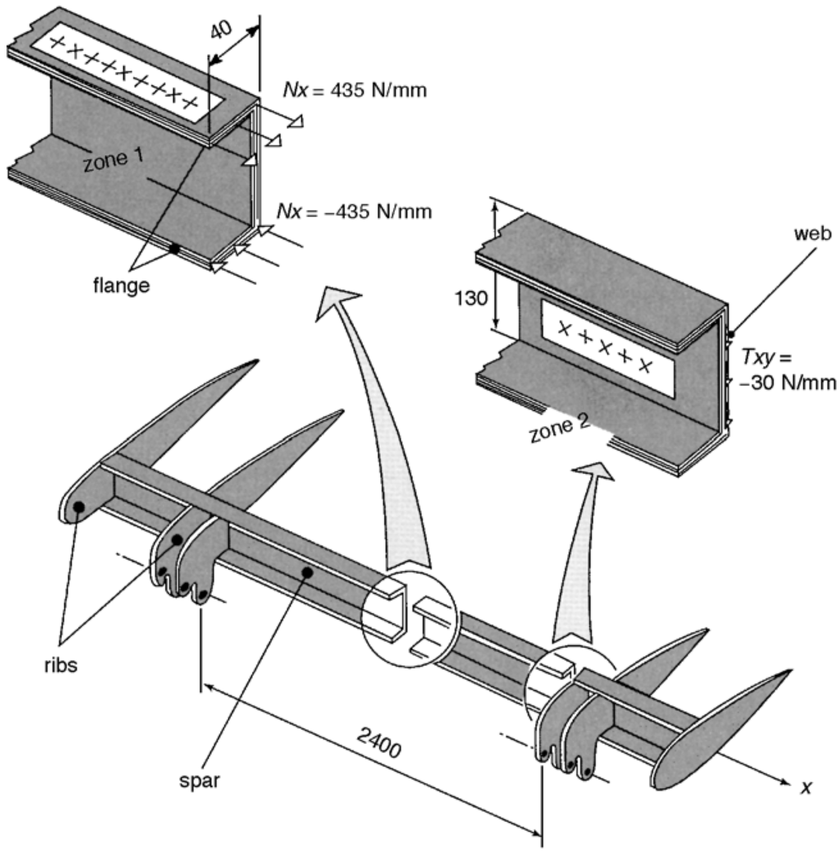
18.2.10 Wing Spar in Carbon/Epoxy

Problem Statement:

Consider an airplane flap with the internal structure (excluding facings) shown schematically in the following figure. It consists of a spar and several ribs. The spar is a laminate of carbon/epoxy fabric with V_f = 45% fiber volume fraction, the composition of which varies with the longitudinal coordinate axis x, in the flange and in the web. A preliminary calculation of the flap in isostatic equilibrium

[36] See also Application 18.2.10.
[37] See Section 5.2.3.

reveals the maximum stress resultants in the two zones of the spar indicated in the figure.



One proposes for each of these zones the compositions indicated in the figure.

1. Evaluate the elastic properties of the laminate in these two zones.
2. Verify the two corresponding laminates:
   a. At rupture.
   b. At buckling.

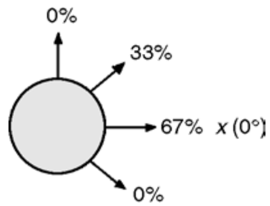
- Thickness of a layer of fabric: 0.24 mm.
- Properties of carbon/epoxy fabric: See Section 3.4.3.

Solution:

1. Elastic properties:
   (a) Zone 1: Composition of the laminate[38]: See sketch below.

[38] See Section 5.2.3 and remark at the end of previous Exercise 18.2.9.

Calculation of elastic moduli (Equations 12.7, 12.8, 12.9, and 11.8):



$$\bar{E}_{11}^{0°} = \bar{E}_\ell; \quad \bar{E}_{12}^{0°} = \nu_{t\ell}\bar{E}_\ell; \quad \bar{E}_{33}^{0°} = G_{\ell t}$$

$$\bar{E}_{11}^{45°} = \frac{\bar{E}_\ell + \bar{E}_t}{4} + \frac{1}{2}(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t}); \quad \bar{E}_{12}^{45°} = \frac{\bar{E}_\ell + \bar{E}_t}{4} - G_{\ell t} + \frac{1}{2}\nu_{t\ell}\bar{E}_\ell$$

$$\bar{E}_{33}^{45°} = \frac{\bar{E}_\ell + \bar{E}_t}{4} - \frac{1}{2}\nu_{t\ell}\bar{E}_\ell$$

with (Section 3.4.3):

$$\bar{E}_\ell = \bar{E}_t = E_x(1 - \nu_{xy} \times \nu_{yx}); \quad E_x = 54\ 000 \text{ MPa}; \quad \nu_{xy} = \nu_{yx} = 0.045$$

$$G_{\ell t} = G_{xy} = 4000$$

Then:

$$\bar{E}_{11}^{0°} = 54{,}100 \text{ MPa}; \quad \bar{E}_{12}^{0°} = 2435 \text{ MPa}; \quad \bar{E}_{33}^{0°} = 4000 \text{ MPa}$$

$$\bar{E}_{11}^{45°} = 32{,}270 \text{ MPa}; \quad \bar{E}_{12}^{45°} = 24{,}270 \text{ MPa}; \quad \bar{E}_{33}^{45°} = 25{,}840 \text{ MPa}$$

One deduces from there:

$$\frac{1}{h}A_{11} = \bar{E}_{11}^{0°} \times 0.67 + \bar{E}_{11}^{45°} \times 0.33 = 46{,}900 \text{ MPa} = \frac{1}{h}A_{22}$$

$$\frac{1}{h}A_{12} = \bar{E}_{12}^{0°} \times 0.67 + \bar{E}_{12}^{45°} \times 0.33 = 9640 \text{ MPa}$$

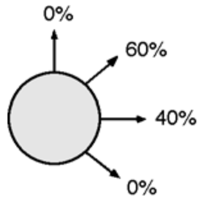
$$\frac{1}{h}A_{33} = \bar{E}_{33}^{0°} \times 0.67 + \bar{E}_{33}^{45°} \times 0.33 = 11{,}210 \text{ MPa}$$

After the calculation of $h[A]^{-1}$, one obtains the law for the behavior in zone 1:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{44{,}920} & -\frac{0.2}{44{,}920} & 0 \\ -\frac{0.2}{44{,}920} & \frac{1}{44{,}920} & 0 \\ 0 & 0 & \frac{1}{11{,}210} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} \qquad [1]$$

(b) Zone 2: Composition of the laminate:

Following the same method as above:



$$\frac{1}{h}A_{11} = \frac{1}{h}A_{22} = \bar{E}_{11}^{0^\circ} \times 0.4 + \bar{E}_{11}^{45^\circ} \times 0.6 = 41{,}010 \text{ MPa}$$

$$\frac{1}{h}A_{12} = 15{,}540 \text{ MPa}$$

$$\frac{1}{h}A_{33} = 17{,}100 \text{ MPa}$$

then, after inversion of the behavior law in zone 2:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{35{,}120} & -\frac{0.38}{35{,}120} & 0 \\ -\frac{0.38}{35{,}120} & \frac{1}{35{,}120} & 0 \\ 0 & 0 & \frac{1}{17{,}100} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} \qquad [2]$$

2. (a) Verification of non rupture:

   - Zone 1: Compression in the lower skin: $N_x = -435$ N/mm, then with 9 layers of fabric of thickness 0.24 mm:

$$\sigma_{ox} = -202 \text{ MPa}$$

from which the strains are (Equation [1] above):

$$\varepsilon_{ox} = -4.497 \times 10^{-3}; \quad \varepsilon_{oy} = 9 \times 10^{-4}; \quad \gamma_{oxy} = 0$$

- Layers at 0°/90°: (Equation 11.8):

$$\sigma_x^{0^\circ} = \bar{E}_{11}^{0^\circ} \times \varepsilon_{ox} + \bar{E}_{12}^{0^\circ} \times \varepsilon_{oy} = -241 \text{ MPa} = \sigma_\ell^{0^\circ}$$

$$\sigma_y^{0^\circ} = \bar{E}_{21}^{0^\circ} \times \varepsilon_{ox} + \bar{E}_{22}^{0^\circ} \times \varepsilon_{oy} = 38 \text{ MPa} = \sigma_t^{0^\circ}$$

$$\tau_{xy}^{0^\circ} = 0 = \tau_{\ell t}^{0^\circ}$$

The Hill–Tsai expression: (Section 5.3.2 and Chapter 14)[39]:

$$\frac{-241^2}{360^2}+\frac{38^2}{420^2}-\frac{-241\times 38}{360^2} = (0.72)^2<1$$

Factor of safety (Section 14.2.3.): $\frac{1}{0.72}-1 = 38\%$.

- Layers at 45°/–45°: One finds by an analogous calculation a much weaker value for the Hill–Tsai expression: $(0.49)^2$. The layers 0°/90° fail first.

■ Zone 2: With a shear stress resultant T_{xy} = –30 N/mm and 5 layers of fabric with 0.24 mm thickness, one has:

$$\tau_{oxy} = -25 \text{ MPa}.$$

From which the strains are (Equation [2] above):

$$\varepsilon_{ox} = 0;\ \varepsilon_{oy} = 0;\ \gamma_{oxy} = -1.46\times 10^{-3}$$

- Layers at 45°/–45° (Equation 11.8):

$$\sigma_x^{45°} = \sigma_y^{45°} = 0; \quad \tau_{xy}^{45°} = -38\,\text{MPa}$$

Equation 11.4:

$$\sigma_\ell^{45°} = -\tau_{xy}^{45°} = 38 \text{ MPa} = -\sigma_t^{45°}; \quad \tau_{\ell t}^{45°} = 0$$

Hill–Tsai expression:

$$\frac{38^2}{420^2}+\frac{-38^2}{360^2}-\frac{-38\times 38}{420^2} = (0.17)^2$$

corresponding to a factor of safety of $\frac{1}{0.17}-1 = 500\%$

- Layers at 0°/90°: One finds a smaller value for the Hill–Tsai expression: $(0.1)^2$. It is the 45°/–45° layers that fail first.

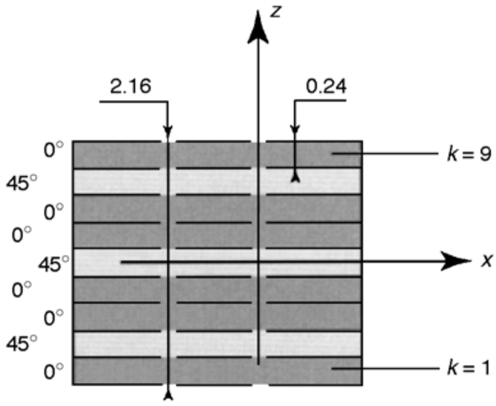
2. (b) Verification for buckling: This is done starting from the graphs of Appendix 2. In this view, one has to evaluate the constants C_{11}, C_{22}, C_{12}, C_{33} that appear in the law of the bending behavior (Equation 12.16):

[39] As recalled from Section 14.2.2 (note 4 at bottom of the page), a balanced fabric is not transversely isotropic. The Hill-Tsai (Equation 14.6) can be rewritten in this case as:

$$\frac{\sigma_\ell^2}{\sigma_{\ell\ \text{rupt}}^2}+\frac{\sigma_t^2}{\sigma_{t\ \text{rupt}}^2}-\sigma_\ell\sigma_t\left(\frac{2}{\sigma_{\ell\ \text{rupt}}^2}-\frac{1}{\sigma_{z\ \text{rupt}}^2}\right)+\frac{\tau_{\ell t}^2}{\tau_{\ell t\ \text{rupt}}^2}<1$$

Without knowing $\sigma_{z\ \text{rupture}}$ and taking into account the weak influence of the modified term, one uses the Equation 14.6.

■ Zone 1:



$$C_{ij} = \sum_{k=1^{st}\,ply}^{9^{th}\,ply} \bar{E}_{ij}^{k} \left(\frac{z_k^3 - z_{k-1}^3}{3}\right)$$

ply $n°k$	1	2	3	4	5	6	7	8	9
$\left(\frac{z_k^3 - z_{k-1}^3}{3}\right)$	0.2223	0.1256	0.0564	0.0150	1.152 E-3	0.0150	0.0564	0.1256	0.223

from which:

$$C_{11} = C_{22} = 39{,}930 \text{ N mm}$$

$$C_{12} = C_{21} = 7555 \text{ N mm}$$

$$C_{33} = 8870 \text{ N mm}$$

Then:

$$[C] = \begin{bmatrix} 39{,}930 & 7555 & 0 \\ 7555 & 39{,}930 & 0 \\ 0 & 0 & 8870 \end{bmatrix} (\text{N} \times \text{mm})$$

Consider the unfavorable case of a plate simply supported at two of its sides, clamped along the third side, and free on the fourth side (see figure in the Problem Statement). Using the Plate 16 in Appendix 2 with the values:

$$C = \frac{C_{21} + 2C_{33}}{\sqrt{C_{11} \times C_{22}}} = \frac{25{,}295}{39{,}930} = 0.63; \quad \frac{a}{b}\left(\frac{C_{22}}{C_{11}}\right)^{1/4} \gg 1$$

one obtains

$$k \# 1.15$$

from which the critical compressive stress resultant is

$$N_{x\,\text{critical}} = 1.15\ \pi^2\ (39930/40^2)$$

$$N_{x\,\text{critical}} = 283\ \text{N/mm} < 435\ \text{N/mm applied}$$

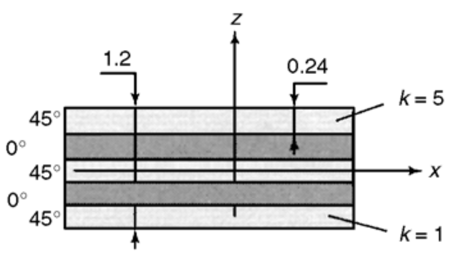
There is a risk of buckling, and one must reinforce the wing in the central part of the spar where the compressive stress resultant is maximum by means of exterior layers at 0°/90° in such a way to augment C_{11} and C_{22}. For example, with a supplementary external layer on either side:

$$C'_{22} = 77{,}475\ \text{N/mm};\quad C'_{21} = 9245\ \text{N/mm};\quad C'_{33} = 11{,}646\ \text{N/mm}.$$

from which $C = 0.42$, $k \# 1$, and:

$$N'_{x\,\text{critical}} = 477\ \text{N/mm} > 435\ \text{N/mm applied.}$$

- Zone 2:



$$C_{ij} = \sum_{k=1^{\text{st}}\text{ ply}}^{k=5^{\text{th}}\text{ ply}} \bar{E}_{ij}^{k} \left(\frac{z_k^3 - z_{k-1}^3}{3} \right)$$

One obtains after calculation:

$$[C] = \begin{bmatrix} 5300 & 2840 & 0 \\ 2840 & 5300 & 0 \\ 0 & 0 & 3065 \end{bmatrix} (\text{N} \times \text{mm})$$

In the unfavorable case of a plate simply supported on four sides (see figure in the Problem Statement), one uses Plate 18 of Appendix 2 with the values:

$$C = 1.7;\quad \frac{a}{b}\left(\frac{C_{22}}{C_{11}}\right)^{1/4} \gg 1$$

One obtains

$$k \# 7$$

from which the critical shear stress resultant is

$$T_{xy\ \text{critical}} = 7\ \pi^2\ (5300/130^2)$$

$$T_{xy\ \text{critical}} = 21\ \text{N/mm} < 30\ \text{N/mm applied}$$

There is then a risk of buckling and one must reinforce the web in this part of the spar where the shear force is maximum. A supplementary external layer at 0°/90° on either side of this web gives

$$C'_{22} = 18{,}890\ \text{N/mm};\quad C'_{21} = 3450\ \text{N/mm};\quad C'_{33} = 4070\ \text{N/mm}.$$

From which: $C = 0.6$, k # 4.3, and:

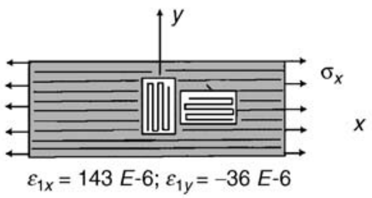
$$T_{xy\ \text{critical}} = 47\ \text{N/mm} > 30\ \text{N/mm applied.}$$

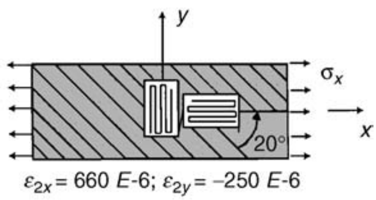
18.2.11 Determination of the Elastic Characteristics of a Carbon/Epoxy Unidirectional Layer from Tensile Test

Problem Statement:

Consider a unidirectional plate of carbon/epoxy, from which one cuts the two samples shown below. They are tested in a testing machine. One measures the strains using strain gages arranged as shown. The strains obtained under different loads are linearized. One shows their values corresponding to a uniform tensile stress σ_x equal to 20 MPa.

Calculate the elastic constants of the unidirectional layer subject to in-plane loading.





Solution:

One can use the Equation 11.5:

- Sample No. 1: The axes x and y are ,coincident with the axes ℓ and t ($\theta =$ 0). From which:

$$\varepsilon_{1x} = \frac{\sigma_x}{E_x} = \frac{\sigma_x}{E_\ell} \rightarrow E_\ell = \frac{20}{143\text{ E-6}} = 139{,}860\ \text{MPa}$$

$$\varepsilon_{1y} = -\frac{\nu_{xy}}{E_x}\times\sigma_x = -\frac{\nu_{\ell t}}{E_\ell}\times\sigma_x \rightarrow \nu_{\ell t} = 0.25$$

- Sample No. 2: The axes x and y make an angle of $\theta = 20°$ with the axes l and t, from which[40]:

$$\varepsilon_{2x} = \frac{\sigma_x}{E_x} = \left\{\frac{c^4}{E_\ell} + \frac{s^4}{E_t} + c^2 s^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{\ell t}}{E_\ell}\right)\right\} \times \sigma_x$$

$$\varepsilon_{2y} = -\frac{\nu_{xy}}{E_x} \times \sigma_x = -\left\{\frac{\nu_{\ell t}}{E_\ell}(c^4 + s^4) - c^2 s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} - \frac{1}{G_{\ell t}}\right)\right\} \times \sigma_x$$

leading to

$$\begin{cases} \dfrac{1}{G_{\ell t}} + \dfrac{0.1325}{E_t} = 2.69 \text{ E-4} \\ \dfrac{1}{G_{\ell t}} - \dfrac{1}{E_t} = 1.144 \text{ E-4} \end{cases}$$

from which: $E_t = 7320$ MPa; $G_{\ell t} = 3980$ MPa
In summary:

$$\boxed{\begin{array}{l} E_\ell = 139{,}860 \text{ MPa} \\ E_t = 7320 \text{ MPa} \\ \nu_{\ell t} = 0.25; \quad \nu_{t\ell} = 0.013 \\ G_{\ell t} = 3980 \text{ MPa} \end{array}}$$

18.2.12 Sailboat Shell in Glass/Polyester

Consider a siding of a laminated shell for a sailboat made of glass/polyester. It is made up of a stack of layers of balanced fabric and glass mat. The reinforcements, in "E" glass, are in the following form:

Balanced fabric: $V_f = 20\%$, mass of the glass per square meter: $m_{of} = 500$ g.
Mat: $V_f = 15\%$, mass of glass per square meter: $m_{of} = 300$ g.

Problem Statement:

1. Calculate:
   (a) The thickness of one layer of fabric of glass/polyester.
   (b) The thickness of a layer of mat of glass/polyester.
2. Given the composition of the laminated siding as follows:

$$[\text{M/F/M/F}]_s \ (\text{M} \leftrightarrow \text{Mat; F} \leftrightarrow \text{Fabric})$$

[40] One has $\frac{\nu_{t\ell}}{E_t} = \frac{\nu_{\ell t}}{E_\ell}$ See Exercise 18.1.2

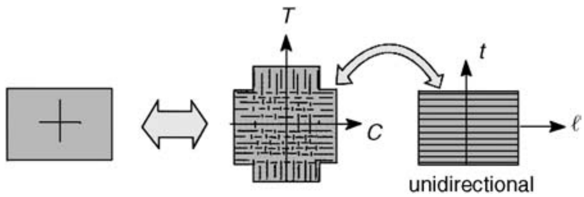


Figure 18.9

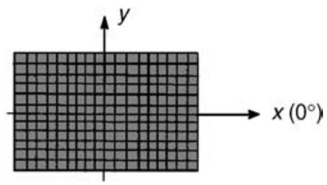


Figure 18.10

what is the total thickness, denoted as h, of the laminated siding?

3. Elastic characteristics of a fabric layer: One considers a layer of balanced fabric to be equivalent to two series of unidirectional plies crossed at 90°, each series possessing half of the total thickness of the fabric layer (see Figure 18.9).

The elastic characteristics of these unidirectional plies are as follows:

$$E_\ell = 18{,}000 \text{ MPa};\ E_t = 4900 \text{ MPa};\ G_{\ell t} = 1850 \text{ MPa};\ \nu_{\ell t} = 0.3.$$

Calculate the elastic characteristics (moduli, Poisson coefficients) of a layer of fabric in axes C,T.

4. The layers of mat are considered as isotropic in their planes, with the elastic characteristics:

$$E_{\text{Mat}} = 8350 \text{ MPa};\ \nu_{\text{Mat}} = 0.3.$$

Figure 18.10 represents one planar portion of the laminated siding. All the fabric plies are oriented at 0° – 90°. Calculate the global elastic characteristics (moduli, Poisson coefficients) of the siding working in its plane.

Remark: Tests done on samples made of this material indicate a modulus of elasticity along the x direction to be equal to 9200 MPa. What can be said about this?

5. Rupture: The rupture strengths, considered to be equal in tension and in compression, are as follows:

- Fabric layer: along C or T: $\sigma_{\text{rupture fabric}} = 139$ MPa.
- Mat layer: $\sigma_{\text{rupture Mat}} = 113$ MPa.

(a) Calculate the maximum stress σ_{ox} leading to first ply failure of the siding. What are the layers that fracture?

(b) Apply the maximum stress σ_{ox}. In the previous layers that fractured, the glass fibers are supposed all broken. What happens to the laminate?

Solution:

1. The thickness of a layer denoted as h is such that (see Section 3.2.4):

$$h = \frac{m_{of}}{V_f \rho_f}$$

The specific mass of "E" glass is (see Section 1.6): $\rho = 2600$ kg/m^3, from which:

$$h_{\text{fabric}} = 0.96 \text{ mm}; \; h_{\text{Mat}} = 0.77 \text{ mm}.$$

2. The siding is constituted of the following stacking sequence:

The total thickness is

$$h = 0.77 \times 4 + 0.96 \times 3 = 5.96 \text{ mm}.$$

3. Elastic characteristics of a fabric layer: The moduli and Poisson coefficients can be evaluated starting from the simplified relations of Section 3.4.2. One obtains, with $k = 0.5$ (balanced fabric):

$$E_C = E_T = 11{,}450 \text{ MPa}$$

$$G_{CT} = 1850 \text{ MPa}; \; \nu_{CT} = \nu_{TC} = 0.128$$

A more precise calculation of these characteristics requires to establish the matrix h$[A]^{-1}$ of Section 12.1.2. (Equation (12.9)). We calculate at first $\frac{1}{h}[A]$ (Equation 12.8):

$$\frac{1}{h}A_{ij} = (\bar{E}_{ij}^{0^\circ})(0.5) + (\bar{E}_{ij}^{90^\circ})(0.5)$$

The terms $\bar{E}_{ij}$ are given by the Equation 11.8. One will have, for example:

$$\frac{1}{h}A_{11} = (\bar{E}_\ell)(0.5) + (\bar{E}_t)(0.5) = \frac{1}{2}\,\frac{E_\ell + E_t}{1 - \nu_{\ell t}\nu_{t\ell}}$$

with

$$\nu_{t\ell} = \nu_{\ell t}\frac{E_t}{E_\ell};$$

One obtains

$$\frac{1}{h}[A] = \begin{bmatrix} 11{,}737 & 1507 & 0 \\ 1507 & 11{,}737 & 0 \\ 0 & 0 & 1850 \end{bmatrix} \text{(MPa)}$$

$$h[A]^{-1} = \begin{bmatrix} \frac{1}{11{,}540} & -\frac{0.128}{11{,}540} & 0 \\ -\frac{0.128}{11{,}540} & \frac{1}{11{,}540} & 0 \\ 0 & 0 & \frac{1}{1850} \end{bmatrix}$$

from which:

$$\boxed{\begin{aligned} E_C &= E_T = 11{,}540 \text{ MPa} \\ G_{CT} &= 1850 \text{ MPa} \\ \nu_{CT} &= \nu_{TC} = 0.128 \end{aligned}}$$

(The difference between the values obtained above is small).

4. Elastic characteristics of the siding: These are deduced from the matrix $h[A]^{-1}$ (Equation 12.9) calculated for all the laminate.

We calculate at first $\frac{1}{h}[A]$ (Equation 12.8):

$$\frac{1}{h}A_{ij} = (\bar{E}_{ij}^{\text{fabric}})(p^{\text{fabric}}) + (\bar{E}_{ij}^{\text{Mat}})(p^{\text{Mat}})$$

with

$$p^{\text{fabric}} = \frac{3 \times 0.96}{5.96} = 0.483; \quad p^{\text{Mat}} = 0.517.$$

$$\bar{E}_{11}^{\text{fabric}} = \bar{E}_{22}^{\text{fabric}} = \bar{E}_C = \frac{E_C}{1 - \nu_{CT}^2} \qquad \text{(see Equation 11.8)}$$

$$\bar{E}_{12}^{\text{fabric}} = \nu_{CT}\,\bar{E}_C; \quad \bar{E}_{33}^{\text{fabric}} = G_{CT}$$

$$\bar{E}_{11}^{\text{Mat}} = \bar{E}_{22}^{\text{Mat}} = \frac{E_{\text{Mat}}}{1 - \nu_{\text{Mat}}^2}; \quad \bar{E}_{12}^{\text{Mat}} = \frac{\nu_{\text{Mat}}E_{\text{Mat}}}{1 - \nu_{\text{Mat}}^2}$$

$$\bar{E}_{33}^{\text{Mat}} = G_{\text{Mat}} = \frac{E_{\text{Mat}}}{2(1 + \nu_{\text{Mat}})}$$

We obtain

$$\frac{1}{h}[A] = \begin{bmatrix} 10{,}410 & 2149 & 0 \\ 2149 & 10{,}410 & 0 \\ 0 & 0 & 2554 \end{bmatrix} [\mathrm{MPa}]$$

$$h[A]^{-1} = \begin{bmatrix} \frac{1}{9966} & -\frac{0.206}{9966} & 0 \\ -\frac{0.206}{9966} & \frac{1}{9966} & 0 \\ 0 & 0 & \frac{1}{2554} \end{bmatrix}$$

then:

$$\boxed{\begin{aligned} E_x = E_y &= 9966 \text{ MPa} \\ G_{xy} &= 2554 \text{ MPa} \\ \nu_{xy} = \nu_{yx} &= 0.206 \end{aligned}}$$

Remark: The real modulus (measured) 9200 MPa is a bit smaller than the one calculated. In effect, due to the curvature of fibers from weaving, a fabric layer is softer than the stacking of unidirectionals that are crossed at 90°. However, the approximation obtained by calculation is suitable (difference < 10%).

5. Fracture of the siding:
   (a) One subjects the siding to a stress σ_{ox}. The strains of this siding are given by the Equation 12.9 as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \sigma_{ox} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma_{ox}/9966 \\ -0.206\ \sigma_{ox}/\ 9966 \\ 0 \end{Bmatrix}$$

These strains give rise to the following stresses:

- In the layers of fabric (see results from Question 3):

$$\begin{Bmatrix} \sigma_C \\ \sigma_T \\ \tau_{CT} \end{Bmatrix} = \begin{Bmatrix} 11{,}737 & 1507 & 0 \\ 1507 & 11{,}737 & 0 \\ 0 & 0 & 1850 \end{Bmatrix} \begin{Bmatrix} \sigma_{ox}/9966 \\ -0.206\ \sigma_{ox}/9966 \\ 0 \end{Bmatrix} \cdots$$

$$\cdots = \begin{Bmatrix} 1.15\ \sigma_{ox} \\ -0.09\ \sigma_{ox} \\ 0 \end{Bmatrix}$$

The Hill–Tsai criterion in these layers is satisfied for a stress σ_{ox} such that:

$$\left(\frac{1.15\sigma_{ox}}{139}\right)^2 + \left(\frac{-0.09\sigma_{ox}}{139}\right)^2 - \frac{-0.09 \times 1.15\sigma_{ox}^2}{139^2} = 1$$

as:

$$\sigma_{ox} = 116 \text{ MPa}$$

- In the layers of Mat, with the values of the coefficients $\bar{E}_{ij}^{\text{Mat}}$ of Question 4, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{E}^{\text{Mat}}] \begin{Bmatrix} \sigma_{ox}/9\,966 \\ -0.206\ \sigma_{ox}/9\,966 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.86\sigma_{ox} \\ 0.087\sigma_{ox} \\ 0 \end{Bmatrix}$$

The Hill–Tsai criterion in the mat layers is satisfied for a stress σ_{ox} such that[41]:

$$\left(\frac{0.86\sigma_{ox}}{113}\right)^2 + \left(\frac{0.087\sigma_{ox}}{113}\right)^2 - \frac{0.86 \times 0.087\sigma_{ox}^2}{113^2} = 1$$

then:

$$\boxed{\sigma_{ox} = 138 \text{ MPa}}$$

The fabric layers are the first to fail, for a stress of

$$\sigma_{ox\,\max} = 116 \text{ MPa}$$

(b) This stress being applied, the rupture of the fabric layers translates into the rupture of the glass fibers. The stress resultant corresponding to this constraint as:

$$N_x = \sigma_{ox\,\max} \times h = 116 \times 5.96 = 691 \text{ N/mm}$$

is then completely taken up by the layers of Mat. The stress in these layers is then:

$$\sigma_{ox\,\text{Mat}} = \frac{N_x}{4 \times h_{\text{Mat}}} = \frac{691}{4 \times 0.77} = 224 \text{ MPa}$$

[41] A mat layer does not have transverse isotropy in the axes y, z (or x, z). The Hill-Tsai expression is then modified. We use however the Equation 14.6 here (see remark 39 at the bottom of page in Application 18.2.10).

It exceeds the rupture strength of the Mat (113 MPa). Then this latter layer fractures.

The siding is then completely broken under the stress:

$$\sigma_{ox\,\max} = 116 \text{ MPa}$$

18.2.13 Determination of the In-Plane Shear Modulus of a Balanced Fabric Ply

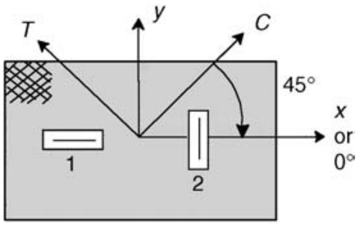
Problem Statement:

Consider a sample cut from a laminated panel made of identical layers of balanced fabric, all oriented along the axes C (warp direction) and T (fill direction) in the following figure.

The sample is in a state of simple tension in its plane along the x axis as shown in the figure.

$$\sigma_{ox} \neq 0; \qquad \sigma_{oy} = \tau_{oxy} = 0$$

Two strain gages are bonded (see figure). These are denoted as 1 and 2:



- From gage 1, one reads a strain ε_{ox}.
- From gage 2, one reads a strain ε_{oy}.

1. Noting that $\gamma_{oxy} = 0$, give the expression for the distortion γ_{CT} in the axes C and T as a function of ε_{ox} and ε_{oy}.
2. Give the expression for the stress τ_{CT} in the axes C and T as a function of σ_{ox}.
3. Deduce from the previous answer the shear modulus G_{CT} as a function of ε_{ox}, ε_{oy}, and σ_{ox}.

Solution:

1. Equation 11.7 allows one to write

$$\begin{Bmatrix} \varepsilon_C \\ \varepsilon_T \\ \gamma_{CT} \end{Bmatrix} = \begin{Bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & (c^2 - s^2) \end{Bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}$$

Here, we have a balanced laminate with midplane symmetry, loaded in its axes x,y. Then (Equation 12.9): $\gamma_{oxy} = 0$, from which:

$$\gamma_{CT} = 2cs\ \varepsilon_{ox} - 2cs\ \varepsilon_{oy} \text{ with } c = \frac{1}{\sqrt{2}};\quad s = -\frac{1}{\sqrt{2}}$$

$$\gamma_{CT} = -\varepsilon_{ox} + \varepsilon_{oy}$$

2. According to Equation 11.4:

$$\begin{Bmatrix} \sigma_C \\ \sigma_T \\ \tau_{CT} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ sc & -sc & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ 0 \\ 0 \end{Bmatrix}$$

then:

$$\tau_{CT} = sc\ \sigma_{ox} = -\frac{\sigma_{ox}}{2}$$

3. The constitutive behavior of the fabric in its axes can be written, starting from the Equation 11.5, as:

$$\begin{Bmatrix} \varepsilon_C \\ \varepsilon_T \\ \gamma_{CT} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{E_c} & -\frac{\nu_{CT}}{E_c} & 0 \\ -\frac{\nu_{CT}}{E_c} & \frac{1}{E_c} & 0 \\ 0 & 0 & \frac{1}{G_{CT}} \end{Bmatrix} = \begin{Bmatrix} \sigma_C \\ \sigma_T \\ \tau_{CT} \end{Bmatrix}$$

$$\gamma_{CT} = \frac{\tau_{CT}}{G_{CT}}$$

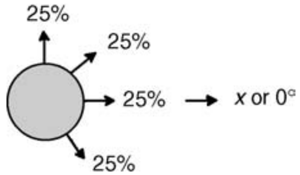
From which:

$$\boxed{G_{CT} = \frac{\sigma_{ox}}{2(\varepsilon_{ox} - \varepsilon_{oy})}}$$

18.2.14 Quasi-Isotropic Laminate

Problem Statement:

Consider a laminate made up of a number of identical unidirectional plies, with midplane symmetry and the following composition:



The elastic characteristics of a ply in its axes ℓ and t are denoted as:

$$E_\ell,\ E_t,\ G_{\ell t},\ \nu_{\ell t},\ \nu_{t\ell}$$

One examines the behavior of this laminate under in-plane loading, following the law (Equation 12.9):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix}$$

1. Calculate the coefficients of the matrix $\frac{1}{h}[A]$.
2. By inversion, deduce from there the elastic moduli of the laminate.
3. What comment can one make? Deduce from there the law for the behavior of the laminate under in-plane loading in the axes (X,Y) derived from the (x,y) axes by a rotation angle θ.

Solution:

1. Coefficients $\frac{1}{h}A_{ij}$ are given by the Equation 12.8 as:

$$\frac{1}{h}A_{ij} = \frac{1}{4}\left[\bar{E}_{ij}^{0^\circ} + \bar{E}_{ij}^{90^\circ} + \bar{E}_{ij}^{+45^\circ} + \bar{E}_{ij}^{-45^\circ}\right]$$

The stiffness coefficients $\bar{E}_{ij}^{0^\circ}$ are obtained from the behavior of a ply (Equation 11.8). In using this relation for $\theta = 0°$, 90°, +45°, –45°, one obtains

$$\frac{1}{h}A_{11} = \frac{1}{h}A_{22} = \frac{1}{4}\left[\frac{3}{2}(\bar{E}_\ell + \bar{E}_t) + \nu_{t\ell}\ \bar{E}_\ell + 2G_{\ell t}\right]$$

$$\frac{1}{h}A_{12} = \frac{1}{4}\left[\frac{1}{2}(\bar{E}_\ell + \bar{E}_t) + 3\nu_{t\ell}\ \bar{E}_\ell - 2G_{\ell t}\right]$$

$$\frac{1}{h}A_{33} = \frac{1}{4}\left[\frac{1}{2}(\bar{E}_\ell + \bar{E}_t) - \nu_{t\ell}\ \bar{E}_\ell + 2G_{\ell t}\right]$$

$$\frac{1}{h}A_{13} = \frac{1}{h}A_{23} = 0$$

where one recalls that: $\bar{E}_\ell = \dfrac{E_\ell}{1-\nu_{\ell t}\nu_{t\ell}}$; $\quad \bar{E}_t = \dfrac{E_t}{1-\nu_{\ell t}\nu_{t\ell}}$

The matrix $\frac{1}{h}[A]$ reduces to

$$\frac{1}{h}\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{11} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

2. From the above, the modulus of elasticity of the laminate in directions x and y, and the associated Poisson coefficient are

$$\frac{1}{E} = \frac{\frac{1}{h}A_{11}}{\frac{1}{h^2}(A_{11}^2 - A_{12}^2)}; \qquad -\frac{\nu}{E} = -\frac{\frac{1}{h}A_{12}}{\frac{1}{h^2}(A_{11}^2 - A_{12}^2)}$$

One obtains after calculation:

$$E = \frac{[2(\bar{E}_\ell + \bar{E}_t) + 4\nu_{t\ell}\ \bar{E}_\ell][\bar{E}_\ell + \bar{E}_t - 2\nu_{t\ell}\ \bar{E}_\ell + 4G_{\ell t}]}{4\left[\frac{3}{2}(\bar{E}_\ell + \bar{E}_t) + \nu_{t\ell}\ \bar{E}_\ell + 2G_{\ell t}\right]}$$

$$\nu = \frac{\frac{1}{2}(\bar{E}_\ell + \bar{E}_t) + 3\nu_{t\ell}\ \bar{E}_\ell - 2G_{\ell t}}{\frac{3}{2}(\bar{E}_\ell + \bar{E}_t) + \nu_{t\ell}\ \bar{E}_\ell + 2G_{\ell t}}$$

The shear modulus is written as:

$$G = \frac{1}{4}\left[\frac{1}{2}(\bar{E}_\ell + \bar{E}_t) - \nu_{t\ell}\ \bar{E}_\ell + 2G_{\ell t}\right]$$

3. One can remark that:

$$G = \frac{E}{2(1+\nu)}$$

This leads to an isotropic elastic behavior of the laminate in its plane. As a result, in all coordinate systems (X,Y) derived from (x,y) by any rotation angle, the constitutive behavior of the laminate is unchanged and is written as:

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} = \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix}$$

Remark: This result generalizes to other groups of orientations for plies such as:

$$\left[0, \frac{\pi}{3}, \frac{2\pi}{3}\right]; \quad \left[0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}\right]$$

and so on.

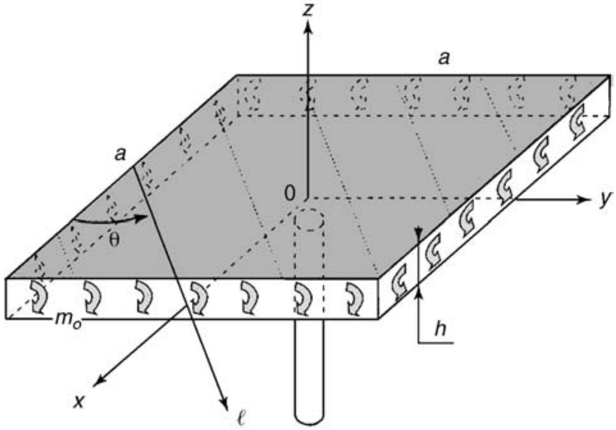
More generally, a laminate made up of n orientations (a whole number $n > 2$), having the values of $\frac{\pi}{n}(p-1)$, with $p = 1,\ldots, n$ and with the same proportion of plies along each orientation as $(1/n)$ is elastically isotropic. One shows also that for all these laminates, E and ν are invariable.[42]

18.2.15 Orthotropic Plate in Pure Torsion

Problem Statement:

Consider a square plate $(a \times a)$ of unidirectional glass/epoxy ($V_f = 60\%$), thickness h, welded at the center of its lower face on a support. It is subjected to a uniform and constant torsional moment density m_o (N mm/mm) along its perimeter.[43]

The directions ℓ, t of the unidirectional form an angle θ with the axes x,y of the plate (see figure).



1. Assuming that all stress resultants in the plate are zero (except the torsional moment), determine the bending displacement at all points of the mid-plane.
2. Determine the state of stresses in the axes (x,y) then in the axes (ℓ,t) of the unidirectional layer.
3. Numerical application: $\theta = 45°$; $a = 1$ m; $h = 5$ mm; $m_o = -10$ N mm/mm.

[42] For more details, see Bibliography at the end of the book: "Stiffness isotropy and resistance quasi-isotropy of laminates with periodic orientations."

[43] The practical importance of such a load is very limited. It is better to consider this example as a means to validate a computer program using finite elements. It is one of the "patches" issued from the work "Computer programs for composite structures: Examples of reference for validation" (see Bibliography at the end of the book).

Solution:

1. In the constitutive Equation 12.16, one has

$$C_{ij} = \bar{E}_{ij}\int_{-h/2}^{h/2} z^2 dz = \bar{E}_{ij}\frac{h^3}{12}$$

then:

$$[C] = \frac{h^3}{12}[\bar{E}]$$

where $[\bar{E}]$ is the matrix shown in details in Equation 11.8.

By inverting the Equation 12.16 and noting that:

$$[C]^{-1} = \frac{12}{h^3}[\bar{E}]^{-1} = \frac{12}{h^3}\left[\frac{1}{E}\right]$$

where $\left[\frac{1}{E}\right]$ is the matrix shown in details in Equation 11.5, one has

$$\begin{Bmatrix} -\dfrac{\partial^2 w_o}{\partial x^2} \\ -\dfrac{\partial^2 w_o}{\partial y^2} \\ -2\dfrac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} = \frac{12}{h^3}\left[\frac{1}{E}\right]\begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} \qquad [1]$$

Assuming the unit stress resultants all to be zero except M_{xy},[44] one has

$$N_x = N_y = T_{xy} = M_x = M_y = 0; \quad M_{xy} = m_o$$

There remains (see Equation 11.5):

$$\frac{\partial^2 w_o}{\partial x^2} = \frac{12}{h^3}\frac{\eta_{xy}}{G_{xy}}m_o; \quad \frac{\partial^2 w_o}{\partial y^2} = \frac{12}{h^3}\frac{\mu_{xy}}{G_{xy}}m_o; \quad 2\frac{\partial^2 w_o}{\partial x \partial y} = \frac{12}{h^3}\frac{1}{G_{xy}}m_o$$

Therefore one can write $w_o(x, y)$ in the form:

$$w_o = \frac{12}{h^3}\frac{m_o}{G_{xy}}(Ax^2 + By^2 + Cxy + Dx + Ey + F)$$

[44] With this hypothesis, equations of equilibrium, constitutive equation, and boundary conditions are verified.

At the center of the plate: $w_o = 0$; $\dfrac{\partial w_o}{\partial x} = \dfrac{\partial w_o}{\partial y} = 0$
from which: $D = E = F = 0$. And by identification with the second derivatives:

$$2A = \eta_{xy}; \quad 2B = \mu_{xy}; \quad 2C = 1$$

The out-of-plane displacement takes the form:

$$\boxed{w_o = \frac{6m_o}{h^3 G_{xy}}(\eta_{xy}x^2 + \mu_{xy}y^2 + xy)} \qquad [2]$$

2. State of stresses: The strain field in the axes of the plate is written as (Equation 12.12, taking into account [1]):

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = z \times \begin{Bmatrix} -\dfrac{\partial^2 w_o}{\partial x^2} \\ -\dfrac{\partial^2 w_o}{\partial y^2} \\ -2\dfrac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} = z \times \frac{12}{h^3}\left[\frac{1}{E}\right]\begin{Bmatrix} 0 \\ 0 \\ -m_o \end{Bmatrix}$$

from which one can write the stresses in the axes (x, y) using Equation 11.8:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{E}]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = z \times \frac{12}{h^3}[\bar{E}]\left[\frac{1}{E}\right]\begin{Bmatrix} 0 \\ 0 \\ -m_o \end{Bmatrix} = z \times \frac{12}{h^3}\begin{Bmatrix} 0 \\ 0 \\ -m_o \end{Bmatrix}$$

then:

$$\sigma_x = 0; \quad \sigma_y = 0; \quad \tau_{xy} = -z \times \frac{12}{h^3} m_o$$

- Stresses in the axes of the unidirectional: These are obtained by using the Equation 11.4, which is[45]

$$\begin{aligned} \sigma_\ell &= -2cs\ \tau_{xy} &&= z \times cs \times \frac{24}{h^3} m_o \\ \sigma_t &= 2cs\ \tau_{xy} &&= -z \times cs \times \frac{24}{h^3} m_o \\ \tau_{\ell t} &= (c^2 - s^2)\tau_{xy} &&= -z(c^2 - s^2) \times \frac{12}{h^3} m_o \end{aligned}$$

[45] Note that here the angle $\theta = (\vec{\ell} \cdot \vec{x})$ since the Equation 11.4 is written with $\theta = (\vec{x} \cdot \vec{\ell})$.

3. Numerical application:

One has (Section 3.3.3) for the glass/epoxy:

$$E_\ell = 45{,}000 \text{ MPa}; \quad E_t = 12{,}000 \text{ MPa};$$

$$G_{\ell t} = 4500 \text{ MPa}; \quad \nu_{\ell t} = 0.3 \ (\nu_{t\ell} = 0.08)$$

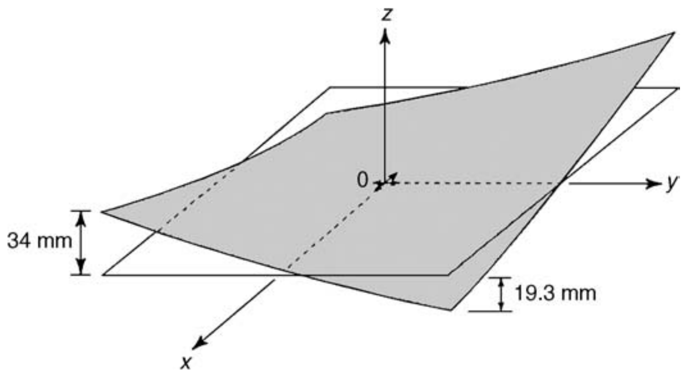
from which (Equation 11.5 with $\theta = -45°$):

$$\frac{\eta_{xy}}{G_{xy}} = \frac{\mu_{xy}}{G_{xy}} = -\frac{0.1375}{4500}$$

and w_o takes the form:

$$w_o = -\frac{1}{9\,375}[xy - 0.1375(x^2 + y^2)]$$

The deformed configuration is shown in the figure below:



The stresses (in MPa) are written as:

$$\sigma_x = \sigma_y = 0; \quad \tau_{xy} = 0.96 \times z \text{ (mm)}.$$

$$\sigma_\ell = -\sigma_t = 0.96 \times z \text{ (mm)}; \quad \tau_{\ell t} = 0$$

18.2.16 Plate Made by Resin Transfer Molding (R.T.M.)

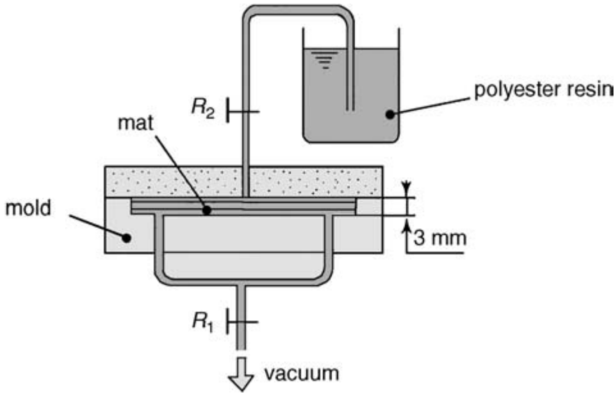
Problem Statement:

First part:

A roll of mat of carbon fibers has the following characteristics:

Areal mass density: $m_{of} = 30$ g/m^2
Specific mass: $\rho_f = 1{,}750$ kg/m^3

One deposits 21 layers of this mat over a plate in a rectangular mold. The mold is then closed and sealed, as shown in the figure below:



R_1 and R_2 represent two valves.

- R_2 is closed, R_1 is open. The mold is vacuumed.
- R_2 is open, R_1 is open. Polyester resin is filled into the cavity of the mold. Then resin begins to flow out through valve R_1.
- R_1 and R_2 are closed.

The mold is then heated, and the resin polymerizes. After demolding, one obtains a plate of mat/polyester.

1. Calculate the fiber volume fraction V_f (%).
2. Calculate the modulus of elasticity along the longitudinal and transverse directions, denoted respectively as E_ℓ and E_t, of a unidirectional of carbon/polyester, that would have the same amount of fiber volume fraction. The following is given

$$E_{f\ell} = 230{,}000 \text{ MPa}; \quad E_{ft} = 15{,}000 \text{ MPa (Section 3.3.1, Table 3.3)}$$

$$E_{\text{resin}} = 4000 \text{ MPa (Section 1.6)}$$

3. Starting from the relation in Section 3.5.1 giving the modulus of elasticity of mat (which is assumed to be isotropic in the plane of the plate), deduce from there the value of E_{mat}. Assume that $\nu_{\text{mat}} = 0.3$.

Second part:
One polymerizes on each face of the previous plate two plies of preimpregnated carbon/epoxy unidirectionals with $V_f = 60\%$. (see characteristics given in Section 3.3.3). Each ply has a thickness of 0.13 mm. The four plies (two above, two below) are oriented in the same direction denoted as x (or 0°). The midplane of the laminated plate coincides with axes x and y.

1. Write numerically for the unidirectional and for the mat the constitutive relation in the x, y axes in the form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{E}]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

2. Calculate in the axes x, y the coefficients of the in-plane constitutive relation of the laminated plate (matrix [A], Section 12.1.1). Deduce from there the moduli of elasticity and the Poisson coefficients of the plate.
3. Calculate in the axes x, y the coefficients for the bending behavior of the laminated plate (matrix [C], Section 12.1.4). Deduce from there the apparent bending moduli along the directions x and y.
4. This laminated plate is submitted to a tensile stress resultant along the x direction denoted as N_x (N/mm). The tensile rupture strength of mat is 100 MPa. Calculate the value of the stress resultant N_x that leads to first-ply failure of the laminate. In which component (unidirectional or mat) will this failure occur? This component is supposed to be completely broken (i.e., its mechanical characteristics are reduced to zero). What then is the state of stress in the other component? Make a conclusion.

Solution:

First part:

1. Fiber volume fraction of carbon:

$$V_f = \frac{\text{vol. fibers}}{\text{total volume}}$$

If s is the rectangular surface at the base of the mold, the volume of a layer of mat is

$$s\frac{m_{of}}{\rho_f}$$

from which, for 21 layers:

$$V_f = \frac{21 \times s \times m_{of}/\rho_f}{s \times 3 \times 10^{-3}} = 12\%$$

2. Moduli of elasticity (see Section 3.3.1): One has

$$E_\ell = E_f V_f + E_m V_m = 31{,}120 \text{ MPa}$$

$$E_t = E_m\left[\frac{1}{V_m + \frac{E_m}{E_{ft}} V_f}\right] = 4386 \text{ MPa}$$

3. One has (see Section 3.5.1)

$$E_{\text{mat}} = \frac{3}{8}E_\ell + \frac{5}{8}E_t = 14{,}410 \text{ MPa}$$

Second part:

1. Constitutive behavior:

- Unidirectional:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{pmatrix} \frac{1}{134{,}000} & -\frac{0.25}{134{,}000} & 0 \\ -\frac{0.25}{134{,}000} & \frac{1}{7000} & 0 \\ 0 & 0 & \frac{1}{4200} \end{pmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

After inversion:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 134{,}440 & 1756 & 0 \\ 1756 & 7023 & 0 \\ 0 & 0 & 4200 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

- Mat:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{14{,}410} & -\frac{0.3}{14{,}410} & 0 \\ -\frac{0.3}{14{,}410} & \frac{1}{14{,}410} & 0 \\ 0 & 0 & \frac{2(1+0.3)}{14{,}410} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

After inversion:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 15{,}835 & 4750 & 0 \\ 4750 & 15{,}835 & 0 \\ 0 & 0 & 5542 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

2. Membrane behavior of the laminated plate:

$$A_{ij} = \sum_{\text{ply } n^\circ 1}^{n} \bar{E}_{ij}^{(k)} e^{(k)}$$

$$A_{11} = 134{,}440 \times 4 \times 0.13 + 15{,}835 \times 3 = 117{,}408 \text{ (MPa mm)}$$

$$A_{22} = 7023 \times 4 \times 0.13 + 15{,}835 \times 3 = 51{,}151 \text{ (MPa mm)}$$

$$A_{12} = 1756 \times 4 \times 0.13 + 4{,}750 \times 3 = 15{,}163 \text{ (MPa mm)}$$

$$A_{13} = A_{23} = 0$$

$$\mathrm{A}_{33} = 4200 \times 4 \times 0.13 + 5542 \times 3 = 18{,}810 \text{ (MPa mm).}$$

From this, and with a total thickness of the plate of

$$h = 3 + 4 \times 0.13 = 3.52 \text{ mm}$$

we have

$$[A] = \begin{bmatrix} 117{,}408 & 15{,}163 & 0 \\ 15{,}163 & 51{,}151 & 0 \\ 0 & 0 & 18{,}810 \end{bmatrix};$$

$$h[A]^{-1} = \begin{bmatrix} \frac{1}{32{,}078} & -\frac{0.13}{13{,}975} & 0 \\ -\frac{0.3}{32{,}078} & \frac{1}{13{,}975} & 0 \\ 0 & 0 & \frac{1}{5344} \end{bmatrix}$$

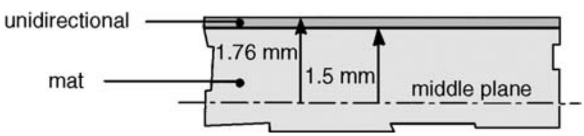
then for the moduli of elasticity of the plate:

$$\bar{E}_x = 31{,}078 \text{ MPa}; \quad \nu_{xy} = 0.3; \quad \bar{G}_{xy} = 5344 \text{ MPa}$$

$$\bar{E}_y = 13{,}975 \text{ MPa}; \quad \nu_{yx} = 0.13$$

3. Bending behavior of the laminated plate:

$$C_{ij} = \sum_{\text{ply } n^\circ 1}^{n} \bar{E}_{ij}^{(k)} \left(\frac{z_k^3 - z_{k-1}^3}{3} \right)$$



$$C_{11} = 134{,}440 \times \frac{1.76^3 - 1.5^3}{3} \times 2 + 15{,}835 \times \frac{1.5^3}{3} \times 2 = 221{,}763 \text{ MPa} \times \text{mm}^3$$

$$C_{22} = 7023 \times \frac{1.76^3 - 1.5^3}{3} \times 2 + 15{,}835 \times \frac{1.5^3}{3} \times 2 = 45{,}352 \text{ MPa} \times \text{mm}^3$$

$$C_{12} = 1756 \times \frac{1.76^3 - 1.5^3}{3} \times 2 + 4750 \times \frac{1.5^3}{3} \times 2 = 13{,}119 \text{ MPa} \times \text{mm}^3$$

$$C_{13} = C_{23} = 0$$

$$C_{33} = 4200 \times \frac{1.76^3 - 1.5^3}{3} \times 2 + 5542 \times \frac{1.5^3}{3} \times 2 = 18{,}284 \text{ MPa} \times \text{mm}^3$$

from which (see Section 12.1.6):

$$[C]^{-1} = \begin{bmatrix} \frac{1}{217{,}968} & -\frac{1}{753{,}509} & 0 \\ -\frac{1}{753{,}509} & \frac{1}{44{,}576} & 0 \\ 0 & 0 & \frac{1}{18{,}284} \end{bmatrix} = \begin{bmatrix} \frac{1}{\overline{\overline{EI}}_{11}} & \frac{1}{\overline{\overline{EI}}_{12}} & 0 \\ \frac{1}{\overline{\overline{EI}}_{21}} & \frac{1}{\overline{\overline{EI}}_{22}} & 0 \\ 0 & 0 & \frac{1}{\overline{\overline{EI}}_{33}} \end{bmatrix}$$

Apparent bending modulus in the x direction:

$$\frac{1}{\overline{\overline{EI}}_{11}} = \frac{1}{E_{fx} \times \frac{h^3}{12}} \rightarrow E_{fx} = 59{,}972 \text{ MPa}$$

The apparent bending modulus in the y direction:

$$\frac{1}{\overline{\overline{EI}}_{22}} = \frac{1}{E_{fy} \times \frac{h^3}{12}} \rightarrow E_{fy} = 12{,}264 \text{ MPa}$$

4. Rupture: For a stress resultant N_x, the plate is deformed in its plane according to the relation:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix}$$

then with the values found for $[A]^{-1}$:

$$\varepsilon_x = 8.856 \times 10^{-6} \times N_x; \quad \varepsilon_y = -2.66 \times 10^{-6} \times N_x; \quad \gamma_{xy} = 0$$

One then has for the stresses:

- In the unidirectional layer:

$$\sigma_\ell = \sigma_x = 134{,}440\ \varepsilon_x + 1756\ \varepsilon_y = 1.183\ N_x$$

$$\sigma_t = \sigma_y = 1756\ \varepsilon_x + 7023\ \varepsilon_y = -0.003\ N_x$$

$$\tau_{\ell t} = \tau_{xy} = 0.$$

- In the mat layer:

$$\sigma_x = 15{,}835\ \varepsilon_x + 4750\ \varepsilon_y = 0.128\ N_x$$

$$\sigma_y = 4750\ \varepsilon_x + 15{,}835\ \varepsilon_y = 5.5 \times 10^{-5} \times N_x$$

$$\tau_{xy} = 0$$

From which the failure criteria are (see Section 14.2.3)

- In the unidirectional layer:

$$\frac{(1.183 N_x)^2}{1270^2} + \frac{(-0.003 N_x)^2}{141^2} - \frac{1.183 \times -0.003 N_x^2}{1270^2} < 1$$

Failure will not occur when: $N_x < 1072$ N/mm.

- In the mat layer:

$$\frac{(0.128 N_x)^2}{100^2} + \frac{(5.5\ E\text{-}5 \times N_x)^2}{100^2} - \frac{0.128 \times 5.5\ E\text{-}5 \times N_x^2}{100^2} < 1$$

Failure will not occur when $N_x < 781$ N/mm.

Failure will first occur in the mat layer (first-ply rupture). The mat is supposed to be completely broken. The stress resultant $N_x = 781$ N/mm leads to a state of uniaxial stress in the laminate such that:

$$\sigma_\ell = \sigma_x = \frac{N_x}{4 \times 0.13} = \frac{781}{0.52} = 1502\ \text{MPa} > \sigma_{\ell\ \text{rupture}}$$

The fibers in the unidirectional layer are broken.
Conclusion: The first-ply failure leads to ultimate rupture of the laminate.

18.2.17 Thermoelastic Behavior of a Balanced Fabric Ply

Problem Statement:

Consider a layer of balanced fabric made of carbon/epoxy ($V_f = 60\%$). The configuration of a unit cell ($a \times a$) is shown in Figure 18.11. One considers the layer of fabric as equivalent to two layers, each with a thickness e.

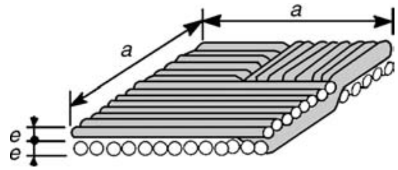


Figure 18.11

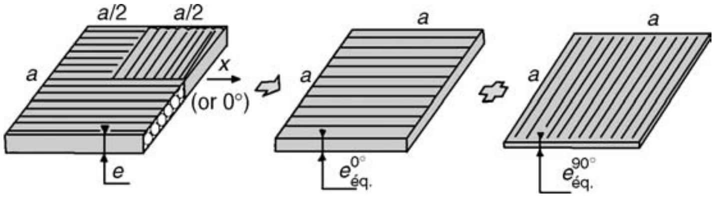


Figure 18.12

First part: Upper layer

We study the upper layer as shown schematically in Figure 18.12. Assume that this upper layer behaves as if it consists of two equivalent unidirectional layers ($a \times a$) crossed at 0° and 90°. These layers have equivalent thicknesses denoted respectively as:

$$e^{0°}_{\text{equi}} \quad \text{and} \quad e^{90°}_{\text{equi}}$$

1. Show that $e^{0°}_{\text{equi}} = \frac{3}{4}e$; $\quad e^{90°}_{\text{equi}} = \frac{1}{4}e$
2. Deduce from the above the stiffness matrix $\frac{1}{h}[A]$ of this upper layer made up of the two previous unidirectionals, with the values of the moduli and Poisson coefficients for the unidirectional indicated in Section 3.3.3.
3. Deduce from the above the moduli of elasticity and Poisson coefficients of this upper layer, denoted as E_x, E_y, G_{xy}, ν_{xy}.[46]
4. The coefficients of thermal expansion of the unidirectional are denoted as α_ℓ and α_t (see values in Section 3.3.3). What are the values of the coefficients of thermal expansion α_{ox}, α_{oy}, α_{oxy} of this layer? (One will at first calculate the terms denoted as $<\alpha E h>_i$ of Section 12.1.7).

[46] Note here that:

$$\frac{1}{h}[A] \# \begin{bmatrix} E_x & \nu_{yx}E_x & 0 \\ \nu_{xy}E_y & E_y & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}$$

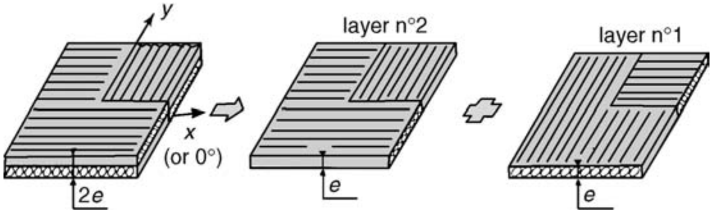


Figure 18.13

Second part: Complete fabric layer

Now we consider the complete fabric ply (thickness $2e$, see Figure 18.11) as the result of a simple superposition of two layers like the one that was studied in the previous part, these two layers being crossed at 0° (upper layer no. 2) and at 90° (lower layer no. 1).

One retains in the following $e = 0.14$ mm.

1. Write numerically with the previous results the in-plane constitutive behavior for layer no. 2, then for layer no. 1 in Figure 18.13 in the form $\{\sigma\} = [\bar{E}]\{\varepsilon\}$.
2. Calculate the coefficients $\overline{\alpha E_i}$ (see Section 11.3.2) of layer no. 2, then of layer no. 1.
3. Calculate the matrix [A] characterizing the in-plane behavior of the double layer in Figure 18.13 (layer no. 1 + layer no. 2).

Third part: (Independent of the two previous parts until Question 9)

We consider a laminate which consists of two orthotropic plies noted as 2 and 1, each with a thickness e, crossed at 0° (or x) and at 90°, respectively. We give below the respective thermomechanical behavior of these layers in axes x and y, which are written as:

Ply no. 1 (lower ply):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_1 = \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \Delta T \begin{Bmatrix} f \\ g \\ 0 \end{Bmatrix}$$

Ply no. 2 (upper ply):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_2 = \begin{bmatrix} b & c & 0 \\ c & a & 0 \\ 0 & 0 & d \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \Delta T \begin{Bmatrix} g \\ f \\ 0 \end{Bmatrix}$$

Recalling that the thermomechanical behavior of a laminate is written as:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \\ M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ -\dfrac{\partial^2 w_o}{\partial x^2} \\ -\dfrac{\partial^2 w_o}{\partial y^2} \\ -2\dfrac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} - \Delta T \begin{bmatrix} \langle \alpha Eh \rangle_x \\ \langle \alpha Eh \rangle_y \\ \langle \alpha Eh \rangle_{xy} \\ \langle \alpha Eh^2 \rangle_x \\ \langle \alpha Eh^2 \rangle_y \\ \langle \alpha Eh^2 \rangle_{xy} \end{bmatrix}$$

1. Write the literal expression of matrix [A].
2. Write the literal expression of matrix [C].
3. Write the literal expression of matrix [B].
4. Calculate the terms $<\alpha Eh>_x$, $<\alpha Eh>_y$, $<\alpha Eh>_{xy}$, $<\alpha Eh^2>_x$, $<\alpha Eh^2>_y$, $<\alpha Eh^2>_{xy}$.
5. Write the thermomechanical behavior equation.
6. This plate is not externally loaded. It is subjected to a variation in temperature ΔT. Deduce from item 5 the corresponding system of equations.
7. Give the values of γ_{oxy} and $\dfrac{\partial^2 w_o}{\partial x \partial y}$.
8. Write the equations that allow the calculation of other strains.
9. Taking into account the results obtained in the second part, write numerically this system of equations with $\Delta T = -160°C$. Give the corresponding values of strains. Comment.

Solution:

1.1. Volume of fibers at 0°:

$$v^{0°} = \frac{3a^2}{4} \times e = a^2 \times e^{0°}_{\text{équiv.}}$$

Volume of fibers at 90°:

$$v^{90°} = \frac{a^2}{4} \times e = a^2 \times e^{90°}_{\text{équiv.}}$$

from which:

$$e^{0°}_{\text{équiv.}} = \frac{3e}{4}; \qquad e^{90°}_{\text{équiv.}} = \frac{e}{4}$$

1.2. Stiffness matrix $\frac{1}{h}[A]$: According to the Equation 11.8 and the values in Section 3.3.3:

$$\bar{E}_{11}^{0°} = \bar{E}_\ell = 134{,}439 \text{ MPa}; \quad \bar{E}_{12}^{0°} = \nu_{t\ell}\, \bar{E}_\ell = 1756 \text{ MPa}$$

$$\bar{E}_{22}^{0°} = \bar{E}_t = 7023 \text{ MPa}; \quad \bar{E}_{33}^{0°} = G_{\ell t} = 4200 \text{ MPa}$$

$$\bar{E}_{11}^{90°} = 7023 \text{ MPa}; \quad \bar{E}_{12}^{90°} = 1756 \text{ MPa}; \quad \bar{E}_{22}^{90°} = 134{,}439 \text{ MPa}; \quad \bar{E}_{33}^{90°} = 4200 \text{MPa}$$

$$A_{11} = \bar{E}_{11}^{0°} \times \frac{3e}{4} + \bar{E}_{11}^{90°} \times \frac{e}{4} = 102{,}585 \times e \text{ (MPa.mm)}$$

$$A_{22} = \bar{E}_{22}^{0°} \times \frac{3e}{4} + \bar{E}_{22}^{90°} \times \frac{e}{4} = 38{,}877 \times e \text{ (MPa.mm)}$$

$$A_{12} = 1756 \text{ MPa}; \quad A_{33} = 4200 \times e \text{ (MPa.mm)}$$

$$\frac{1}{h}[A] = \begin{bmatrix} 102{,}585 & 1756 & 0 \\ 1756 & 38{,}877 & 0 \\ 0 & & 4200 \end{bmatrix} \text{(MPa)}$$

1.3. One has, according to Equation 12.9:

$$h[A]^{-1} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix}$$

from which:

$$E_x = 102{,}506 \text{ MPa}$$

$$\text{E}_y = 38{,}847 \text{ MPa}$$

$$\nu_{yx} = 0.017; \; \nu_{xy} = 0.045$$

$$G_{xy} = 4200 \text{ MPa}$$

One then can verify that:

$$\frac{1}{h}[A] \# \begin{bmatrix} E_x & \nu_{yx} E_x & 0 \\ \nu_{xy} E_y & E_y & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}$$

1.4. One has (Equation 12.18):

$$\begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \frac{1}{h}\langle \alpha E h \rangle_x \\ \frac{1}{h}\langle \alpha E h \rangle_y \\ \frac{1}{h}\langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

With (Equations 12.17 and 11.10):

$$\langle \alpha E h \rangle_x = \overline{\alpha E_1}^{0°} \times \frac{3}{4}e + \overline{\alpha E_1}^{90°} \times \frac{e}{4} = \cdots$$

$$\cdots \bar{E}_\ell(\alpha_\ell + \nu_{t\ell}\alpha_t) \times \frac{3}{4}e + \bar{E}_t(\nu_{\ell t}\alpha_\ell + \alpha_t) \times \frac{e}{4}$$

with (Section 3.3.3): $\alpha_\ell = -0.12 \times 10^{-5}$; $\alpha_t = 3.4 \times 10^{-5}$.

$$\frac{1}{h}\langle \alpha E h \rangle_x = -1726 \times 10^{-5}. \text{ Then:}$$

$$\frac{1}{h}\langle \alpha E h \rangle_y = 15{,}203 \times 10^{-5}; \quad \frac{1}{h}\langle \alpha E h \rangle_{xy} = 0$$

One then deduces:

$$\alpha_{ox} = -2.3 \times 10^{-7}; \quad \alpha_{oy} = 39 \times 10^{-7}; \quad \alpha_{oxy} = 0$$

2.1. Constitutive behavior: $\{\sigma\} = [\bar{E}]\{\varepsilon\}$: According to Equation 11.8

Layer no. 2:

$$\bar{E}_{11}^{(2)} = \bar{E}_x = \frac{E_x}{1 - \nu_{yx}\ \nu_{xy}} = 102{,}584 \text{ MPa etc.}$$

$$[\bar{E}]^{(2)} = \begin{bmatrix} 102{,}584 & 1744 & 0 \\ 1744 & 38{,}877 & 0 \\ 0 & 0 & 4200 \end{bmatrix}$$

Layer no. 1:

$$[\bar{E}]^{(1)} = \begin{bmatrix} 38{,}877 & 1744 & 0 \\ 1744 & 102{,}584 & 0 \\ 0 & 0 & 4200 \end{bmatrix}$$

2.2. Coefficients $\overline{\alpha E_i}$:

Layer no. 2:

$$\overline{\alpha E}_1^{(2)} = \bar{E}_x(\alpha_{ox} + \nu_{yx}\alpha_{oy}) = -0.0168$$
$$\overline{\alpha E}_2^{(2)} = 0.1512; \quad \overline{\alpha E}_3^{(2)} = 0$$

Layer no. 1 (rotation of 90°):

$$\overline{\alpha E}_1^{(1)} = 0.1512; \quad \overline{\alpha E}_2^{(1)} = -0.0168; \quad \overline{\alpha E}_3^{(1)} = 0$$

2.3. In-plane behavior of the double layer:

$$A_{11} = \bar{E}_{11}^{(1)} \times e + \bar{E}_{11}^{(2)} \times e = (102{,}584 + 38{,}877) \times 0.14, \text{etc.}$$

$$[A] = \begin{bmatrix} 19{,}804 & 488 & 0 \\ 488 & 19{,}804 & 0 \\ 0 & 0 & 1176 \end{bmatrix} (\text{MPa. mm})$$

3.1. Matrix [A]:

$$[A] = \begin{bmatrix} (a+b)e & 2ce & 0 \\ 2ce & (a+b)e & 0 \\ 0 & 0 & 2de \end{bmatrix}$$

3.2. Matrix [C]:

$$C_{11} = a\left(\frac{0-(-e)^3}{3}\right) + b\left(\frac{e^3 - 0}{3}\right) = (a+b)\frac{e^3}{3}, \text{etc.}$$

$$[C] = \begin{bmatrix} (a+b)\frac{e^3}{3} & 2c\frac{e^3}{3} & 0 \\ 2c\frac{e^3}{3} & (a+b)\frac{e^3}{3} & 0 \\ 0 & 0 & 2d\frac{e^3}{3} \end{bmatrix}$$

3.3. Matrix [B]:

$$B_{11} = a\left(\frac{0-(-e)^2}{2}\right) + b\left(\frac{e^2-0}{2}\right) = (b-a)\frac{e^2}{2}, \text{ etc.}$$

$$[B] = \begin{bmatrix} (b-a)\frac{e^2}{2} & 0 & 0 \\ 0 & (a-b)\frac{e^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.4. Terms $<\alpha Eh>_i$ and $<\alpha Eh^2>_i$:

$$\langle \alpha Eh \rangle_x = fe + ge = (f+g)e$$

$$\langle \alpha Eh \rangle_y = (f+g)e; \quad (\alpha Eh)_{xy} = 0$$

$$\langle \alpha Eh^2 \rangle_x = (g-f)\frac{e^2}{2}$$

$$\langle \alpha Eh^2 \rangle_y = (f-g)\frac{e^2}{2}; \quad (\alpha Eh^2)_{xy} = 0$$

3.5. Thermomechanical behavior:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \\ M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = \begin{bmatrix} (a+b)e & 2ce & 0 & (b-a)\frac{e^2}{2} & 0 & 0 \\ 2ce & (a+b)e & 0 & 0 & (a-b)\frac{e^2}{2} & 0 \\ 0 & 0 & 2de & 0 & 0 & 0 \\ (b-a)\frac{e^2}{2} & 0 & 0 & (a+b)\frac{e^3}{3} & 2c\frac{e^3}{3} & 0 \\ 0 & (a-b)\frac{e^2}{2} & 0 & 2c\frac{e^3}{3} & (a+b)\frac{e^3}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2d\frac{e^3}{3} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ -\frac{\partial^2 w_o}{\partial x^2} \\ -\frac{\partial^2 w_o}{\partial y^2} \\ -2\frac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \cdots$$

$$\cdots - \Delta T \begin{Bmatrix} (f+g)e \\ (f+g)e \\ 0 \\ (g-f)\frac{e^2}{2} \\ (f-g)\frac{e^2}{2} \\ 0 \end{Bmatrix}$$

3.6. Variation of temperature ΔT:
One has here:

$$N_x = N_y = T_{xy} = M_x = M_y = M_{xy} = 0$$

from which we have

$$\begin{bmatrix} (a+b)e & 2ce & 0 & (b-a)\frac{e^2}{2} & 0 & 0 \\ 2ce & (a+b)e & 0 & 0 & (a-b)\frac{e^2}{2} & 0 \\ 0 & 0 & 2de & 0 & 0 & 0 \\ (b-a)\frac{e^2}{2} & 0 & 0 & (a+b)\frac{e^3}{3} & 2c\frac{e^3}{3} & 0 \\ 0 & (a-b)\frac{e^2}{2} & 0 & 2c\frac{e^3}{3} & (a+b)\frac{e^3}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2d\frac{e^3}{3} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ -\frac{\partial^2 w_o}{\partial x^2} \\ -\frac{\partial^2 w_o}{\partial y^2} \\ -2\frac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} = \Delta T \begin{Bmatrix} (f+g)e \\ (f+g)e \\ 0 \\ (g-f)\frac{e^2}{2} \\ (f-g)\frac{e^2}{2} \\ 0 \end{Bmatrix}$$

3.7. One can note that:

$$\gamma_{oxy} = 0; \quad \frac{\partial^2 w_o}{\partial xdy} = 0$$

3.8. There remains

$$\begin{bmatrix} (a+b)e & 2ce & (b-a)\frac{e^2}{2} & 0 \\ 2ce & (a+b)e & 0 & (a-b)\frac{e^2}{2} \\ (b-a)\frac{e^2}{2} & 0 & (a+b)\frac{e^3}{3} & 2c\frac{e^3}{3} \\ 0 & (a-b)\frac{e^2}{2} & 2c\frac{e^3}{3} & (a+b)\frac{e^3}{3} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ -\frac{\partial^2 w_o}{\partial x^2} \\ -\frac{\partial^2 w_o}{\partial^2 y^2} \end{Bmatrix} = \Delta T \begin{Bmatrix} (f+g)e \\ (f+g)e \\ (g-f)\frac{e^2}{2} \\ (f-g)\frac{e^2}{2} \end{Bmatrix}$$

Remark: According to the model studied, one truly must have

$$\varepsilon_{ox} = \varepsilon_{oy}; \quad \frac{\partial^2 w_o}{\partial x^2} = -\frac{\partial^2 w_o}{\partial y^2}$$

It is worthy to note that with this hypothesis one obtains two identical systems of equations which are written as:

$$\begin{bmatrix} (a+b+2c)e & (b-a)\frac{e^2}{2} \\ (b-a)\frac{e^2}{2} & (a+b-2c)\frac{e^3}{3} \end{bmatrix} \begin{bmatrix} \varepsilon_{ox} \\ -\frac{\partial^2 w_o}{\partial x^2} \end{bmatrix} = \Delta T \begin{bmatrix} (f+g)e \\ (g-f)\frac{e^2}{2} \end{bmatrix}$$

3.9. With the results of the second part, and $\Delta T = -160$°C (corresponding to the cooling in the autoclave after the polymerization of the resin), one has (units: N and mm):

$$(a+b)e = 19{,}804; \quad 2ce = 488; \quad (a+b)\frac{e^3}{3} = 129; \quad 2c\frac{e^3}{3} = 3.2$$

$$(b-a)\frac{e^2}{2} = 624; \quad (f+g)e = 0.0188; \quad (g-f)\frac{e^2}{2} = -0.00164$$

from which we obtain the strains and curvatures:

$$\varepsilon_{ox} = \varepsilon_{oy} = -1.7 \times 10^{-4}$$

$$\frac{\partial^2 w_o}{\partial x^2} = -\frac{\partial^2 w_o}{\partial y^2} = -8.6 \times 10^{-4}$$

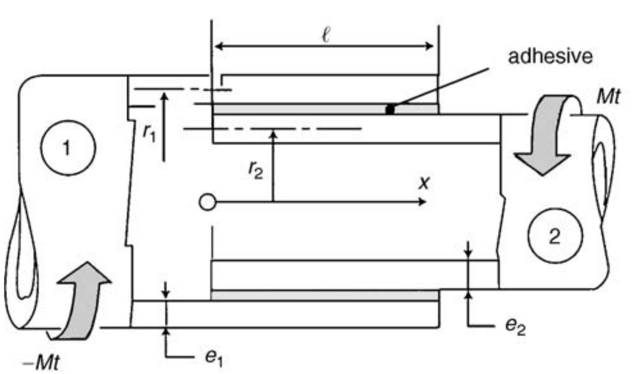
We can conclude that during the cooling the layer of balanced fabric not only contracts but also, due to its weave, takes the form of a double curvature surface along the warp and fill directions; that is, the form of a horse saddle.

18.3 LEVEL 3

18.3.1 Cylindrical Bonding

Problem Statement:

We propose to study, in a simplified approach, a bonded assembly of two cylindrical tubes (figure below). The shear moduli of the materials are denoted along with the figure:

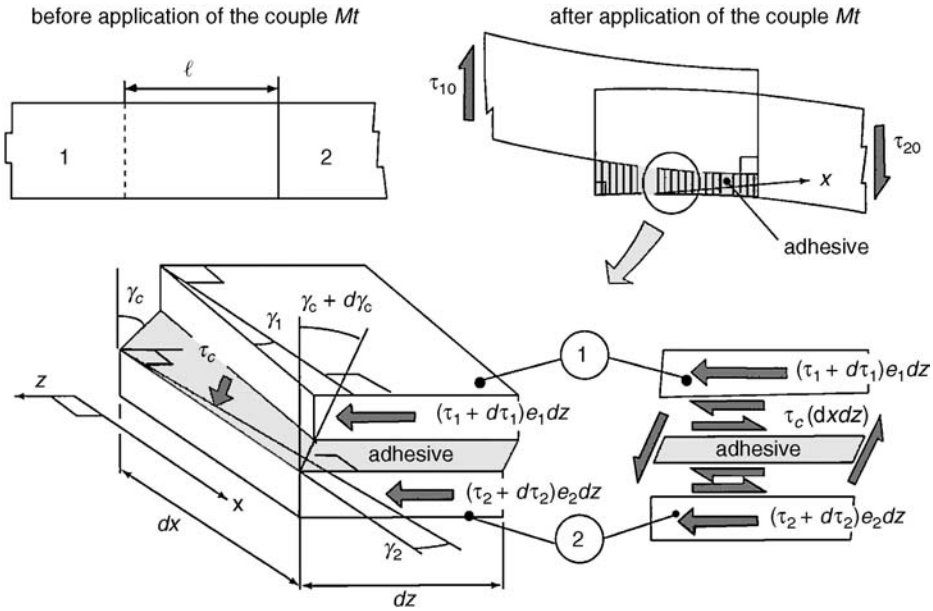


G_1 for material 1

G_2 for material 2

G_c for adhesive

The deformed configuration of the generator of each of the tubes viewed from above is shown in the following figure, with the shear stresses τ_{10} and τ_{20} that are assumed to be uniform across the thickness of each tube. Also shown is the bonding element.



1. Find the distribution of the shear stresses in the adhesive layer, denoted as τ_C in the previous figure.
2. Numerical application:

$G_1 = 28{,}430$ MPa;	$G_2 = 79{,}000$ MPa;	$G_C = 1700$ MPa;
$e_1 = e_2 = 12$ mm;	$e_C = 0.2$ mm;	$M_t = 300$ m.daN;
$r_1 = 63.5$ mm;	$r_2 = 51.5$ mm;	$\ell = 44$ mm.

3. Calculate the maximum shear stress in the particular case where the materials 1 and 2 are identical and have the same thickness, denoted as e, which is small compared with the radii.

Solution:

1. Shear stresses in the adhesive layer: In the previous figure that represents the bonding element, one reads the following equilibrium:
   - Equilibrium of material element 1:

$$d\tau_1 e_1\, dz + \tau_c\, dx\, dz = 0 \rightarrow \frac{d\tau_1}{dx} e_1 + \tau_c = 0 \qquad \text{[a]}$$

- Equilibrium of material element 2:

$$d\tau_2 e_2\, dz - \tau_c\, dx\, dz = 0 \rightarrow \frac{d\tau_2}{dx} e_2 - \tau_c = 0 \qquad \text{[b]}$$

The shear stresses are proportional to the angular distortions, denoted here as γ_1 for material 1, γ_2 for material 2, and γ_C for the adhesive, from which:

$$\gamma_1 = \frac{\tau_1}{G_1}; \quad \gamma_2 = \frac{\tau_2}{G_2}; \quad \gamma_c = \frac{\tau_c}{G_c}$$

In addition one has the following geometric relation, by approximating the tangents and angles (tg $\theta \cong \theta$; see figure):

$$(\gamma_c + d\gamma_c) - \gamma_c \; \# \; \frac{-\gamma_1\, dx + \gamma_2\, dx}{e_c}$$

as:

$$\frac{d\gamma_c}{dx} = \frac{\gamma_2 - \gamma_1}{e_c}$$

In substituting the stresses:

$$\frac{d\tau_c}{dx}\frac{e_c}{G_c} = \frac{\tau_2}{G_2} - \frac{\tau_1}{G_1} \qquad \text{[c]}$$

One then obtains the 3 relations [a], [b], [c], from the unknowns τ_1, τ_2, τ_C.

Eliminating τ_1 and τ_2 yields

$$\frac{d^2\tau_c}{dx^2}\frac{e_c}{G_c} = \frac{\tau_c}{e_2 G_2} + \frac{\tau_c}{e_1 G_1}$$

then:

$$\boxed{\frac{d^2\tau_c}{dx^2} - \lambda^2 \tau_c = 0 \quad \text{with} \quad \lambda^2 = \frac{G_c}{e_c}\left(\frac{1}{e_2 G_2} + \frac{1}{e_1 G_1}\right)}$$

The general solution for the above differential equation is:

$$\tau_C = \mathrm{A\ ch\ }\lambda \mathrm{x} + \mathrm{B\ sh\ }\lambda \mathrm{x}.$$

- Boundary conditions:
  For $x = 0$: It is the free edge of material 2, where $\gamma_2 = 0$ and $\gamma_1 = \tau_{10}/G_1$.

from which: $$\left.\frac{d\gamma_c}{dx}\right|_{x=0} = \frac{\gamma_2 - \gamma_1}{e_c} = -\frac{\tau_{10}}{e_c G_1}$$

then: $$\left.\frac{d\tau_c}{dx}\right|_{x=0} = -\frac{\tau_{10} G_c}{e_c G_1} \qquad \text{[d]}$$

For $x = \ell$: It is the free edge of material 1, where $\gamma_1 = 0$ and $\gamma_2 = \tau_{20}/G_2$

$$\text{from which:} \quad \left.\frac{d\gamma_c}{dx}\right|_{x=\ell} = \frac{\gamma_2 - \gamma_1}{e_c} = \frac{\tau_{20}}{e_c G_2}$$

$$\text{then:} \quad \left.\frac{d\tau_c}{dx}\right|_{x=\ell} = \frac{\tau_{20} G_c}{e_c G_2} \qquad \text{[e]}$$

The boundary conditions [d] and [e] allow the calculation of the constants A and B of the general solution. We obtain

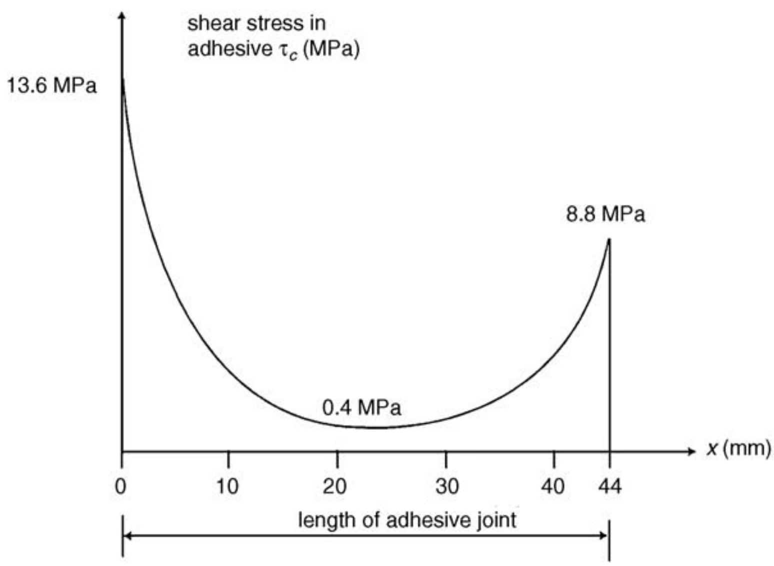
$$\boxed{\tau_c = \frac{G_c}{e_c \lambda}\left\{\left(\frac{\tau_{10}}{G_1}\frac{1}{\text{th}\ \lambda\ell} + \frac{\tau_{20}}{G_2}\frac{1}{\text{sh}\ \lambda\ell}\right)\text{ch}\ \lambda x - \frac{\tau_{10}}{G_1}\text{sh}\ \lambda x\right\}}$$

2. Numerical application:

$$\tau_{10} = \frac{M_t}{2\pi r_1^2 e_1} = 9.86 \ \text{MPa};$$

$$\tau_{20} = \frac{M_t}{2\pi r_2^2 e_2} = 15 \ \text{MPa}$$

One obtains for the shear stress τ_C the following distribution, where the stress concentrations at the extremities of the assembly can be noted.



This explains that one should not design such a bonding assembly by basing on the average shear stress, which does not exist in reality.

Note: The proposed numerical values correspond here to those of Application 18.1.4 relative to the design of a transmission shaft in carbon/epoxy. One can note that the rupture strength of araldite, taken to be 15 MPa, is not effectively reached at the location of stress concentrations.

3. Particular case:

$$G_1 = G_2 = G; \quad e_1 = e_2 = e; \quad e/r_1 \# e/r_2.$$

The comparison:

$$\tau_{10} = \frac{M_t}{2\pi r_1^2 e} \quad \text{and} \quad \tau_{20} = \frac{M_t}{2\pi r_2^2 e}$$

allows one to write approximately:

$$\tau_{10} \# \tau_{20}$$

from which:

$$\tau_c = \frac{G_c}{\lambda e_c G}\tau_o\left\{\left(\frac{1}{\text{th } \lambda\ell} + \frac{1}{\text{sh } \lambda\ell}\right)\text{ch } \lambda x - \text{sh } \lambda x\right\}$$

One notes the presence of peaks of identical stress at $x = 0$ and $x = \ell$ as:

$$\tau_{c\ \max} = \frac{G_c}{\lambda e_c G}\tau_o\frac{\text{ch } \lambda\ell + 1}{\text{sh } \lambda\ell} = \frac{G_c}{\lambda e_c G}\tau_o\frac{1}{\text{th } \frac{\lambda\ell}{2}}$$

Taking into account that:

$$\lambda^2 = \frac{2G_c}{e_c G e}:$$

$$\tau_{c\ \max} = \tau_o\frac{\lambda^2 e}{2\lambda}\frac{1}{\text{th } \frac{\lambda\ell}{2}} = \tau_o e\frac{\lambda/2}{\text{th } \lambda\ell/2}$$

reveals the average stress in the adhesive (fictitious notion as mentioned above):

$$\tau_{\text{average}} = \frac{M_t}{2\pi r^2 \ell} = \frac{M_t}{2\pi r^2 e}\frac{e}{\ell} = \tau_o\frac{e}{\ell}$$

from which:

$$\tau_{c\ \max} = \tau_{\text{average}}\frac{\lambda\ell/2}{\text{th } \lambda\ell/2}$$

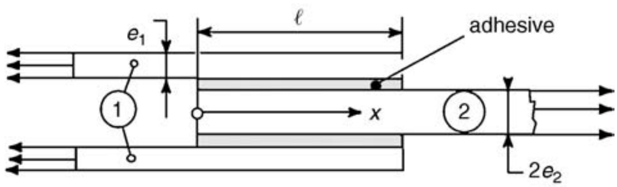
In setting $\lambda\ell/2 = a$, one finds again the relation of Section 6.2.3:

$$\boxed{\tau_{max} = \frac{a}{\text{th } a} \times \tau_{average}; \qquad \text{with} \quad a = \sqrt{\frac{G_c \ell^2}{2\text{Gee}_c}}}$$

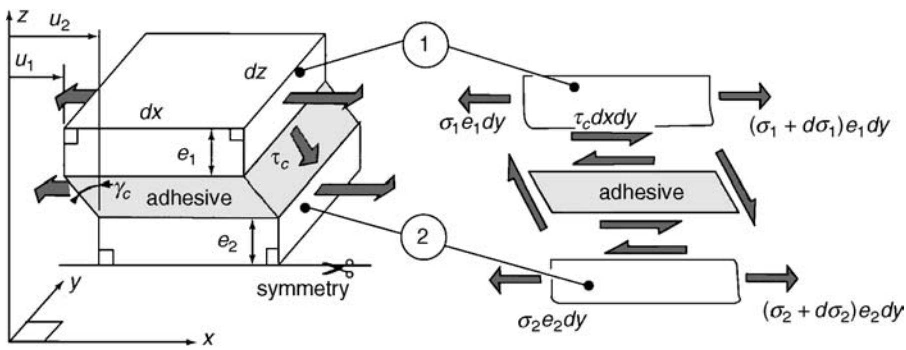
18.3.2 Double Bonded Joint

Problem Statement:

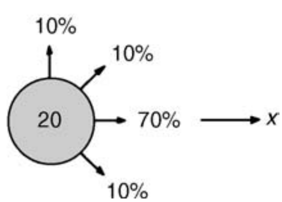
Shown below is an assembly consisting of two identical plates of material 1 bonded to a central plate of material 2. This joint provides a plane of symmetry ($x - y$). We will study approximately the shear stress in the adhesive. For that, assume that the stresses are just functions of x.



The configuration of a bonding element of length dx is shown below. The moduli of the materials are denoted as: E_1 for material 1, E_2 for material 2, G_C for the adhesive.



1. Determine the distribution of shear stresses in the adhesive, denoted as $\tau_C(x)$.
2. Numerical application: The two external plates are made of titanium alloy (TA 6V), with thickness 1.5 mm. The intermediate plate is a laminate of carbon/epoxy, with V_f = 60% fiber volume fraction and the following composition:



The thickness of one ply is 0.125 mm. The rupture strength of the adhesive (araldite) is taken to be 15 MPa. Its thickness is 0.2 mm. What length of bond ℓ will allow the bonding assembly to transmit a stress resultant of 20 daN/mm of width?

3. Calculate the maximum shear stress in the particular case where the materials 1 and 2 are identical and where $e_1 = e_2 = e$.

Solution:

1. Shear stress in the adhesive: In the previous figure showing an element of the bond, one reads the following equilibrium:
   - Equilibrium of element of material 1:

$$d\sigma_1 e_1\,dy + \tau_c\,dx\,dy = 0 \rightarrow \frac{d\sigma_1}{dx}e_1 + \tau_c = 0 \qquad [a]$$

   - Equilibrium of element of material 2:

$$d\sigma_2 e_2\,dy - \tau_c\,dx\,dy = 0 \rightarrow \frac{d\sigma_2}{dx}e_2 - \tau_c = 0 \qquad [b]$$

In addition, one also has the following geometric relation in approximating the tangents and angles:

$$\gamma_c \;\#\; \frac{u_2 - u_1}{e_c}$$

then with the constitutive relations:

$$\gamma_c = \frac{\tau_c}{G_c}; \quad \frac{du_1}{dx} = \frac{1}{E_1}\sigma_1; \quad \frac{du_2}{dx} = \frac{1}{E_2}\sigma_2$$

$$\frac{\tau_c}{G_c} \;\#\; \frac{u_2 - u_1}{e_c}$$

$$\frac{e_c}{G_c}\frac{d\tau_c}{dx} = \frac{\sigma_2}{E_2} - \frac{\sigma_1}{E_1} \qquad [c]$$

One obtains three relations [a], [b], [c] for the three unknowns σ_1, σ_2, τ_C. One can write:

$$\frac{1}{E_1}\frac{d\sigma_1}{dx} = -\frac{\tau_c}{e_1E_1}; \quad \frac{1}{E_2}\frac{d\sigma_2}{dx} = \frac{\tau_c}{e_2E_2}$$

$$\frac{1}{E_1}\frac{d\sigma_1}{dx} - \frac{1}{E_2}\frac{d\sigma_2}{dx} = -\tau_c\left(\frac{1}{e_1E_1} + \frac{1}{e_2E_2}\right)$$

Taking into account the relation [c]:

$$\frac{d^2}{dx^2}\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right) = \frac{G_c}{e_c}\left(\frac{1}{e_1E_1}+\frac{1}{e_2E_2}\right)\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right)$$

$$\boxed{\frac{d^2}{dx^2}\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right)-\lambda^2\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right) = 0; \quad \text{with} \quad \lambda^2 = \frac{G_c}{e_c}\left(\frac{1}{e_1E_1}+\frac{1}{e_2E_2}\right)}$$

The solution of the differential equation can be written as:

$$\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2} = A \ \text{ch} \ \lambda x + B \ \text{sh} \ \lambda x$$

- Boundary conditions:

$$\text{for } x = 0;\ \sigma_1 = \sigma_{10} \quad \text{and} \quad \sigma_2 = 0 \text{ then:} \left.\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right)\right|_{x=0} = \frac{\sigma_{10}}{E_1}$$

$$\text{for } x = \ell;\ \sigma_1 = 0 \quad \text{and} \quad \sigma_2 = \sigma_{20} \text{ then:} \left.\left(\frac{\sigma_1}{E_1}-\frac{\sigma_2}{E_2}\right)\right|_{x=\ell} = -\frac{\sigma_{20}}{E_2}$$

from which we can write the constant values:

$$A = \frac{\sigma_{10}}{E_1};\quad B = -\left(\frac{\sigma_{20}}{E_2 \ \text{sh} \ \lambda\ell}+\frac{\sigma_{10}}{E_1 \ \text{th} \ \lambda\ell}\right)$$

In addition (relation [a] + [b]):

$$\frac{d\sigma_1}{dx}e_1+\frac{d\sigma_2}{dx}e_2 = 0$$

$$\text{where:} \qquad \frac{d\sigma_1}{dx}\left[\frac{1}{E_1}+\frac{e_1}{e_2E_2}\right] = A\lambda \ \text{sh} \ \lambda x + B\lambda \ \text{ch} \ \lambda x$$

That is, according to [a]

$$-\tau_c\left[\frac{1}{e_1E_1}+\frac{1}{e_2E_2}\right] = A\lambda \ \text{sh} \ \lambda x + B\lambda \ \text{ch} \ \lambda x$$

$$\boxed{\tau_c = \frac{G_c}{e_c\lambda}\left\{\left(\frac{\sigma_{10}}{E_1}\frac{1}{\text{th} \ \lambda\ell}+\frac{\sigma_{20}}{E_2}\frac{1}{\text{sh} \ \lambda\ell}\right)\text{ch} \ \lambda x - \frac{\sigma_{10}}{E_1} \ \text{sh} \ \lambda x\right\}} \qquad \text{[d]}$$

Remarks:

- One obtains in this manner only an approximation for the shear stress τ_C. It should be possible to deduce directly from relations [a], [b], [c] a differential equation in τ_C. However, its integration will reveal at the limits $x = 0$ and $x = \ell$ the zero values of τ_C (free surface of the adhesive) making it impossible to obtain a nonzero solution. At the inverse, the expression found here for τ_C does not become zero for $x = 0$ and $x = \ell$. This contradicts with reality.

One can conclude from the above that the unidimensional approximation for the stresses σ_1, σ_2, τ_C is unwarranted. However, the form found here for τ_C gives an acceptable order of magnitude for this stress, except at the immediate vicinity of the free edge. Numerical modeling of the phenomenon (finite element method) shows in effect that the shear stress τ_C increases very rapidly from the free edge, up to a peak value very close to the value here. Apart from this particularity, there is a good correlation with the values given in relation [d].

- It also appears in the adhesive normal peel stresses that are confined to a peak zone close to the free edge. They constitute another factor that is not taken into account in this study.

2. Numerical application:
   Longitudinal modulus of titanium (see Section 1.6): E_1 = 105,000 MPa.
   Shear modulus of the adhesive (araldite): G_C = 1,700 MPa.
   Longitudinal modulus of the laminate: With the proportions of the previous plies along the directions 0°, 90°, ±45°, one finds (Table 5.4 in Section 5.4.2): E_2 = 100,590 MPa.
   Thickness of the laminate: $2e_2 = 20$ plies × 0.125 mm = 2.5 mm from which e_2 = 1.25 mm.

A stress resultant of 20 daN/mm corresponds to the stresses:

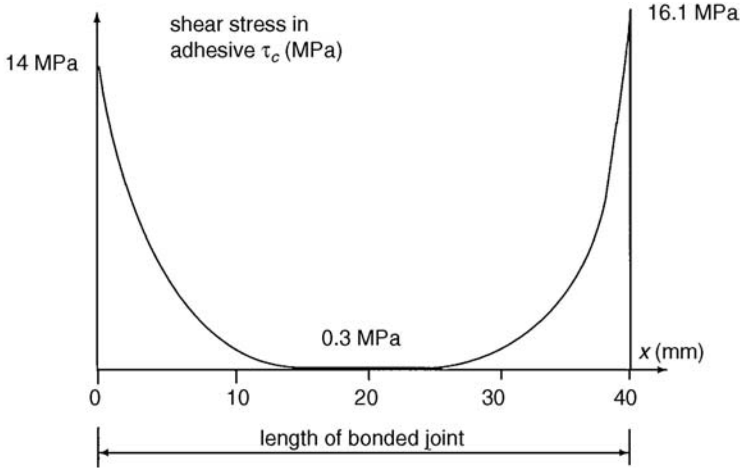
- In the titanium:

$$\sigma_{10} = \frac{200}{2 \times 1.5} = 66.66 \text{ MPa}$$

- In the laminate:

$$\sigma_{20} = \frac{200}{2.5} = 80 \text{ MPa}$$

A numerical calculation of expression [d], for example, with a programmable calculator, allows one to verify easily the rupture criterion of the adhesive for a length of ℓ = 40 mm, as shown in the following:



3. Particular case: The materials are identical: $e_1 = e_2 = e$. Then $\sigma_{10} = \sigma_{20} = \sigma_0$ and:

$$\tau_c = \frac{G_c}{\lambda e_c E}\sigma_0\left\{\left(\frac{1}{\text{th }\lambda\ell} + \frac{1}{\text{sh }\lambda\ell}\right)\text{ch }\lambda x - \text{sh }\lambda x\right\}$$

One notes identical peak values of stress for $x = 0$ or $x = \ell$ as:

$$\tau_{c\,\max} = \frac{G_c}{\lambda e_c E}\sigma_0\frac{\text{ch }\lambda\ell + 1}{\text{sh }\lambda\ell} = \frac{G_c}{\lambda e_c E}\sigma_0\frac{1}{\text{th }\frac{\lambda\ell}{2}}$$

Taking into account that:

$$\lambda^2 = \frac{2G_c}{e_c e E}$$

$$\tau_{c\,\max} = \sigma_0\frac{e\lambda^2}{2\lambda}\frac{1}{\text{th }\lambda\ell/2}$$

Introducing an average shear stress in the adhesive, which is a fictitious stress as one can consider in the previous figure:

$$\tau_{\text{average}} = \sigma_0\frac{e}{\ell}$$

then:

$$\tau_{c\,\max} = \frac{\lambda\ell/2}{\text{th }\lambda\ell/2}\tau_{\text{average}}$$

in posing:

$$\lambda\ell/2 = a$$

$$\boxed{\tau_{c\,\max} = \frac{a}{\text{th } a} \times \tau_{\text{average}}; \quad \text{with } a = \sqrt{\frac{G_c \ell^2}{2\text{Eee}_c}}}$$

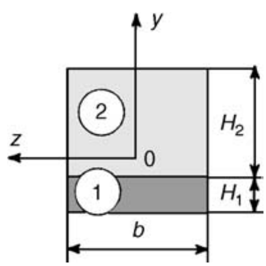
18.3.3 Composite Beam with Two Layers

Problem Statement:

A composite beam is made up of two different materials, denoted as 1 and 2, that are bonded together. The cross section of the beam is shown in the figure below. The thickness of the adhesive is neglected. The materials are isotropic and elastic. The longitudinal and shear moduli of the two materials are denoted as E_1, G_1 and E_2, G_2.

The elements that allow the study of the bending behavior of this beam in its plane of symmetry are summarized in Table 15.16.

1. Determine the location of the elastic center denoted as O.



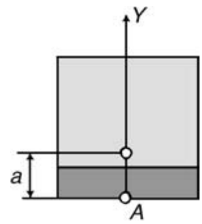
2. Write the expression for the equivalent stiffnesses (do not provide details for the shear coefficient k).
3. The shear force along the y direction for the considered section is denoted as T. Calculate the shear stress distribution τ_{xy}. Deduce from that the shear stress in the adhesive at the interface between the two materials.

Solution:

1. Elastic Center: This is determined such that:

$$\int_{\text{section}} E_i y \, dS = 0$$

(see Equation 15.16). Let A be an arbitrary origin defining ordinate denoted as Y. The point O to be found is such that:



$$\int_{\text{section}} E_i(Y-a)\,dS = 0$$

then:

$$\boxed{a = \frac{\int_{\text{section}} E_i Y\,dS}{\int_{\text{section}} E_i\,dS}}$$

$$\int_{\text{section}} E_i Y\ dS = \int_0^{H_1} E_1 Yb\ dY + \int_{H_1}^{H_1+H_2} E_2 Yb\ dY$$

One finds after calculation:

$$\boxed{a = \frac{1}{2}\left\{\frac{(E_1-E_2)H_1^2 + E_2(H_1+H_2)^2}{E_1H_1+E_2H_2}\right\}}$$

2. Equivalent stiffnesses:
   - Extensional stiffness:

$$\langle ES\rangle = \sum_i E_iS_i = b(E_1H_1+E_2H_2)$$

   - Shear stiffness:

$$\frac{\langle GS\rangle}{k} = \sum_i \frac{G_iS_i}{k} = \frac{b}{k}(G_1H_1+G_2H_2)$$

   - Bending stiffness:

$$\langle EI\rangle = \sum_i E_iI_i$$

$$\langle EI \rangle = bE_1 \int_{-a}^{H_1-a} y^2 dS + bE_2 \int_{H_1-a}^{H_1+H_2-a} y^2 dS$$

$$\langle EI \rangle = \frac{b}{3}\{E_1[(H_1-a)^3 + a^3] + E_2[(H_1+H_2-a)^3 - (H_1-a)^3]\}$$

3. Subject to a shear force T along the y direction, the shear stresses are assumed to be limited to the component τ_{xy}, given in the material "i" by the relation (see Equation 15.16):

$$\tau_{xy} = G_i \frac{T}{\langle GS \rangle} \frac{dg_{oi}}{dy}$$

in which $g_o(y)$ is the warping function due to shear and solution of the problem:

$$\begin{cases} \dfrac{d^2 g_o}{dy^2} = -\dfrac{E_i}{G_i}\dfrac{\langle GS \rangle}{\langle EI \rangle} y & \text{throughout the cross section} \\ \dfrac{dg_o}{dy} = 0 \text{ for } y = -a \text{ and } y = H_1 + H_2 - a & \text{(free boundaries)} \end{cases}$$

The uniqueness of the function $g_o(y)$ is assured by the condition:

$$\int_{\text{section}} E_i g_o \, dS = 0.$$

One finds in material 1:

$$\frac{dg_{o1}}{dy} = \frac{1}{2}\frac{E_1}{G_1}\frac{\langle GS \rangle}{\langle EI \rangle}(a^2 - y^2)$$

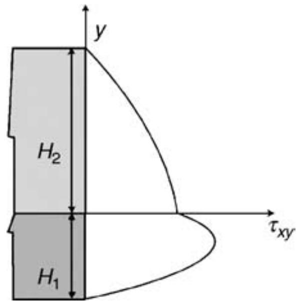
and in material 2:

$$\frac{dg_{o2}}{dy} = \frac{1}{2}\frac{E_2}{G_2}\frac{\langle GS \rangle}{\langle EI \rangle}[(H_1 + H_2 - a)^2 - y^2]$$

from which one finds for shear the following parabolic distribution along the height of the beam

$$\boxed{\begin{aligned} -a \le y \le H_1 - a:\ \tau_{xy} &= \frac{T}{2}\frac{E_1}{\langle EI \rangle}(a^2 - y^2) \\ H_1 - a \le y \le H_1 + H_2 - a:\ \tau_{xy} &= \frac{T}{2}\frac{E_2}{\langle EI \rangle}[(H_1 + H_2 - a)^2 - y^2] \end{aligned}}$$

The corresponding variations are shown below. At the junction between the two materials ($y = H_1 - a$), one finds the shear in the adhesive:

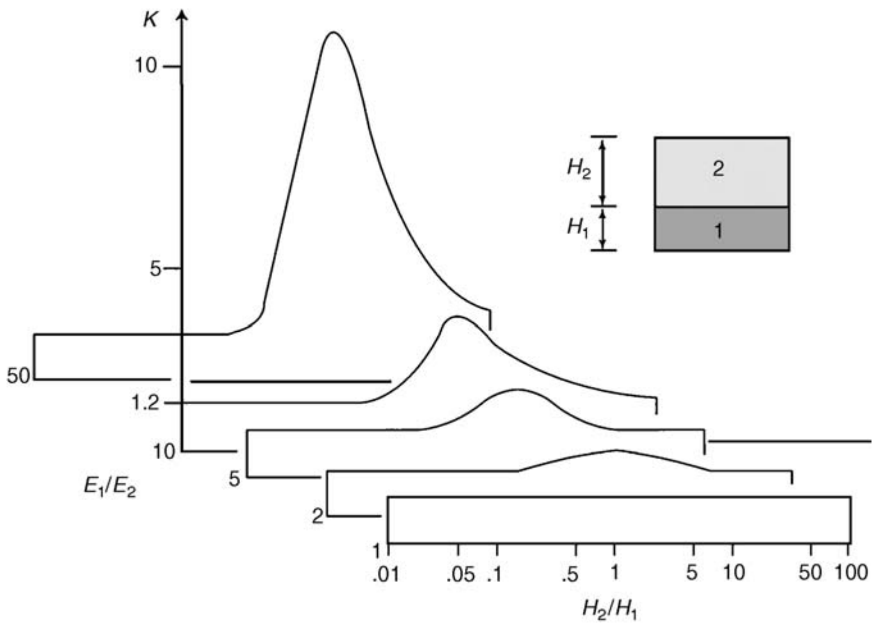


$$\boxed{\begin{array}{l}\tau_{xy} = \dfrac{T}{2}\dfrac{E_1}{\langle EI\rangle}H_1(2a-H_1) \\ \text{adhesive}\end{array}}$$

Remark: The integration of the function $g_o(y)$ allows the calculation of the shear coefficient k by Equation 15.16:

$$k = \frac{1}{\langle EI\rangle}\int_{\text{section}} E_i\, g_o\, y\, dS$$

The calculation is long but does not present any particular difficulty. The numerical values of k are shown in the following figure for different ratios of E_1/E_2 and H_2/H_1, for the particular case of identical Poisson coefficients.

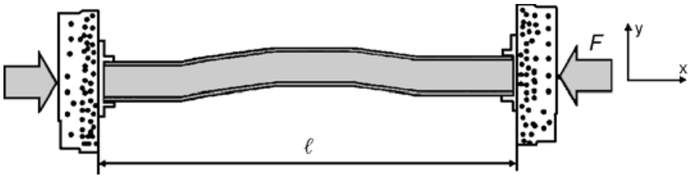


18.3.4 Buckling of a Sandwich Beam

Problem Statement:

A sandwich beam is compressed at its two ends by two opposite forces F. The two ends are constrained so that there is no rotation.

1. For what value of F, denoted as F_{critical}, can we obtain a deformed configuration for the beam other than the straight configuration, for example, the configuration shown in the figure below (adjacent-equilibrium)?



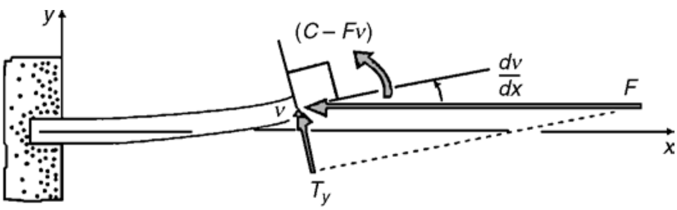
2. What error will be caused by neglecting shear deformation of the beam? (Assume that the dimensions and the material constitutive relations are known.)

Solution:

1. With the notation convention of Chapter 15 (bending of composite beams), recall the behavior equations for the beam Equation 15.16.

$$T_y = \frac{\langle GS\rangle}{k}\left(\frac{dv}{dx} - \theta_z\right); \quad M_z = \langle EI_z\rangle \frac{d\theta_z}{dx}$$

Referring to the figure below, one can write the following relations, in which C represents the moment due to the constraint on the right-hand side.



$$T_y \# F\frac{dv}{dx}; \quad M_z = C - Fv$$

from which, by substituting in the constitutive relations:

$$F\frac{dv}{dx} = \frac{\langle GS\rangle}{k}\left(\frac{dv}{dx} - \theta_z\right); \quad C - Fv = \langle EI_z\rangle \frac{d\theta_z}{dx}$$

Elimination of θ_z between these two relations leads to the following equation:

$$\boxed{\frac{d^2v}{dx^2} + \lambda^2 v = \lambda^2\frac{C}{F} \quad \text{with} \quad \lambda^2 = \frac{F}{\langle EI_z\rangle}\frac{1}{\left(1 - \frac{kF}{\langle GS\rangle}\right)}} \qquad [a]$$

from which the general solution can be written as: (with the condition that $F < \frac{\langle GS\rangle}{k}$)

$$v(x) = A\cos\lambda x + B\sin\lambda x + \frac{C}{F}$$

- Boundary conditions:
  For $x = 0$ one notes $v(0) = 0$ and $\theta_z(0) = 0$. This last condition leads to

$$\left.\frac{dv}{dx}\right|_{x=0} = 0$$

Due to:

$$\theta_z = \left(1 - F\frac{k}{\langle GS\rangle}\right)\frac{dv}{dx}$$

One then finds that:

$$B = 0; \quad A = -\frac{C}{F}$$

from which: $$v(x) = \frac{C}{F}(1 - \cos\lambda x)$$

For $x = \ell$ one notes $v(\ell) = 0$ and $\theta_z(\ell) = 0$.

$$\cos\lambda\ell = 1$$

from which:

$$\lambda\ell = 2n\pi$$

one obtains for $v(x)$ the form:

$$\boxed{v(x) = \frac{C}{F}\left(1 - \cos 2n\pi\frac{x}{\ell}\right)} \qquad \text{[b]}$$

The critical value F_{critical} is given by:

$$\lambda^2 = \frac{4n^2\pi^2}{\ell^2}$$

where λ^2 has the form [a], leading to:

$$F_{\text{critical}} = \frac{4n^2\pi^2\langle EI_z\rangle}{\ell^2\left(1 + \frac{4n^2\pi^2\langle EI_z\rangle k}{\ell^2\langle GS\rangle}\right)}$$

The smallest value of F is obtained for $n = 1$ as:

$$\boxed{F_{\text{critical}} = \frac{4\pi^2\langle EI_z\rangle}{\ell^2\left(1+\dfrac{4\pi^2\langle EI_z\rangle}{\ell^2}\dfrac{k}{\langle GS\rangle}\right)}}$$

Remarks:

- One can verify that the value of F_{critical} is less than

$$\frac{\langle GS\rangle}{k}$$

  the form of the general solution $v(x)$ written above is therefore legitimate.
- It is convenient to note that the deformed $v(x)$ written in [b] is only defined by a multiplication factor, because the constraining couple C is **indeterminate**. One can find this property by writing explicitly as a function of $v(x)$ the relation:

$$C = M_z(\ell) = \langle EI_z\rangle \left.\frac{d\theta_z}{dx}\right|_{x=\ell}$$

2. Neglecting shear effect the assumed undeformability under shear leads to zero corresponding energy of deformation (Equation 15.16). In this case, one has: $k = 0$.

The critical force then takes the value:

$$F'_{\text{critical}} = \frac{4\pi^2\langle EI_z\rangle}{\ell^2}$$

The error relative to its previous value is then:

$$\text{Error} = \frac{F'_{\text{critical}}}{F_{\text{critical}}} - 1$$

$$\boxed{\text{Error} = \frac{4\pi^2\langle EI_z\rangle k}{\ell^2\langle GS\rangle}}$$

For numerical value, we calculate this error for the beam in Section 4.2.2 (beam made of polyurethane foam and aluminum, 1 meter long). One has

$$\langle EI_z\rangle = 475 \times 10^2; \quad \frac{\langle GS\rangle}{k} = 650 \times 10^2$$

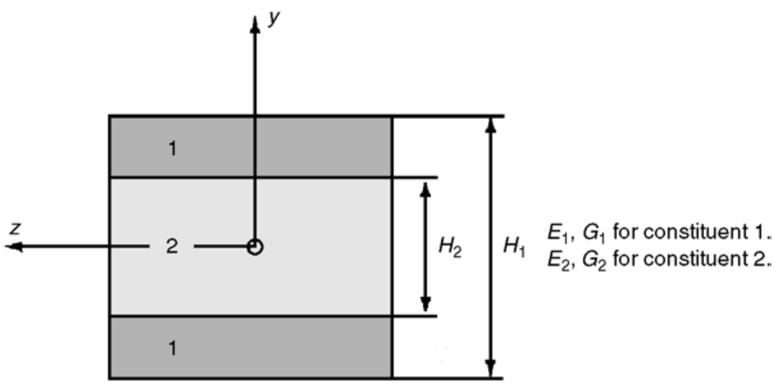
The error committed is spectacular:

$$\text{Error} = 28.84 = 2884\%!$$

18.3.5 Shear Due to Bending in a Sandwich Beam

Problem Statement:

One considers the cross section of a sandwich beam as shown in the following figure. The components, assumed to be isotropic (or transversely isotropic), are denoted as 1 and 2. They are perfectly bonded to each other with an adhesive with negligible thickness. The beam has a unit width. The moduli of elasticity are denoted as shown.



Using the formulation in Equation 15.16 for bending of composite beams:

1. Study the warping function g_o for this cross section.
2. Deduce from there the shear stress distribution.
3. Calculate the shear coefficient for bending in the plane xy, as well as the deformed configuration of a cross section under the effect of shear. Numerical application: Give the value of k for a beam made of polystyrene foam with thickness of 80.2 mm ($E_2 = 21.5$ MPa; $G_2 = 7.7$ MPa) and with aluminum skins with thickness of 2.15 mm ($E_1 = 65{,}200$ MPa; $G_1 = 24{,}890$ MPa).

Solution:

1. Warping due to bending:
   This is the solution of the problem described in Equation 15.16. Assuming here that g_o does not vary with the variable z:

$$\begin{cases} \dfrac{d^2 g_o}{dy^2} = -\dfrac{E_i}{G_i}\dfrac{\langle GS\rangle}{\langle EI_z\rangle} \times y \text{ in the domain of the section} \\ \dfrac{dg_o}{dy} = 0 \text{ for } y = \pm H_1/2. \end{cases}$$

in which both g_o and $G_i\, dg_o/dy$ are continuous as one crosses from material 1 to material 2.

Taking into account the antisymmetry of the function g_o with respect to variable y, one obtains

$$
\begin{aligned}
H_2/2 \le y \le H_1/2 \quad &: \quad g_{o1} = -\frac{E_1}{G_1}\frac{a}{6}y^3 + A_1 y + B_1 \\
-H_2/2 \le y \le H_2/2 \quad &: \quad g_{o2} = -\frac{E_2}{G_2}\frac{a}{6}y^3 + A_2 y \\
-H_1/2 \le y \le -H_2/2 \quad &: \quad g_{o3} = -\frac{E_1}{G_1}\frac{a}{6}y^3 + A_1 y - B_1
\end{aligned}
$$

with:

$$
\begin{aligned}
a &= \frac{\langle GS \rangle}{\langle EI_z \rangle} = 12\frac{G_2 H_2 + G_1(H_1 - H_2)}{E_2 H_2^3 + E_1(H_1^3 - H_2^3)} \\
A_1 &= \frac{E_1}{G_1}\frac{a}{2}\frac{H_1^2}{4} \\
B_1 &= a\frac{H_2}{16}\left\{\left(\frac{1}{G_2} - \frac{1}{G_1}\right)E_1 H_1^2 - \left(\frac{E_1 - E_2}{G_2}\right)H_2^2 - \left(\frac{E_2}{G_2} - \frac{E_1}{G_1}\right)\frac{H_2^2}{3}\right\} \\
A_2 &= A_1 + \frac{2B_1}{H_2} + \frac{a}{3}\frac{H_2^2}{8}\left(\frac{E_2}{G_2} - \frac{E_1}{G_1}\right)
\end{aligned}
$$

2. Shear stresses due to bending:
   These are given by the relation (see Equation 15.16):

$$
\vec{\tau} = G_i \frac{T_y}{\langle GS \rangle}\, \overrightarrow{\text{grad}}\; g_o
$$

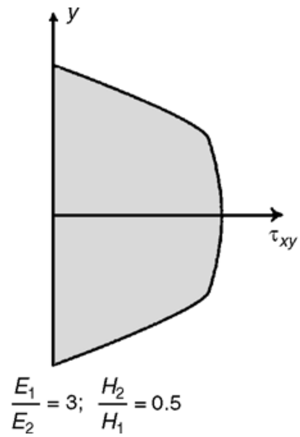
then:

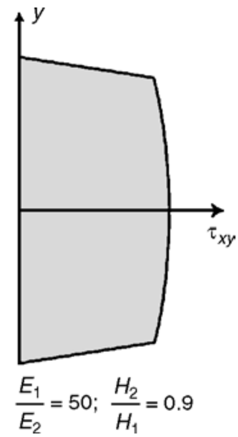
$$
\tau_{xy} = G_i \frac{T_y}{\langle GS \rangle}\frac{\partial g_o}{\partial y}; \quad \tau_{xz} = G_i \frac{T_y}{\langle GS \rangle}\frac{\partial g_o}{\partial z} = 0
$$

One obtains

$$
\boxed{
\begin{aligned}
0 \le y \le H_2/2 \quad &: \quad \tau_{xy} = \frac{1}{2}\frac{T_y}{\langle EI_z \rangle}\left\{E_2\left(\frac{H_2^2}{4} - y^2\right) + E_1\left(\frac{H_1^2}{4} - \frac{H_2^2}{4}\right)\right\} \\
H_2/2 \le y \le H_1/2 \quad &: \quad \tau_{xy} = \frac{1}{2}\frac{T_y}{\langle EI_z \rangle}E_1\left(\frac{H_1^2}{4} - y^2\right)
\end{aligned}
}
$$

The corresponding distribution is illustrated below for two distinct designs of the components 1 and 2.[47]





3. Shear coefficient:
   The calculation of k is done without difficulty starting from Equation 15.16:

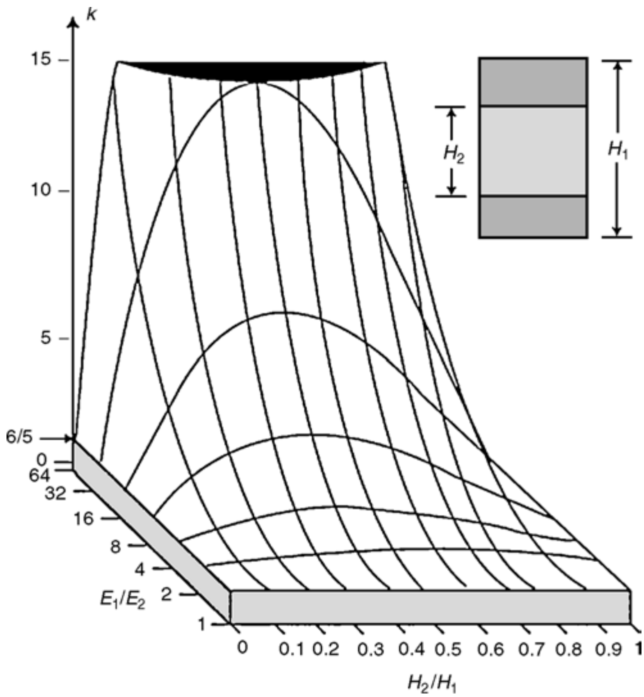
$$k = \frac{1}{\langle EI_z \rangle} \int_D E_i g_o y \; dS$$

One obtains

$$\boxed{\begin{aligned} k &= \frac{a}{8[E_2 H_2^3 + E_1(H_1^3 - H_2^3)]} \left\{ \frac{E_2}{G_2} H_2^3 \left[E_1 H_1^2 + \left(\frac{4}{5} E_2 - E_1 \right) H_2^2 \right] \cdots \right. \\ &\qquad \cdots \left. + \frac{E_1^2}{G_1} \left(\frac{4}{5} H_1^5 + \frac{H_2^5}{5} - H_1^2 H_2^3 \right) \right\} + \frac{3bE_1(H_1^2 - H_2^2)}{E_2 H_2^3 + E_1(H_1^3 - H_2^3)} \\ \text{with:} \quad a &= 12 \frac{G_2 H_2 + G_1(H_1 - H_2)}{E_2 H_2^3 + E_1(H_1^3 - H_2^3)} \\ b &= \frac{a}{16} H_2 \frac{E_1}{G_1} \left\{ \frac{H_2^2}{3} + H_1^2 \left(\frac{G_1}{G_2} - 1 \right) - H_2^2 \frac{G_1}{G_2} \left(1 - \frac{2}{3} \frac{E_2}{E_1} \right) \right\} \end{aligned}}$$

The evolution of the shear coefficient k is represented in the following figure for different values of the ratios E_1/E_2 and with the same Poisson coefficient (0.3) when varying thickness of the skins.

[47] Observation of the evolution of τ_{xy} for the beam with thin skins justifies the simplification proposed in Application 18.2.1, "Sandwich Beam: Simplified Calculation of the Shear Coefficient."

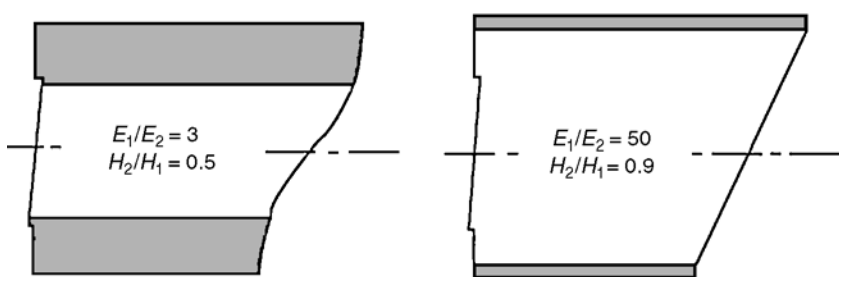


Remarks:

- The limiting cases $E_2 = E_1$; $H_2 = H_1$; $H_2 = 0$ correspond to a homogeneous beam with rectangular cross section for which one finds again the classical value $k = 6/5$ (or 1.2).
- The expression for the k coefficient written above is long. One can obtain a more simplified expression for easier manipulation if the skins are thin relative to the total thickness of the beam. One can refer to Application 18.2.1.
- Deformed configuration of a cross section: The displacement of each point of the cross section out of its initial plane is obtained starting from the function g_o by the relation (see Equations 15.12 and 15.15):

$$\eta_x = \frac{T_y}{\langle GS \rangle}(g_o - k \times y)$$

It is described graphically for two distinct sets of properties of components 1 and 2 in the following figure:



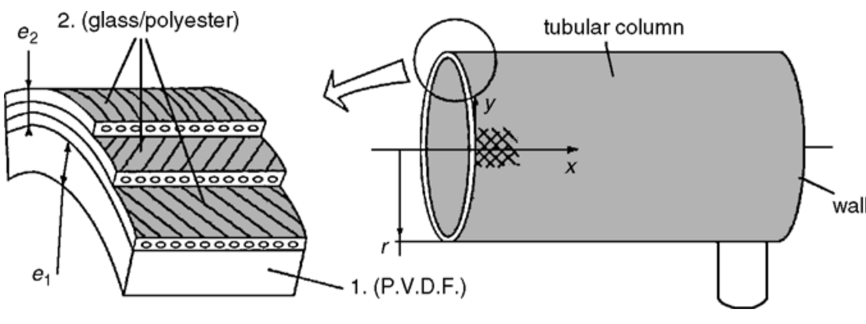
- Numerical application:
  One finds: $k = 165.7$. Note that for this type of beam, the shear coefficient can have very high values compared with those that characterize the homogeneous beams.

18.3.6 Column Made of Stretched Polymer

Problem Statement:

Consider a cylindrical column or revolution designed for use in the chemical industry (temperature can be high, and it may contain corrosive fluid under pressure) made of polyvinylidene fluoride (PVDF). It is reinforced on the outside by a filament-wound layer of "E" glass/polyester. The characteristics of the two layers of materials are as follows:

- Internal layer in PVDF: thickness e_1, isotropic material, modulus of elasticity E_1, Poisson coefficient ν_1.
- External layer in glass/polyester: To simplify the calculation, one will neglect the presence of the resin. As a consequence, E_t, $\nu_{t\ell}$, $G_{\ell t}$ (see Chapter 10) are neglected. The total thickness of the glass/polyester layer E_2 consists of a thickness denoted as h^{90} of windings along the 90° direction relative to the direction of the generator of the cylinder, and a thickness denoted as $h^{\pm 45}$ of balanced windings along the +45° and −45° direction (as many fibers along the +45° as along the −45° direction). One then has $e_2 = h^{90} + h^{\pm 45}$ (see figure below).



The longitudinal modulus of elasticity of the glass/polyester layer is denoted as E_ℓ. The thicknesses e_1 (internal) and e_2 (external) will be considered to be small relative to the average radius of the column.

1. The plane that is tangent to the midplane of the glass/polyester laminate is denoted as x,y (see figure). Calculate the equivalent moduli $\bar{E}_x$ and $\bar{E}_y$, the equivalent coefficients $\bar{\nu}_{yx}$ and $\bar{\nu}_{xy}$ of the reinforcement glass/polyester as function of E_ℓ, h^{90}, and $h^{\pm 45}$.
2. One imposes a pressure of p_0 inside the column at room temperature (creep of the materials not considered). The resulting stresses are denoted

in the axes x,y:

σ_{1x} and σ_{1y} in the internal layer of PVDF
σ_{2x} and σ_{2y} in the external layer of glass/polyester

(a) Write the equilibrium relation and the constitutive equation resulting from the assembly that is assumed to be perfectly bonded. Deduce from there the system that allows the calculation of σ_{1x} and σ_{2x}.
(b) Numerical applications: Internal pressure $p_0 = 3$ MPa (30 bars); $r = 100$ mm. PVDF: $E_1 = 260$ MPa; $\nu_1 = 0.3$; $e_1 = 10$ mm. Glass/polyester: $E_\ell = 74{,}000$ MPa; $e_2 = 0.75$ mm; $h^{90} = h^{\pm 45}/3$. Calculate σ_{1x}, σ_{1y}, σ_{2x}, σ_{2y}.
(c) Deduce from the previous results the stresses σ_ℓ^{90} in the glass fibers at 90°, and $\sigma_\ell^{\pm 45}$ in the fibers at ±45°. Comment.

3. We desire to modify the ratio $h^{90}/h^{\pm 45}$ such that the stresses are identical in the fibers at 90° and in the fibers at ±45° ("isotensoid" external layer).
(a) What are the relations that $h^{90}/h^{\pm 45}$, σ_{2x}, σ_{2y} have to verify?
(b) Indicate an iterative method that allows, starting from the results of Question 2b, the calculation of the suitable ratio $h^{90}/h^{\pm 45}$. Give the composition of the glass/polyester with the corresponding real thicknesses (use a mixture with $V_f = 25\%$ fiber volume fraction).

Solution:

1. Equivalent moduli:
The constitutive law of the laminate in the axes x,y is written as (see Equation 12.4):

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \gamma_{oxy} \end{Bmatrix} \quad \text{with} \quad A_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{\text{ij}}^{\text{k}} e_k$$

The coefficients $\bar{E}_{ij}^k$ are given by Equation 11.8 as, neglecting E_t, $\nu_{t\ell}$, $G_{\ell t}$:

- For the plies at 90°:

$$\bar{E}_{11}^{90} = \bar{E}_{12}^{90} = \bar{E}_{33}^{90} = \bar{E}_{23}^{90} = \bar{E}_{13}^{90} = 0$$
$$\bar{E}_{22}^{90} = E_\ell$$

- For the plies at +45°:

$$\bar{E}_{11}^{+45} = \bar{E}_{22}^{+45} = \bar{E}_{33}^{+45} = \bar{E}_{12}^{+45} \cdots$$
$$\cdots = -\bar{E}_{13}^{+45} = -\bar{E}_{23}^{+45} = E_\ell/4$$

■ For the plies at −45°:

$$\bar{E}_{11}^{-45} = \bar{E}_{22}^{-45} = \bar{E}_{33}^{-45} = \bar{E}_{12}^{-45} \cdots$$
$$\cdots = \bar{E}_{13}^{-45} = \bar{E}_{23}^{-45} = E_\ell/4$$

from which one can find the coefficients A_{ij}. For example, one has

$$A_{11} = \bar{E}_{11}^{90} h^{90} + \bar{E}_{11}^{+45} h^{+45} + \bar{E}_{11}^{-45} h^{-45} = \frac{E_\ell}{4} h^{\pm 45}$$

$$A_{12} = \bar{E}_{12}^{90} h^{90} + \bar{E}_{12}^{+45} h^{+45} + \bar{E}_{12}^{-45} h^{-45} = \frac{E_\ell}{4} h^{\pm 45}$$

and so forth. One obtains

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \frac{E_\ell}{4} h^{\pm 45} \begin{bmatrix} 1 & 1 & 0 \\ 1 & \left(1 + 4\dfrac{h^{90}}{h^{\pm 45}}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}$$

In inverting and in denoting for the average stresses (fictitious) in the external laminated layer (index 2): $\sigma_{2x} = N_x/e_2$; $\sigma_{2y} = N_y/e_2$; $\tau_{2xy} = T_{xy}/e_2$

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \frac{e_2}{E_\ell h^{90}} \begin{bmatrix} \left(1 + 4\dfrac{h^{90}}{h^{\pm 45}}\right) & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{2x} \\ \sigma_{2y} \\ \tau_{2xy} \end{Bmatrix}$$

The above relation can be also interpreted as follows (see Equation 12.9):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\bar{E}_x} & -\dfrac{\bar{\nu}_{yx}}{\bar{E}_y} & 0 \\ -\dfrac{\bar{\nu}_{xy}}{\bar{E}_x} & \dfrac{1}{\bar{E}_y} & 0 \\ 0 & 0 & \dfrac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{2x} \\ \sigma_{2y} \\ \tau_{2xy} \end{Bmatrix}$$

where the equivalent moduli of the laminate appear. From this, by identification one has

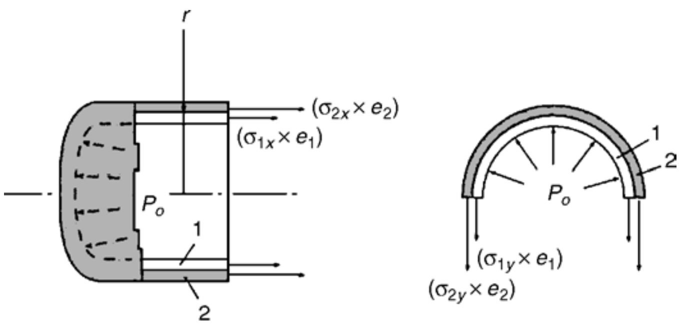
$$\boxed{\begin{aligned} \bar{E}_x &= \frac{E_\ell}{e_2\left(\dfrac{1}{h^{90}} + \dfrac{4}{h^{\pm 45}}\right)}; \quad \bar{E}_y = E_\ell \frac{h^{90}}{e_2} \\ \bar{\nu}_{xy} &= \frac{1}{\left(1 + 4\dfrac{h^{90}}{h^{\pm 45}}\right)}; \quad \bar{\nu}_{yx} = 1 \end{aligned}} \qquad [1]$$

Comment:

The obtained results are simple because:

- The polyester resin is not taken into account. The fibers work only in their direction.
- The voluntary decoupling between the external layer (glass/resin) and the internal layer (PVDF) is preferred to the consideration of a "global" laminate consisting of plies of glass/resin at 90°, +45°, –45° and a ply of PVDF, isotropic, with thickness e_1.

2. (a) Equilibrium relation:
 The isolation of the portions of the column shown below allows one to write



$$2\pi r(e_1\sigma_{1x} + e_2\sigma_{2x}) = \pi r^2 p_0$$
$$1 \times 2(e_1\sigma_{1y} + e_2\sigma_{2y}) = 1 \times 2r \times p_0$$

from which one has the equilibrium relations:

$$\boxed{\begin{aligned} e_1\sigma_{1x} + e_2\sigma_{2x} &= p_0\frac{r}{2} \quad [2] \\ e_1\sigma_{1y} + e_2\sigma_{2y} &= p_0 r \quad [3] \end{aligned}}$$

- Constitutive relations:
 The elastic behavior of the internal layer of PVDF is described by the classical isotropic equation:

$$\begin{Bmatrix} \varepsilon_{1x} \\ \varepsilon_{1y} \\ \gamma_{1xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_1}{E_1} & 0 \\ -\frac{\nu_1}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_1} \end{bmatrix} \begin{Bmatrix} \sigma_{1x} \\ \sigma_{1y} \\ \tau_{1xy} \end{Bmatrix}$$

The behavior of the external layer in composite is described by the relation obtained in the previous question as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & 0 \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & 0 \\ 0 & 0 & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{2x} \\ \sigma_{2y} \\ \tau_{2xy} \end{Bmatrix}$$

Equality of strains under the action of stresses is written as:

$$\varepsilon_{1x} = \varepsilon_{ox}; \quad \varepsilon_{1y} = \varepsilon_{oy}$$

and leads to the relation:

$$\boxed{\begin{aligned} \frac{1}{E_1}\sigma_{1x} - \frac{\nu_1}{E_1}\sigma_{1y} &= \frac{1}{\bar{E}_x}\sigma_{2x} - \frac{\bar{\nu}_{yx}}{\bar{E}_y}\sigma_{2y} \qquad [4] \\ -\frac{\nu_1}{E_1}\sigma_{1x} + \frac{1}{E_1}\sigma_{1y} &= -\frac{\bar{\nu}_{xy}}{\bar{E}_x}\sigma_{2x} + \frac{1}{\bar{E}_y}\sigma_{2y} \qquad [5] \end{aligned}}$$

Relations [2], [3], [4], [5] constitute a system of four equations for the four unknowns σ_{1x}, σ_{1y}, σ_{2x}, σ_{2y}. Performing the subtraction [4] – [5], one obtains

$$\sigma_{1x}\left(\frac{1+\nu_1}{E_1}\right) - \sigma_{1y}\left(\frac{1+\nu_1}{E_1}\right) = \sigma_{2x}\left(\frac{1+\bar{\nu}_{xy}}{\bar{E}_x}\right) - \sigma_{2y}\left(\frac{1+\bar{\nu}_{yx}}{\bar{E}_y}\right)$$

In performing the addition [4] + [5], one obtains

$$\sigma_{1x}\left(\frac{1-\nu_1}{E_1}\right) - \sigma_{1y}\left(\frac{1-\nu_1}{E_1}\right) = \sigma_{2x}\left(\frac{1-\bar{\nu}_{xy}}{\bar{E}_x}\right) - \sigma_{2y}\left(\frac{1-\bar{\nu}_{yx}}{\bar{E}_y}\right)$$

and with [2] and [3], by substitution, one obtains a system that allows the calculation of σ_{1x} and σ_{1y} as:

$$\boxed{\begin{aligned} &\sigma_{1x}\left[\left(\frac{1+\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1+\bar{\nu}_{xy}}{\bar{E}_x}\right)\right] - \sigma_{1y}\left[\left(\frac{1+\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1+\bar{\nu}_{yx}}{\bar{E}_y}\right)\right]\cdots \\ &\qquad\cdots = \frac{p_0 r}{e_2}\left[\left(\frac{1+\bar{\nu}_{xy}}{2\bar{E}_x}\right) - \left(\frac{1+\bar{\nu}_{yx}}{\bar{E}_y}\right)\right] \\ &\sigma_{1x}\left[\left(\frac{1-\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1-\bar{\nu}_{xy}}{\bar{E}_x}\right)\right] + \sigma_{1y}\left[\left(\frac{1-\nu_1}{E_1}\right) + \frac{e_1}{e_2}\left(\frac{1-\bar{\nu}_{yx}}{\bar{E}_y}\right)\right]\cdots \\ &\qquad\cdots = \frac{p_0 r}{e_2}\left[\left(\frac{1-\bar{\nu}_{xy}}{2\bar{E}_x}\right) + \left(\frac{1-\bar{\nu}_{yx}}{\bar{E}_y}\right)\right] \end{aligned}} \qquad [6]$$

(b) Numerical application:
One has $h^{90} = h^{\pm 45}/3$, from which we have

$$e_2 = h^{90} + h^{\pm 45} = 0.75 \text{ mm}; \quad h^{\pm 45} = 0.56 \text{ mm}; \quad h^{90} = 0.19 \text{ mm}.$$

Following the results [1], one finds

$$\bar{E}_x = 7953 \text{ MPa}; \quad \bar{E}_y = 18747 \text{ MPa}; \quad \bar{\nu}_{xy} = 0.42$$

The system [6] has for solutions:

$$\boxed{\sigma_{1x} = 1.71 \text{ MPa}; \quad \sigma_{1y} = 3.07 \text{ MPa}}$$

Relations [4] and [5] allow the calculation of σ_{2x} and σ_{2y}. One finds

$$\boxed{\sigma_{2x} = 188 \text{ MPa}; \quad \sigma_{2y} = 386 \text{ MPa}}$$

(c) Stresses in the fibers:
Following Equation 11.8, one has for any ply k in the external layer:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^k = E_\ell \begin{bmatrix} c^4 & c^2 s^2 & -c^3 s \\ c^2 s^2 & s^4 & -cs^3 \\ -c^3 s & -cs^3 & c^2 s^2 \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} \qquad [7]$$

The strains ε_{ox} and ε_{oy} are obtained by means of the previous results (see Question 2a), for example:

$$\varepsilon_{ox} = \varepsilon_{1x} = \frac{\sigma_{1x}}{E_1} - \frac{\nu_1}{E_1}\sigma_{1y} = 3.03 \times 10^{-3}$$

$$\varepsilon_{oy} = \varepsilon_{1y} = -\frac{\nu_1}{E_1}\sigma_{1x} + \frac{\sigma_{1y}}{E_1} = 9.85 \times 10^{-3}$$

If one inverts Equation 11.4, taking into account that the only nonzero stress in the axes ℓ, t of the ply is σ_ℓ:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -sc & sc & (c^2 - s^2) \end{bmatrix}^k \begin{Bmatrix} \sigma_\ell \\ 0 \\ 0 \end{Bmatrix} \qquad [8]$$

One then has

- In the fibers at 90°:
  Following [7], $\sigma_x^{90} = 0$; $\sigma_y^{90} = E_\ell\, \varepsilon_{oy}$
  Following [8], $\sigma_x^{90} = 0$; $\sigma_y^{90} = \sigma_\ell^{90}$

from which:

$$\boxed{\begin{aligned}\sigma_\ell^{90} &= E_\ell \varepsilon_{oy} \\ \sigma_\ell^{90} &= 729 \text{ MPa}\end{aligned}}$$

- In the fibers at +45°:
  Following [7]: $\sigma_x^{+45} = \sigma_y^{+45} = \frac{E_\ell}{4}(\varepsilon_{ox} + \varepsilon_{oy})$
  Following [8]: $\sigma_x^{+45} = \sigma_y^{+45} = \frac{1}{2}\sigma_\ell^{+45}$
  from which one obtains

$$\boxed{\begin{aligned}\sigma_\ell^{+45} &= \frac{E_\ell}{2}(\varepsilon_{ox} + \varepsilon_{oy}) \\ \sigma_\ell^{+45} &= 477 \text{ MPa}\end{aligned}}$$

One obtains an identical stress in the fibers at −45°. Note the disparity of the stresses in the fibers at 90° and at ±45°. In fact, the external layer is not suitably designed, because it is desirable to make all fibers work equally in order to obtain a uniform extension in the glass fibers.

3. (a) One desires that $\sigma_\ell^{90} = \sigma_\ell^{\pm 45}$:
   Referring to the results of the previous question, this equality is also written as:

$$E_\ell \varepsilon_{o\ell} = \frac{E_\ell}{2}(\varepsilon_{ox} + \varepsilon_{oy})$$

   as:

$$\varepsilon_{oy} = \varepsilon_{ox}$$

   The constitutive relation of the laminate (Question 1 and relation [1]) indicates then:

$$\frac{\sigma_{2x}}{\bar{E}_x} - \frac{\bar{\nu}_{yx}}{\bar{E}_y}\sigma_{2y} = -\frac{\bar{\nu}_{xy}}{\bar{E}_x}\sigma_{2x} + \frac{\sigma_{2y}}{\bar{E}_y}$$

   Then after calculation:

$$\boxed{\frac{h^{90}}{h^{\pm 45}} = \frac{\sigma_{2y} - \sigma_{2x}}{\sigma_{2x}}} \qquad [9]$$

   (b) With the results of numerical application 2(b), relation [9] above indicates

$$\frac{\sigma_{2y} - \sigma_{2x}}{\sigma_{2x}} = 0.53$$

If one adopts this new value for the ratio $h^{90}/h^{\pm 45}$, one obtains for new results:

- $h^{90}/h^{\pm45} = 0.53$;

$$\bar{E}_x = 8216 \text{ MPa}; \quad \bar{E}_y = 25{,}653 \text{ MPa}; \quad \bar{\nu}_{xy} = 0.32; \quad \bar{\nu}_{yx} = 1$$
$$\sigma_{1x} = 2.42 \text{ MPa}; \quad \sigma_{1y} = 2.72 \text{ MPa};$$
$$\sigma_{2x} = 167 \text{ MPa}; \quad \sigma_{2y} = 364 \text{ MPa};$$

Relation [9] then indicates

$$\frac{\sigma_{2y} - \sigma_{2x}}{\sigma_{2x}} = 0.587$$

that one adopts for the new ratio $h^{90}/h^{\pm45}$:

- $h^{90}/h^{\pm45} = 0.587$:

$$\bar{E}_x = 8166 \text{ MPa}; \quad \bar{E}_y = 27{,}627 \text{ MPa}; \quad \bar{\nu}_{xy} = 0.29; \quad \bar{\nu}_{yx} = 1$$
$$\sigma_{1x} = 2.63 \text{ MPa}; \quad \sigma_{1y} = 2.69 \text{ MPa};$$
$$\sigma_{2x} = 165 \text{ MPa}; \quad \sigma_{2y} = 364 \text{ MPa}.$$

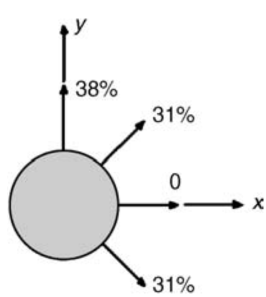
Relation [9] then indicates

$$\frac{\sigma_{2y} - \sigma_{2x}}{\sigma_{2x}} = 0.6$$

that is, a relative variation of 2% with respect to the value of the ratio $(h^{90}/h^{\pm45})$ taken to carry out the calculations. The iterative procedure then converges rapidly. One will obtain the external isotensoid layer and an internal layer of PVDF in biaxial tension for a ratio of

$$h^{90}/h^{\pm45} \# 0.6$$

The composition of the glass/polyester reinforcement will be as follows:



The real thickness of the windings in glass/polyester, taking into account the volume of the resin, will be (with $V_f = 0.25$):

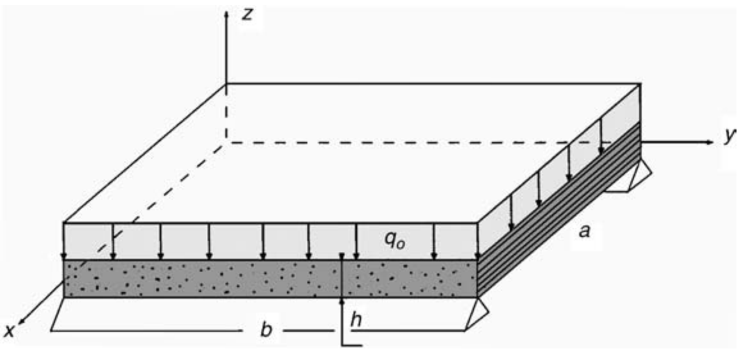
$$e_2' = e_2/0.25$$

$$e_2' = 3 \text{ mm}$$

18.3.7 Cylindrical Bending of a Thick Orthotropic Plate under Uniform Loading

Problem Statement:

Consider a thick rectangular plate $b \times a$, with $b >> a$ made of unidirectional glass/resin (see figure). It is supported at two opposite sides and is loaded by a constant transverse pressure of q_o.



1. Calculate the deflection due to bending at the midline of the plate located at $x = a/2$ (maximum deflection).
2. Indicate the numerical values of the contributions from the bending moment and from transverse shear using the following: $E_x = 40{,}000$ MPa; $G_{xz} = 400$ MPa; $\nu_{xy} = 0.3$; $\nu_{yx} = 0.075$; $q_o = -1$ MPa; $a = 150$ mm; $h = 15$ mm. Comment.

Solution:

1. For the cylindrical bending considered, Equation 17.32 allows one to write

$$\frac{dQ_x}{dx} = -q_0; \quad \frac{dM_y}{dx} = Q_x; \quad M_y = C_{11}\frac{d\theta_y}{dx}; \quad Q_x = \frac{hG_{xz}}{k_x}\left(\frac{dw_o}{dx} + \theta_y\right)$$

Elimination of Q_x, M_y and θ_y leads to

$$\frac{d^4 w_o}{dx^4} = \frac{q_o}{C_{11}}$$

then:

$$w_o = \frac{q_o}{C_{11}}\left(\frac{x^4}{24} + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D\right)$$

The boundary conditions are written as:

$$\left.\begin{matrix} x = 0 \\ x = a \end{matrix}\right\} w_0 = 0; \quad M_y = 0 \Rightarrow \frac{d\theta_y}{dx} = \frac{k_x}{hG_{xz}} q_0 - \frac{d^2 w_o}{dx^2} = 0$$

After calculation of the constants A, B, C, D, one obtains for the deflection at $x = a/2$:

$$w_0\left(\frac{a}{2}\right) = q_0 a^4 \frac{12(1-\nu_{xy}\nu_{yx})}{E_x h^3}\left\{\frac{5}{384} + k_x\left(\frac{h}{a}\right)^2 \frac{E_x}{G_{xz}} \frac{1}{96(1-\nu_{xy}\nu_{yx})}\right\}$$

The calculation of k_x was done in Section 17.7.1 for this type of plate. One has (see Equation 17.34)

$$k_x = 6/5 = 1.2$$

from which:

$$\boxed{w_0\left(\frac{a}{2}\right) = q_0 a^4 \frac{12(1-\nu_{xy}\nu_{yx})}{E_x h^3}\left\{\frac{5}{384} + \left(\frac{h}{a}\right)^2 \frac{E_x}{G_{xz}} \frac{1}{80(1-\nu_{xy}\nu_{yx})}\right\}}$$

The first term in the brackets represents the contribution from the bending moment, and the second term represents that due to transverse shear.

2. Numerical values:

$$w_o(a/2) = \underset{\text{(moment)}}{-0.5727 \text{ mm}} - \underset{\text{(transverse shear)}}{0.5625 \text{ mm}}$$

$$w_o(a/2) = -1.13525 \text{ mm}$$

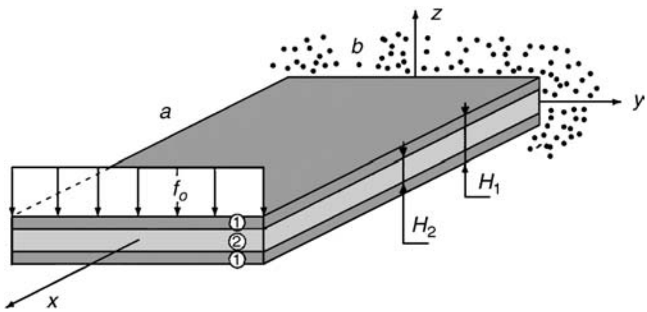
Note that 49.5% of this deflection is due to transverse shear. One can see from the above expression for $w_o(a/2)$ that the influence of transverse shear on the bending deflection increases with the value of the relative thickness h/a (here,

$h/a = 1/10$ corresponds to a thick plate). One also notes the influence of the ratio E_x/G_{xz}.[48]

18.3.8 Bending of a Sandwich Plate

Problem Statement:

A rectangular sandwich plate ($a \times b$) is fixed on side b, and loaded along the opposite side by a constant distributed load f_o (N/mm). The two other sides are free (see figure).



The plate consists of two identical orthotropic skins of material 1, and an orthotropic core made of material 2. The orthotropic axes are parallel to the axes x,y,z.

1. Calculate the deflection of the midplane at the extremity $x = a$ of the x axis, assuming cylindrical bending about y axis for the plate.
2. Numerical application:
   Given
   $f_o = -10$ N/mm
   $a = b = 1000$ mm
   $H_1 = 2H_2 = 100$ mm

   Material 1:
   $E_x^{(1)} = 40{,}000$ MPa
   $G_{xz}^{(1)} = 4000$ MPa

   Material 2:
   $E_x^{(2)} = 40$ MPa
   $G_{xz}^{(2)} = 15$ MPa

   For each of the materials
   $\nu_{xy} = 0.3$
   $\nu_{yx} = 0.075$

[48] This example of thick plate in bending constitutes a test case for the evaluation of computer programs using finite elements. For complementary information on this topic, see bibliography, "Computer programs for Composite Structures: Reference examples and Validation."

(a) Calculate the deflection at the extremity $x = a$ and show the contributions from the bending moment and from the transverse shear.
(b) Calculate the transverse shear stress τ_{xz}:

- On the midplane of the plate.
- At the interface between the core and the upper skin.
- At the midthickness of the upper skin.

Solution:

1. In the case of cylindrical bending, Equation 17.32 allows one to write

$$\frac{dQ_x}{dx} = 0; \quad \frac{dM_y}{dx} = Q_x; \quad M_y = C_{11}\frac{d\theta_y}{dx}; \quad Q_x = \frac{\langle hG_{xz}\rangle}{k_x}\left(\frac{dw_0}{dx} + \theta_y\right)$$

Then $Q_x = f_o$, and elimination of Q_x, M_y, and θ_y leads to

$$\frac{d^3 w_o}{dx^3} = \frac{f_o}{C_{11}}$$

then:

$$w_o = -\frac{f_o}{C_{11}}\left(\frac{x^3}{6} + A\frac{x^2}{2} + Bx + C\right)$$

The boundary conditions are written as:

$$x = 0 : w_O = 0 \text{ et } \theta_y = 0 \Rightarrow k_x\frac{f_o}{\langle hG_{xz}\rangle} - \frac{dw_o}{dx} = 0$$

$$x = a : M_y = 0 \Rightarrow \frac{d\theta_y}{dx} = -\frac{d^2 w_o}{dx^2} = 0$$

After calculation of the constants A, B, C, one obtains the deflection at $x = a$:

$$w_o(a) = \frac{f_o a^3}{3C_{11}} + k_x\frac{f_o a}{\langle hG_{xz}\rangle}$$

with (see Equation 12.16):

$$C_{11} = \bar{E}_{11}^{①}\left(\frac{H_1^3 - H_2^3}{12}\right) + \bar{E}_{11}^{②}\frac{H_2^3}{12}$$

and according to Equation 17.2:

$$C_{11} = \frac{E_x^{①}(H_1^3 - H_2^3) + E_x^{②}H_2^3}{12(1 - \nu_{xy}\nu_{yx})}$$

According to Equation 17.10:

$$\langle hG_{xz}\rangle = G_{xz}^{①}(H_1 - H_2) + G_{xz}^{②}H_2$$

from which one obtains

$$\boxed{w_o(a) = \frac{4(1-\nu_{xy}\nu_{yx})f_o a^3}{E_x^{①}(H_1^3 - H_2^3) + E_x^{②}H_2^3} + \frac{k_x f_o a}{G_{xz}^{①}(H_1 - H_2) + G_{xz}^{②}H_2}}$$

The calculation of k_x was carried out in Section 17.7.2 for this type of plate. It was given by Equation 17.39.

2. Numerical application:
   (a) Deflection: Equation 17.39 gives $k_x = 110.8$
   from which:

$$w_o(a) = \underset{\text{(moment)}}{-1.177\text{ mm}} + \underset{\text{(transverse shear)}}{(-5.519\text{ mm})}$$

$$w_o(a) = -6.636\text{ mm}$$

Note that 83% of this deflection is due to transverse shear, and this happens in spite of very thick skins. This important influence is due to the very large value compared with unity (110.8) of the transverse shear coefficient and due to the fact that the plate is thick ($H_1/a = 1/10$).

   (b) Transverse shear stress τ_{xz} (see Section 17.7.2):

Midplane: ($z = 0$): $\tau_{xz} = 0.1286$ MPa

Interface ($z = H_2/2$): $\tau_{xz} = 0.12855$ MPa

Midthickness of the upper layer: $z = (H_1 + H_2)/4$: $\tau_{xz} = 0.075$ MPa

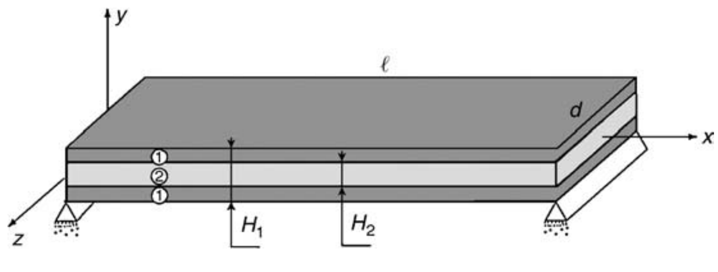
18.3.9 Bending Vibration of a Sandwich Beam[49]

Problem Statement:

Consider a sandwich beam of length ℓ and width d simply supported at its ends (see figure). It consists of two identical skins of material 1 (glass/resin) and a core

[49] This application constitutes a test case for the validation of computer programs using finite elements, see in the bibiography, "Programs for the calculation of Composite Structures, Reference examples and Validation."

of material 2 (foam). These materials are transversely isotropic in the plane y, z.



The elastic characteristics are denoted as:

$$E_x^{①}, G_{xy}^{①}, E_x^{②}, G_{xy}^{②}$$

Specific masses are ρ_1 and ρ_2.

1. Write the equation for the resonant frequencies for bending vibration in the plane of symmetry (x, y) of this beam.
2. Numerical application:

Given:

$$E_x^{①} = 40{,}000 \text{ MPa}; \quad G_{xy}^{①} = 4{,}000 \text{ MPa}; \quad \rho_1 = 2{,}000 \text{ kg/m}^3$$

$$E_x^{②} = 40 \text{ MPa}; \quad G_{xy}^{②} = 15 \text{ MPa}; \quad \rho_2 = 50 \text{ kg/m}^3$$

$$H_1 = 2H_2 = 100 \text{ mm}; \quad \ell = 1000 \text{ mm}; \quad d = 100 \text{ mm}$$

Calculate the first five frequencies in bending vibration.

Solution:

1. Equation for the vibration frequencies:
   At first one establishes the differential equation for the dynamic displacement $v(x, t)$ starting from the Equation 15.18, noting that for the example considered, the elastic center and the center of gravity of section are the same (decoupling between bending vibration and longitudinal vibrations).

$$\frac{\partial T_y}{\partial x} = \langle \rho S \rangle \frac{\partial^2 v}{\partial t^2}; \quad \frac{\partial M_z}{\partial x} + T_y = \langle \rho I_z \rangle \frac{\partial^2 \theta_z}{\partial t^2}$$

$$T_y = \frac{\langle GS \rangle}{k}\left(\frac{\partial v}{\partial x} - \theta_z\right); \quad M_z = \langle EI_z \rangle \frac{\partial \theta_z}{\partial x}$$

Elimination of T_y, M_z, θ_z between these four relations leads to the equation for $v(x, t)$:

$$\frac{\partial^4 v}{\partial x^4} - \frac{\langle \rho I_z \rangle}{\langle EI_z \rangle}(1 + a)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\langle \rho S \rangle}{\langle EI_z \rangle}\frac{\partial^2 v}{\partial t^2} + k\frac{\langle \rho I_z \rangle \langle \rho S \rangle}{\langle GS \rangle \langle EI_z \rangle}\frac{\partial^4 v}{\partial t^4} = 0$$

with

$$a = \frac{\langle \rho S \rangle \langle EI_z \rangle}{\langle GS \rangle \langle \rho I_z \rangle} \times k$$

In assuming that the solution takes the form $v(x, t) = v_o(x) \times \cos(\omega t + \varphi)$ one can rewrite the differential equation that defines the modal deformation $v_o(x)$ in the following nondimensional form:

$$\frac{d^4 \bar{v}_o}{d\bar{x}^4} + \bar{\omega}^2(1+a)\frac{d^2 \bar{v}_o}{d\bar{x}^2} + \bar{\omega}^2\left(a\bar{\omega}^2 - \frac{1}{\bar{r}^2}\right)\bar{v}_o = 0$$

in which

$$\bar{x} = \frac{x}{\ell}; \quad \bar{v}_o = \frac{v_o}{\ell}; \quad \bar{\omega}^2 = \frac{\langle \rho I_z \rangle}{\langle EI_z \rangle}\omega^2 \ell^2; \quad \bar{r}^2 = \frac{\langle \rho I_z \rangle}{\langle \rho S \rangle \ell^2}$$

After writing the characteristic equation, the reduced modal deformation takes the form:

$$\bar{v}_o = A ch X_1 \bar{x} + B sh X_1 \bar{x} + C \cos X_2 \bar{x} + D \sin X_2 \bar{x} \qquad [1]$$

where:

$$\begin{matrix} X_1^2 \\ X_2^2 \end{matrix} = \mp \frac{\bar{\omega}^2(1+a)}{2} + \sqrt{\bar{\omega}^2\left[\bar{\omega}^2\left(\frac{1-a}{2}\right)^2 + \frac{1}{\bar{r}^2}\right]} \qquad [2]$$

The boundary conditions corresponding to simply supported ends are written as:

$$x = 0 \quad \text{or} \quad \ell : v = 0; \quad M_z = \langle EI_z \rangle \frac{\partial \theta_z}{\partial x} = 0 \ \forall t$$

or:

$$\bar{x} = 0 \quad \text{or} \quad 1 : \bar{v}_o = 0; \quad \frac{d^2 \bar{v}_o}{d\bar{x}^2} + a\bar{\omega}^2 \bar{v}_o = 0$$

These four conditions allow one to obtain with [1] a linear and homogeneous system in A,B,C,D. By setting the determinant equal to zero, one obtains an equation for vibrations which reduces to

$$\sin X_2 = 0$$

Then the solution can be written as:

$$X_2 = n\pi, \ (n = 1, 2, 3, \ldots) \qquad [3]$$

2. With the numerical values indicated in the Problem Statement, one can calculate the shear coefficient k, the literal expression for which has been established in Application 18.3.5. One finds k = 110.8. The frequencies can be written starting from the circular frequencies ω_1, ω_2, ω_3,… extracted from equation [3], where X_2 takes the form [2].

$$f_i = \frac{\omega_i}{2\pi}(\mathrm{H}_z)$$

one obtains:[50]

$$\boxed{\begin{aligned} &f_1 = 64.476 \text{ Hz}; \quad f_2 = 131.918 \text{ Hz}; \quad f_3 = 198.734 \text{ Hz} \\ &f_4 = 265.383 \text{ Hz}; \quad f_5 = 331.963 \text{ Hz}. \end{aligned}}$$

[50] One keeps voluntarily the nonsignificant decimal, for the purpose of comparison with values obtained from numerical models.