On the origin of the particles in black hole evaporation

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We present an analytic derivation of Hawking radiation for an arbitrary (spatial) dispersion relation $\omega(k)$ as a model for ultra-high energy deviations from general covariance. It turns out that the Hawking temperature is proportional to the product of the group $d\omega/dk$ and phase ω/k velocities evaluated at the frequency ω of the outgoing radiation far away, which suggests that Hawking radiation is basically a low-energy phenomenon. Nevertheless, a group velocity growing too fast at ultra-short distances would generate Hawking radiation at ultra-high energies ("ultra-violet catastrophe") and hence should not be a realistic model for the microscopic structure of quantum gravity.

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Introduction In the history of physics, unexplained coincidences were often the precursor of striking discoveries. For example, the limiting propagation velocity derived from the properties of coils and capacitors in table-top experiments turned out to be strikingly close to the speed of light measured via planetary motion – which guided the unification of these phenomena by the laws of electrodynamics and ultimately lead to the theory of relativity. Today, a similar riddle is the question of *why* black holes seem to behave like thermal objects [1] and evaporate by emitting Hawking radiation [2, 3, 4].

One way of achieving a better understanding of these links is to study the origin of the particles in black hole evaporation, i.e., the question of *where* they are created. For example, it has been suggested to resolve the black hole "information paradox" (i.e., the apparent contradiction between unitarity and the second law of thermodynamics in this system) by encoding information into the outgoing Hawking particles. Clearly, this (hidden) encoding mechanism should then occur at the origin of the particles (or on their way to infinity).

Since the event horizon marks the "point of no return", the modes containing the Hawking particles originate from the region very close to the horizon, i.e., from very short wavelengths (gravitational red-shift). However, the origin of the *modes* is not necessarily the place where the *particles* are created.

In order to address this problem, we derive Hawking radiation in the presence of a very general dispersion relation $\omega = ck \to \omega = \omega(k)$ associated to the propagating degrees of freedom. Such a modified dispersion relation is inspired by the analogy to condensed matter, i.e., the sonic black hole analogues (silent or "dumb" holes), which rely on the quantitative analogy between quantum fields in curved space-times and phonons (or other quasi-particles) propagating in fluids with a general flow velocity $\boldsymbol{v}(t, \boldsymbol{r})$, cf. [5, 6]. In this case, the phase ω/k and group $d\omega/dk$ velocities vary with wavelength and thus the dependence of Hawking radiation on the dispersion relation should show us which wavenumbers k are most important for particle creation: The emitted radiation at various frequencies ω will "see" different horizons. The question this paper will try to answer is what determines the temperature of the radiation emitted at any particular frequency ω. If the temperature is determined when the wavelengths are very small, and frequencies large – i.e., when the horizon first splits the incoming wave packet (in its vacuum state) into positive pseudo-norm modes outside the horizon (which will turn into the Hawking particles) and negative pseudo-norm modes (their infalling partner particles) inside – then one would expect a universal temperature for all of the low-frequency modes. On the other hand, if it is the low frequency aspects of the modes which determine the temperature, one might expect the properties of the horizon defined via one of the velocities (phase, group, or other) associated with the wave at low frequencies to dictate the temperature (via dv/dr at that horizon location) of that mode.

Previous analytic calculations were restricted to low energies ω and a small vicinity of horizon, see, e.g., [7], while numerical studies necessarily involved a given (mostly sub-luminal) dispersion relation within a restricted parameter range. In the following, we present an analytic derivation of Hawking radiation valid for any frequency ω and almost arbitrary (spatial) dispersion relations $\omega(k)$; the only assumption we make is that the black or "dumb" hole should be large, i.e., macroscopic, and that the velocity profile of the background flow have a specific form.

Dispersion relation Since Hawking/"dumb"-hole radiation is basically a 1+1 dimensional effect, we consider the Painlevé-Gullstrand-Lemaître metric in 1+1 dimensions ($\hbar = c = G_{\rm N} = k_{\rm B} = 1$)

$$ds^2 = \left[1 - v^2(x)\right] dt^2 - 2v(x)\, dt\, dx - dx^2\,, \qquad (1)$$

with the local frame dragging velocity $v(x)$, which corresponds to the flow velocity of the fluid analogue [5].

It determines the position of the horizon via $v(x) = \pm c$ where c is the velocity of the propagating modes, which is assumed to be constant and set to unity here. The propagation of a massless scalar field Φ in this metric (or, equivalently, phonons in the fluid) is governed by the d'Alembertian

$$\begin{aligned} \Box\Phi &= \left((\partial_t + \partial_x v)(\partial_t + v\partial_x) - \partial_x^2\right)\Phi \\ &= (\partial_t + \partial_x v + \partial_x)(\partial_t + v\partial_x - \partial_x)\Phi\,. \end{aligned} \tag{2}$$

Due to the conformal invariance in 1+1 dimensions, the left-moving modes $(\partial_t + \partial_x v - \partial_x)\Phi = 0$ propagating against the frame-dragging velocity are decoupled from the right-moving solutions $\phi = (\partial_t + \partial_x v - \partial_x)\Phi$. However, an arbitrary modification of the dispersion relation would not preserve this decoupling in general. The choice

$$\Box_h = (\partial_t + \partial_x v)(\partial_t + v\partial_x) - h(\partial_x^2)\,, \tag{3}$$

for example, does not factorize for a general function h. On the other hand, if we modify the left and right-moving branch separately with an arbitrary function f via

$$\begin{aligned} \Box_f &= \left(\partial_t + \partial_x[1 + v + f(-\partial_x^2)]\right) \times \\ &\quad \left(\partial_t - [1 - v + f(-\partial_x^2)]\partial_x\right)\,, \end{aligned} \tag{4}$$

the left-moving modes $(\partial_t - [1-v+f(-\partial_x^2)]\partial_x)\Phi = 0$ are again decoupled from the right-moving solutions given by $\phi = (\partial_t - [1-v+f(-\partial_x^2)]\partial_x)\Phi$. The difference between the two options (3) and (4) scales with the derivative of $v(x)$ compared with the characteristic scale of the dispersion relation $[v(x), f(-\partial_x^2)]$, which is negligibly small in the hydrodynamic limit (which corresponds to the black hole being large). The coupling occurs at high frequencies over long scales (the scale of variation of v) which means the response will be adiabatic and left-moving waves will not convert to right-moving. This is the reason why the numerical simulations using the first option (3) never saw a mixing between left and right-movers one might expect from the coupling between the two sets of modes.

Derivation Even though the Schwarzschild metric corresponds to $v(x) = \pm\sqrt{2M/x}$, we consider two different velocity profiles $v(x) = -\lambda/x$ and $v(x) = \kappa x$ in the following, because they admit analytic solutions. The latter of course has no asymptotically flat region, and calling it a black, or "dumb", hole is metaphorical. Let us first study the case $v(x) = -\lambda/x$. After a Fourier-Laplace transformation with $\partial_x \to ik$ and $x \to i\partial_k$ as well as $\partial_t \to -i\omega$, the left-moving solutions satisfy the integral equation (since $\partial_k^{-1} = \int dk$)

$$\left(\omega - k\left[1 + i\lambda\partial_k^{-1} + f(k^2)\right]\right)\phi_\omega(k) = 0\,, \tag{5}$$

which can be solved via separation of variables and differentiation $\phi_\omega(k) = \exp\{-i\lambda\int dk'/g(k')\}/g(k)$ with the spectral function $g(k) = 1 + f(k^2) - \omega/k$. The inverse Fourier-Laplace transformation

$$\phi_\omega(r) = \int \frac{dk}{g(k)} \exp\left\{ikx - i\lambda\int\frac{dk'}{g(k')}\right\}\,, \tag{6}$$

yields the spatial modes $\phi_\omega(x)$ with the integration contour being determined by the boundary conditions. In the following, we shall assume the length scale λ on which $v(x)$ changes to be very large compared with the typical wavenumber of the dispersion relation $f(k^2)$ and the frequency ω. In terms of the fluid analogue, this is precisely the hydrodynamic limit – whereas, for real black holes, is corresponds to demanding that the size of the black hole is much larger than the Planck scale. Note, however, that we do not restrict ω relative to the Planck scale (i.e., ω could be Planckian). Since the exponent in the integral above contains the large numbers x and λ, it is very useful to deform the integration contour into the complex plane (assuming that f is an analytic function), where the leading contributions will be determined by singularities and the associated branch cuts as well as saddle points. The saddle points (stationary phase)

$$x = \frac{\lambda}{g(k)} \rightsquigarrow \omega = k\left[1 - \frac{\lambda}{x} + f(k^2)\right]\,, \tag{7}$$

are solutions of dispersion relation $\omega + vk = k[1 + f(k^2)]$. For large wavenumbers $|k| \gg \omega$, we get pairs of saddle points $k_\pm$ satisfying $f(k_\pm^2) = \lambda/x - 1$. The singularities of the integrand at $g(k) = 0$ correspond to solutions of dispersion relation far away from the black hole $x \to \infty$. Assuming simple poles at k_α only, we may employ the residual expansion

$$\frac{1}{g(k)} = \sum_\alpha \frac{c_\alpha}{k - k_\alpha} \leftrightarrow c_\alpha = \frac{1}{2\pi i}\oint\limits_{\mathfrak{C}_\alpha}\frac{dk}{g(k)} = \frac{k_\alpha}{v_{\rm gr}(k_\alpha)}\,. \tag{8}$$

The contours $\mathfrak{C}_\alpha$ denote small circles around the poles at k_α and the residual coefficients c_α are related to the group velocity at these points. Insertion of the residual expansion (8) into Eq. (6) yields

$$\phi_\omega(x) = \int\frac{dk}{g(k)}\,e^{ikx}\prod_\alpha (k - k_\alpha)^{-i\lambda c_\alpha}\,. \tag{9}$$

Consequently, there are branch cuts starting from each singularity unless $2iMc_\alpha \in \mathbb{Z}$, cf. Fig. 1.

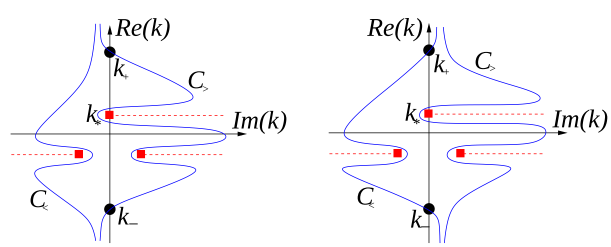

FIG. 1: [Color online] Sketch (not to scale) of integration contours in the complex plane for the sub-luminal (left) and the super-luminal case (right). The dots denote saddle points $k_\pm$ at the real axis and the squares are the singularities k_α with the associated branch cuts (dashed lines).

Sub-luminal case In order to determine the most suitable integration contour in the complex plane, we have to incorporate some assumptions about the function $f(k^2)$. First, we assume that the asymptotic ($x \to \infty$) dispersion relation $\omega = k[1 + f(k^2)]$ is convex, i.e., sub-luminal $f(k^2) < 0$, and always monotonically increasing, i.e., with a positive group velocity. In this case, saddle points in Eq. (7) occur outside the horizon $x > \lambda$ and yield the following contributions to the integral in Eq. (6)

$$\phi_\omega^\pm(x) \approx \sqrt{\frac{2i\pi}{\lambda g'(k_\pm)}}\, e^{ik_\pm x} \prod_\alpha (k_\pm - k_\alpha)^{-i\lambda c_\alpha} \ . \quad (10)$$

In view of their spatial behavior, these are the positive ϕ_ω^+ and negative ϕ_ω^- pseudo-norm in-modes with large wavenumbers $|k_\pm| \gg \omega$ and therefore small group velocities, which are swept towards the horizon. After quantizing the field Φ, these positive/negative pseudo-norm solutions yield the creation/annihilation operators of the in-modes at large k.

In order to close the integration contour, we have to circumvent the branch cuts in the upper complex half plane $\Im(k) > 0$, cf. Fig. 1. In the limit $x \to \infty$, the contributions of the branch cuts starting from the singularities k_α away from the real axis $\Im(k_\alpha) > 0$ are exponentially suppressed and only the branch cut starting at the real axis $\Im(k_*) = 0$ contributes. This wavenumber k_* represents a real solution of the dispersion relation far away from the horizon and just corresponds to the outgoing Hawking radiation with frequency $\omega > 0$. Hence, at $x \to \infty$, the integral in Eq. (6) corresponding to the contour $\mathfrak{C}_>$ in Fig. 1 yields a superposition of the outgoing Hawking modes with k_* and the large-k in-modes in (10) with $k_\pm$. Continuing this solution beyond the horizon $x < \lambda$, the saddle points vanish and the integrand in (10) decays exponentially in the lower complex half plane $\Im(k) < 0$. Thus, we deform the integration contour to $\mathfrak{C}_<$, where the main contribution stems from the branch cut(s) starting at $\Im(k_\alpha) < 0$. Again, for large λ, these contributions are exponentially suppressed.

Therefore, the outgoing Hawking mode with k_* originates entirely from the in-modes in (10). Assuming that the initial quantum state is the ground state of the large-k modes in (10) with respect to the freely falling observer, the amount of created particles is then determined by the mixture between these positive and negative pseudo-norm solutions with large k contained in the outgoing k_*-modes after the immense gravitational red-shift near the horizon. In view of $|k_\alpha| \ll |k_\pm|$, the only difference between positive and negative pseudo-norm modes is caused by branch cut(s)

$$\left|\frac{\phi_\omega^+(x)}{\phi_\omega^-(x)}\right| \approx \exp\left\{\pi\lambda \sum_\alpha (-1)^{s_\alpha} \Re(c_\alpha)\right\} \ , \quad (11)$$

where $(-1)^{s_\alpha}$ is the sign associated with the direction of the branch cut. Since $g(k)$ is a real function, the singularities k_α occur symmetric w.r.t. the real axis $k_\alpha \to k_\alpha^*$ and $c_\alpha \to c_\alpha^*$. Choosing the branch cuts suitably (see Fig. 1), the contributions from the symmetric pairs cancel each other and hence only the singularity at the real axis $k_* = k_\alpha \in \mathbb{R}$ contributes. Together with the unitarity relation $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$, the ratio (11) directly determines the size of the Bogoliubov coefficients via $|\alpha_\omega/\beta_\omega|^2 = |\phi_\omega^+/\phi_\omega^-|^2 = \exp\{\omega/T\}$. Hence we may read off the effective Hawking temperature

$$T_{\rm Hawking}(\omega) = \frac{v_{\rm gr}(k_*) v_{\rm ph}(k_*)}{2\pi\lambda} \, . \quad (12)$$

We observe that the geometric mean of group and phase velocity [8] evaluated at the frequency ω of the outgoing radiation far away $x \to \infty$ determines the Hawking temperature [9]. Therefore, the behavior of the dispersion relation at large k is not relevant – even though the Hawking radiation originates from large-k modes – which indicates that the Hawking effect is basically a low-energy phenomenon. The ω-dependence of the Hawking temperature can be explained by the fact that high-energy wave-packets have a different group velocity than those at low energy and hence the various modes "see" different horizons and thus other values for the surface gravity, i.e., velocity gradient $dv/dx = \lambda/x^2 \propto v^2/\lambda$.

This explanation has been confirmed by numerical simulations [9] and can further be supported by considering the second case $v(x) = \kappa x$, where the velocity gradient κ is constant. In this case, Eq. (6) should be replaced by

$$\phi_\omega(x) = \int dk\, k^{-i\omega/\kappa} \exp\left\{ikx - i\kappa \int dk' f(k'^2)\right\} . \quad (13)$$

Hence the weight of the branch cut starting at $k = 0$ is just determined by the ratio ω/κ and the Hawking temperature does not depend on ω at all because all modes "see" the same surface gravity κ

$$T_{\rm Hawking} = \frac{\kappa}{2\pi} = \text{const} \, . \quad (14)$$

Super-luminal case A super-luminal dispersion relation $f(k^2)$ can be treated in a completely analogous way. As the only difference, the large-k in-modes determined by the saddle points $k_\pm$ originate from inside the horizon $x < \lambda$ and thus the contours in the complex plane ($\mathfrak{C}_>$ for $x > \lambda$ and $\mathfrak{C}_<$ for $x < \lambda$, cf. Fig. 1) are slightly different.

In view of this observation, one might wonder whether the ansatz $v(x) = -\lambda/x$ instead of $v(x) = \pm\sqrt{2M/x}$ is justified. To address this question, let us consider the Schwarzschild geometry in 1+1 dimensions using the Eddington-Finkelstein coordinates (V, r)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dV^2 - 2dV\, dr \, . \quad (15)$$

For a massless scalar field Φ, the wave equation reads

$$\left(2\partial_V \partial_r + \partial_r \left[1 - \frac{2M}{r} + f(-\partial_r^2)\right] \partial_r\right) \Phi = 0 \, , \quad (16)$$

where we have again included a modification $f(k^2)$ of the dispersion relation. Comparison with the previous derivation yields completely the same results for the outgoing solutions $\phi = \partial_r \Phi$ up to the replacement $\omega \to 2\omega$ due to the Eddington-Finkelstein coordinate $t \to V$.

However, in the super-luminal case, an additional complication may arise: According to Eqs. (8) and (11), the thermal Boltzmann factor $\exp\{\omega/(4T)\}$ determining the amount of created particles with frequency ω can be recast into the alternative form $\exp\{2\pi M k_*/v_{\rm gr}(k_*)\}$. Hence, if the group velocity grows slower than linear in k, the number of produced particles decreases with energy. However, if $v_{\rm gr}(k)$ rises too fast in some k-region, the amount of created particles drops at low k (where $v_{\rm gr} = 1$) but later increases again! In such an extremal case, the Hawking radiation could contain a large contribution of ultra-high energy particles ("ultra-violet catastrophe"). Going a step further and taking the dispersion relation seriously as a model for ultra-high energy deviations from general relativity [10], one would exclude such a case in view of our observational evidence for the existence of black holes with macroscopic life-times.

Let us discuss some examples: The dispersion relation $\omega^2 = k^2 + k^4$ which is realized for the sonic black-hole analogues in Bose-Einstein condensates [11], does not generate an "ultra-violet catastrophe" and reproduces Hawking's prediction. In contrast, the expressions $\omega^2 = \exp\{k^2\} - 1$ or $\omega = k/\sqrt{1-k^2}$ grow too fast and hence lead to the aforementioned problems [12].

Note that the condition $v_{\rm gr}(k_*) \geq \mathcal{O}(Mk_*)$ for particle creation obtained from the Boltzmann factor precisely marks the break-down of the saddle-point (i.e., geometric optics) approximation. Writing the integrand in Eq. (6) as $G(k)\exp\{F(k)\}$, the first-order corrections to the saddle-point expansion scale as $G'/G \times F^{(3)}/(F'')^2$, $F^{(4)}/(F'')^2$, and $G''/(GF'')$, evaluated at the saddle point $F' = 0$. In our case (6), we have $F'' = -2iMG'$ and hence inserting $v_{\rm gr}(k_*) \geq \mathcal{O}(Mk_*)$ yields "corrections" of order one – i.e., the saddle-point approximation fails. Even if the modes started out in their ground state (at $k_\pm$), they get excited (at k_*) due to the strong gravitational red-shift. Based on these general adiabaticity arguments, one would expect that the main result remains valid even beyond a sole modification of the dispersion relation: If the outgoing Hawking modes originate from the vicinity of the singularity and the spectral properties of quantum gravity change too fast with energy, one would expect a break-down of adiabaticity at short distances resulting in the emission of high-energy particles. This mechanism is not necessarily restricted to a Planck-length vicinity of the singularity since the effective surface gravity "seen" by the high-energy modes scales with $2M/r^2$ and hence may exceed the Planck temperature already many Planck lengths away from the singularity.

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