**CENTRE DE PHYSIQUE THÉORIQUE**
**CNRS - Luminy, Case 907**
**13288 Marseille Cedex 9**



# Geometries and Forces

**Thomas SCHÜCKER** [1]

**Abstract**

The present status of Connes' noncommutative view at the four forces is reviewed.

PACS-92: 11.15 Gauge field theories, 04.20 General relativity
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[1] schucker@cpt.univ-mrs.fr, also at Université de Provence

# Contents

# Chapter 1

# Two dreisätze for Maxwell

The dream is older than Democrite: describe the universe as a game of Lego: a few 'elementary' particles are held together by a few 'elementary' forces. The universe is complicated, the dream naive. Still, it has seen impressive successes explaining e.g. our solar system, chemical elements, light, the hydrogen atom, nuclear reactions. To avoid misunderstanding, the successes were on purely scientific level, often followed by human failure. The aim of these notes is to review today's version of the game and Connes' attempt to understand its rules as geometry.

## 1.1 A qualitative vocabulary

Today we believe that there are four forces: gravity, electromagnetism, weak and strong forces.

Gravity describes the falling apple, the motion of earth around the sun, the dynamics inside a galaxy and maybe even the dynamics of galaxies. But the last item is the cosmological part of theology. In any case all items are macroscopic phenomena and we do not know of any microscopic manifestation of gravity. What is more, we have so far no consistent quantum theory of gravity.

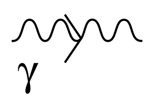


Figure 1.1: A photon from $\gamma$ decay

Electromagnetism describes, on the macroscopic side, e.g. electric generators and motors, light, radio transmission. On quantum level, it is responsible for $\gamma$ decay, bremsstrahlung, and pair creation. The first refers to an unstable nucleus, a bound state of protons and neutrons. The protons and neutrons rearrange by emitting a photon with enormous energy, figure 1.1. This photon is a killer and you better hide behind a solid screen of lead. Bremsstrahlung says that a high energy electron for instance from a $\beta$ decay may slow down by emitting a high energy photon, figure 1.2. To protect yourself against these electrons typically a layer of cheap plexiglass is sufficient. Bremsstrahlung makes radioprotection expensive, before getting stuck in the plexiglass, the electron emits a photon that goes through plexiglass like through butter and you better buy lead. Pair creation is a process where a photon traveling with sufficient energy changes into an electron and a positron, figure 1.3. With it, quantum electrodynamics teaches us two important lessons: even an 'elementary' particle, here the photon, may be unstable, it may change identity or said differently it may *decay*. This makes quantum field theory so complicated. Fortunately the decays are not arbitrary. They are governed by precise laws, e.g. conservation laws for which group theory will play a fundamental role. An important task for physicists is to compute life times and branching ratios from these laws and to confront the numbers with experiment. What is a branching ratio in our example of a decaying photon? If the photon has enough energy it may decay into any pair of a charged particle and antiparticle. The branching ratios are the corresponding probabilities. The word interaction is often used instead of force to underline that now a force not only changes the state of motion of the concerned particles but also their identity, their state in an *internal space*. We owe the second lesson to Dirac who generalized Schrödinger's equation to high energies, that means to special relativity or Minkowskian geometry. This generalization forces the introduction of antimatter. To every particle there must exist an antiparticle, with same mass and spin but with opposite charges. For instance, the antiparticle of the electron is the positron. Electromagnetism is *the* show off theory of physics. It is successful both on macroscopic and quantum level, it operates with clean mathematics and has many applications to every day life. It should be used to set the scale of success in our Lego game.

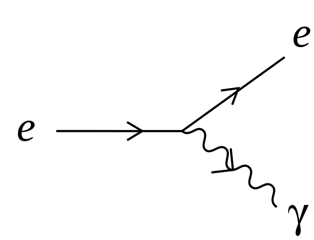


Figure 1.2: Bremsstrahlung

Weak interactions describe the $\beta$ decay and they are popular since Chernobyl. Take an iodine-131 isotope. It is a bound state of 53 protons and 78 neutrons. One of the neutrons changes identity. It decays to a proton, an electron and an anti-neutrino, figure 1.4. The proton is heavy and lazy. It stays in the nucleus which becomes a Xenon-131. The neutrino has zero mass and zero electric charge and is therefore harmless for man. It can pass through the entire earth without losing energy. The damage is done by the electron that deposits its

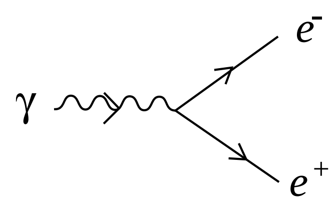


Figure 1.3: Pair production

energy in the immediate vicinity of its point of decay. This is e.g. the thyroid of babies where iodine likes to accumulate [1]. Let us be macabre and note an academic property of the killer electron: its *chirality*. The electron goes at almost the speed of light and it has spin 1/2. Quantum mechanics tells us, that in this situation, there are only two possibilities, the spin is parallel to its velocity, the electron has chirality *left* or the spin is anti-parallel to its velocity, the electron has chirality *right*. Here comes the surprising observation, the electron from $\beta$ decay is always left-handed. The spin is a vector describing the axis of rotation of the electron around itself. Therefore the spin is an *axial* vector, a vector that changes sign under *parity*, space reflection. Weak interactions break parity maximally, you never observe a right-handed electron or neutrino coming out of a $\beta$ decay. The (electric) charge of a particle indicates to what extent it is subject to the electric force. Likewise there is the weak charge called *(weak) isospin*. The left-handed electron and the left-handed neutrino have non-vanishing isospin. The right-handed electron has zero isospin. A right-handed neutrino has never been observed. If the neutrino has no right-handed part then it must be massless, in agreement with observation.

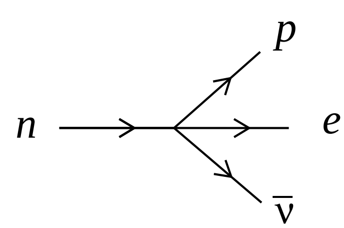


Figure 1.4: A $\beta$ decay

We do not know of any macroscopic manifestation of weak forces. We will try to understand this with the help of spontaneous breaking of gauge symmetry that will be in the center of our discussion. Let us anticipate a little. Maxwell tells us that the electromagnetic force between two charged particles results from another particle being exchanged between them. This particle, the photon or generically the gauge boson, has spin 1 and is massless. The gauge symmetry implies that the gauge boson is massless, which in turn implies that the force is long range, falls off like the inverse square of the distance. Weak interactions are also mediated by gauge bosons, the $W$ or weak boson. In fact, the neutron in the example from Chernobyl first decays into a $W$ and the proton. Then the $W$ decays into a neutrino and an electron like in pair creation from the photon. The $W$ has spin 1 as the photon, but must be very massive to render the weak interactions short range in accordance with experiment.

The strong force was invented to bind protons and neutrons inside the nucleus. Protons have electric charge and according to Coulomb's law they repel each other with an electric force that increases as the inverse square of their distance. The size of the nucleus being only $10^{-15}$ meters, we must invoke a strong force to explain the stability of the nucleus. Once accepted, the strong force also explains $\alpha$ decay, that is the emission of a helium nucleus, two protons and two neutrons, from a heavy nucleus like plutonium-239. Moreover, the strong force explains fusion and fission and thereby the energy production in the sun, in diverse nuclear bombs and in nuclear 'facilities'. Again we have to face the question, why do we see no macroscopic manifestation. The gauge bosons of the strong force are called gluons and they are massless. Nevertheless the strong force is short range because of *confinement*. Confinement has so far resisted every attempt of proof. All we have is clue from perturbation theory indicating that the strong force decreases with energy, *asymptotic freedom*. Extrapolating to low energies we do assume an extremely strong static force law that confines all particles with nonvanishing strong charge, called *colour*. The idea then is that the proton and the neutron are colourless bound states of three coloured quarks. Quarks are supposed elementary. We have the *up* quark with electric charge 2/3 (in units of the absolute value of the electron charge) and the *down* quark of charge $-1/3$. The quarks carry also the strong charge, colour. On the other hand the proton, a *uud* bound state, and the neutron, *udd*, are colourless and therefore they can be isolated. The force tying protons and neutrons to a nucleus are imagined of van der Waals' type and consequently short range. When in our Chernobyl example one of the neutrons inside the iodine nucleus suffers $\beta$ decay to a proton and a $W$, it is in fact one of the two *down* quarks in this neutron that decays to an *up* quark and a $W^-$, figure 1.5. Just as gravity and electromagnetism, the strong force preserves parity, it is *vectorial*.

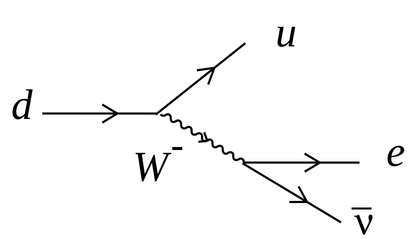


Figure 1.5: Same $\beta$ decay with better resolution

Let us recapitulate our elementary particles and start with the gauge bosons. They mediate the non-gravitational forces and have spin 1. There is the photon $\gamma$. Its mass, electric charge and colour all vanish, but not its isospin. There is the weak boson $W^{\pm}$. It is very massive, it has unit electric charge, non-zero isospin, no colour. A second weak boson, the $Z^0$ was discovered in the seventies. Its quantum numbers are as for the $W$ except for zero electric charge. Finally there are eight gluons, no mass, no charge, no isospin, but non-zero colour. Gravity is also mediated by a boson, the graviton. It is *not* a gauge boson, it has spin 2. It is massless and has no charge, no isospin, no colour. Let us anticipate that we shall need another boson, the Brout-Englert-Higgs scalar, with spin 0, no charge, no colour, but with isospin, and massive. We need it to give masses to bosons and fermions via spontaneous symmetry break down. We need it, but we have not seen it and it is the last missing particle.

All other elementary particles are fermions, they have spin 1/2. They fall into two classes, leptons and quarks. Leptons, from the greek word mild, do not participate in strong interaction, they are colourless. There is the (electronic) neutrino $\nu_e$, purely left-handed and therefore massless, no charge but isospin. The electron $e$ is massive, charge $-1$. Its left-handed part $e_L$ has isospin, its right-handed part $e_R$ has isospin 0. Confinement suggests that quarks (and gluons) will never be observed alone and today we only have indirect measurements of their quantum numbers. All quarks are thought to be massive, the $u$ has charge 2/3, the $d$ has $-1/3$. Their left-handed parts have isospin, the right-handed parts do not.

We are now ready for another mystery of particle physics. More elementary fermions have been observed that are boring copies of the above ones. Let us call the $(u,\ d,\ \nu_e,\ e)$ first generation. Then we have two more generations $(c,\ s,\ \nu_\mu,\ \mu)$ and $(t,\ b,\ \nu_\tau,\ \tau)$. The names of the quarks, *charm, strange, top, bottom*, recount well how the physicists felt about their discovery, estranged, charmed, blasé. Nature has simply copied the quantum numbers of the first generation, except for the masses, that remain a puzzle. The *top* is extremely heavy and consequently was the last to be seen, only two years ago. Experiments also indicate – of course only indirectly – that the *top* is the last, there should not be a fourth generation. Here is today's periodic table of elementary fermions:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

$$\begin{matrix} u_R, & c_R, & t_R, \\ d_R, & s_R, & b_R, \end{matrix} \quad e_R, \quad \mu_R, \quad \tau_R$$

The parentheses indicate *isospin doublets*, i.e. particles that can be produced pairwise from a decaying $W$.

## 1.2 The gauge dreisatz

In this section we want to be a little more quantitative about weak and strong charges. Behind the decay laws, there are conservation laws. Behind conservation laws, there is group theory by Emmy Noether's theorem. This is well known from electromagnetism. Maxwell's equations are invariant under the group $U(1)$. This invariance explains the experimentally well established charge conservation. For instance, the electricly neutral photon can only decay into an electricly neutral pair. Charge conservation also implies that the $W$ has unit charge. Moreover, the electromagnetic gauge group is Abelian. This implies that the photon has zero electric charge, that it does not itself feel the force which it mediates. On the other hand, we expect the weak and strong gauge groups to be non-Abelian. Maxwell's equations are also Lorentz invariant, if we suppose that the conserved electric charge is Lorentz invariant, i.e. does not depend on velocity. Then the photon must have spin 1, and likewise for weak and strong bosons. However, the gravitational 'charge' is the mass or more precisely energy, which is not a Lorentz scalar. Energy is a component of a four-vector in Minkowskian geometry. Therefore the graviton has spin 2. To cut a long story short here is today's credo for playing Lego:

- Elementary particles are orthonormal basis vectors of a unitary group representation. The group $G$ falls from heaven, most of the time.
- The charge parameterizes the choice of the representation.
- Composite particles are obtained from tensor products.

Wigner proposed the credo. His starting point, the Poincaré group or its spin cover, does not fall from heaven. It comes from Minkowskian geometry. The Poincaré group is non-compact and its unitary representations are infinite dimensional. They are characterized by a continuous variable, the mass, and a discrete one, the spin. The spin parameterizes the finite dimensional part of the representation under the compact subgroup $SU(2)$, the cover of the rotation group in three dimensional Euclidean space. We denote by $\underline{2j+1}$ the $2j+1$ dimensional irreducible representation of $SU(2)$. It has spin $j$, $j = 0,\ \frac{1}{2},\ 1, ...$ A composite particle consisting of two spin $\frac{1}{2}$ particles,

$$\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}, \tag{1.1}$$

can have spin 0, the antisymmetric part of the tensor product: the two spin $\frac{1}{2}$ are anti-parallel, or it can have spin 1, the symmetric part of the tensor product: the two spin $\frac{1}{2}$ are parallel. This is the Clebsch-Gordan decomposition and you must know that physicists working at CERN carry a pocket size table with two hundred Clebsch-Gordon coefficients [2].

Motivated from charge conservation and Emmy Noether, let us have $G = U(1)$ fall from heaven. Its irreducible, unitary representations are all one dimensional, $\mathcal{H} = \mathbb{C} \ni \psi$ with $\rho(\exp i\theta)\psi = \exp i(q/e)\theta\, \psi$. $q$ is the electric charge, it is additive under tensor products. Indeed,

$$\begin{aligned}(\rho_1 \otimes \rho_2)(\exp i\theta)(\psi_1 \otimes \psi_2) &= (\rho_1(\exp i\theta)\psi_1) \otimes (\rho_2(\exp i\theta)\psi_2) \\ &= (\exp i(q_1/e)\theta\, \psi_1) \otimes (\exp i(q_2/e)\theta\, \psi_2) = \exp i((q_1 + q_2)/e)\theta\, \psi. \end{aligned} \tag{1.2}$$

Heisenberg found that one dimensional representations are boring and tried $G = SU(2)$ which he called (strong) isospin in order to distinguish it from the spin $SU(2)$. Instead of spin up and spin down, he puts the proton and the neutron – or in today's picture the *up* and *down* quarks – in the $\underline{2}$. Gell-Mann was more successful with $G = SU(3)$ and discovered the first three quarks $(u, d, s)$ sitting in the fundamental representation, $\mathcal{H} = \mathbb{C}^3$, $\rho(g) = g$. Indeed, this hypothesis allowed him to classify the baryons and mesons of his time as bound states of three quarks or of quark-antiquark. Heisenberg's $SU(2)$ of strong isospin and Gell-Mann's $SU(3)$ of flavour should not be confused with the *gauged* $SU(2)$ of weak isospin and the *gauged* $SU(3)$ of coulour. The latter will play a fundamental role and generate the forces. At the same time they will allow to derive the non-gauged ones. Consequently, the non-gauged ones play a secondary role today and we mentioned them for historical reasons. They lead to the discovery of quarks and to the establishment of the credo. A newcomer should be warned however: the confusion is still present today and not only in the terminology isospin.

At the root of this confusion is the gauge miracle. The ungauged $U(1)$ of electric charge conservation can be gauged and its gauging produces electromagnetism.

Here is the story in short. The ungauged $U(1)$ of electric charge conservation does *not* fall from heaven, it is given to us free of charge by quantum mechanics, via the conservation of probability. The representation space of quantum mechanics is $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C})$. Its elements, complex valued, square integrable functions on our Euclidean space $\mathbb{R}^3$, are the wave functions, $\psi(\vec{x})$. A natural group of unitaries acts in this Hilbert space. Its elements are translations $U_{\vec{\xi}}$, rotations $U_R$ and phase transformations $U_{\exp i\theta}$,

$$(U_{\vec{\xi}}\psi)(\vec{x}) \;:=\; \psi(\vec{x} - \vec{\xi}), \tag{1.3}$$

$$(U_R\psi)(\vec{x}) \;:=\; \psi(R^{-1}\vec{x}), \tag{1.4}$$

$$(U_{\exp i\theta}\psi)(\vec{x}) \;:=\; \exp(i(q/e)\theta)\, \psi(\vec{x}). \tag{1.5}$$

Their generators are momentum, angular momentum and electric charge,

$$\frac{\partial}{\partial \vec{x}}, \quad \vec{x} \wedge \frac{\partial}{\partial \vec{x}}, \quad i1. \tag{1.6}$$

The associated conserved quantities have the same names. The free Schrödinger equation follows via the Euler-Lagrange variational principle from the action

$$\int \mathrm{d}t \int \mathrm{d}\vec{x}\ \bar{\psi}(t,\vec{x})\,(i\hbar\,\frac{\partial}{\partial t} + \tfrac{\hbar^2}{2m}\,\frac{\partial^2}{\partial \vec{x}^2}\,)\psi(t,\vec{x}). \tag{1.7}$$

Schrödinger's version of quantum mechanics treats space and time differently, the position $\vec{x}_{\rm op}$ is an observable, a Hermitian operator, $(\vec{x}_{\rm op}\psi)(t,\vec{x}) = \vec{x}\psi(t,\vec{x})$, and as such it has an uncertainty, time $t$ is just a parameter. $\hbar = 1.055 \cdot 10^{-34}$ m$^2$ kg/s is Planck's constant, most of the time we adopt units such that $\hbar = 1$. $m$ denotes the mass of the free 'matter'. Schrödinger's action is obviously invariant under phase transformations, in agreement with the postulate that only $|\psi|^2$ has physical significance: it is the probability density of location. However, the choice of phase must be rigid, constant over the entire universe. One might object that a physicists somewhere in Andromeda should be able to do his quantum mechanical calculations with a phase convention that should not be tied to a phase convention used by a colleague on Earth. This leads us to consider spacetime dependent phase transformations $\exp i\theta(t,\vec{x})$. They form the infinite dimensional gauge group or gauged $U(1)$. Its elements are functions from spacetime into $U(1)$ with pointwise multiplication. How can we render Schrödinger's action gauge invariant? The trick goes by the name of minimal coupling. Postulate the existence of a connection or gauge field $A_\mu$, $\mu = 0,1,2,3$ with the affine transformation law

$$\rho_V(\exp i\theta)A_\mu = A_\mu + i\tfrac{\hbar}{e}\exp i\theta\,\frac{\partial}{\partial x^\mu}\,\exp -i\theta = A_\mu + \tfrac{\hbar}{e}\,\frac{\partial}{\partial x^\mu}\,\theta \tag{1.8}$$

where we have put $x^0 = ct$. The subscript $V$ stands for vector because the gauge field is a vector field, it has spin 1. Now replace all derivatives $\frac{\partial}{\partial x^\mu}$ in the free action by covariant derivatives $\frac{\partial}{\partial x^\mu} + i\frac{q}{\hbar}A_\mu$ and you get a gauge invariant action. Physically the free matter particle, we started from, is now coupled to an electromagnetic field, 'radiation', whose vector potential is $A_\mu$. In a second stroke, we want to make the gauge field dynamical. We look for a kinetic term, i.e. a term involving derivatives of the gauge field, and that is gauge invariant. In lowest order, two derivatives, the answer is unique, it is Maxwell's action with the $1/r^2$ fall off in its static force field. Genesis is rewritten, *Let there be light* is to be replaced by *Let there be gauge* and we can summarize the gauge miracle in form of a *dreisatz* or *regra de três:*

- quantum mechanics + gauge invariance = Maxwell.

Note that this dreisatz works for non-relativistic quantum mechanics, Schrödinger, or relativistic quantum mechanics, Klein-Gordon and Dirac.

We anticipate that the gauge miracle also works for weak and strong interactions. *Let there be non-Abelian gauge.* However, the groups $G = SU(2)$, $SU(3)$, and their representations fall from heaven and we cherish the dream to derive them from first principles just as the $U(1)$ was derived from quantum mechanics. This is precisely what Connes proposes. *Let there be noncommutative geometry.* We end this section with a warning: there is a semantical ambiguity, should a particle be an entire representation space or only a vector therein? In Wigner's point of view, it is the same particle that can have different energies and spin orientations. An applied force can change the energy or spin orientation of a particle without changing its identity. Weak interactions force us to treat the different spin orientations or chiralities of the electron as different particles with different charges and at the same time forces us to allow for interactions that change the spin orientation in spacetime *and* the isospin orientation in an internal space. The latter is the change of identity in pair production.

## 1.3 The Minkowski dreisatz

Our next dreisatz is of even more geometric nature:

- Coulomb + Minkowskian geometry = Maxwell.

This dreisatz does not do justice to the historical development, Maxwell's theory existed when Einstein discovered special relativity and it came as surprise that Maxwell's theory was already Lorentz invariant.

We start with Coulomb's static force law,

$$F = \frac{1}{4\pi\epsilon_0}\,\frac{qQ}{r^2} \tag{1.9}$$

describing the force between two electric charges $q$ and $Q$ at rest at a distance $r$. The proportionality constant $\epsilon_0 = 8.8544 \cdot 10^{-12}\mathrm{s}^2\,\mathrm{C}^2/(\mathrm{m}^3\,\mathrm{kg})$ will be referred to as the inverse square of the *coupling constant*. In the following we will measure electric charge not in Coulomb C but in units of minus the electron charge, $e = 1.6021 \cdot 10^{-19}$ C. Note that this normalization can be changed at will, only the ratio $e^2/\epsilon_0$ is physical. Often we also use units of electric charge such that $\epsilon_0 = 1$. Then $e$ is the coupling constant. Now we perform a Lorentz boost

$$c\bar t = \frac{ct + vx/c}{\sqrt{1 - v^2/c^2}}\,, \quad \bar x = \frac{x + vt}{\sqrt{1 - v^2/c^2}}\,, \quad \bar y = y, \quad \bar z = z, \tag{1.10}$$

with the speed of light $c = 2.9979 \cdot 10^8$ m/s, and the magnetic field pops up. The force involving two time derivatives has a complicated transformation law under the Lorentz group and we take advantage of the fact that Coulomb's force derives from a potential,

$$\vec E = \vec F/q = -\frac{\partial}{\partial \vec x}V, \quad V = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{r} \tag{1.11}$$

In terms of the potential we can write Coulomb's law as the following differential equation

$$\frac{\partial^2}{\partial \vec{x}^2} V = -\rho/\epsilon_0, \tag{1.12}$$

where $\rho$ is the charge density. The Coulomb potential $V$, equation (1.11) is the elementary solution or Green function of the Laplacian for a pointlike source $\rho(\vec{x}) = \frac{Q}{\epsilon_0}\delta(\vec{x})$. The charge being a conserved quantity we must suppose that it is Lorentz invariant. Then the charge density transforms as the zero component $j^0 = c\rho$ of a four vector $j^\mu$ whose spatial components are the current density $\vec{j}$. Consequently, the lhs of the differential equation (1.12) must be a four vector. Therefore we introduce the (four) vector potential $A^\mu$ with $A^0 = V/c$. The force a 'test' charge or matter particle $q$ feels in the electromagnetic field is obtained from the static force law,

$$m \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = q\vec{E}, \tag{1.13}$$

by a Lorentz boost:

$$m \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = qF^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,. \tag{1.14}$$

We use Einstein's summation convention, summing over repeated indices is always understood. We denote by $\tau$ the Lorentz invariant proper time defined only on the trajectory $x^\mu(\tau)$ of the test particle by the implicit equation

$$c\tau = \int_0^\tau \left[ c^2 \left( \frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}} \right)^2 - \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tilde{\tau}} \cdot \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tilde{\tau}} \right]^{1/2} \mathrm{d}\tilde{\tau}, \tag{1.15}$$

or infinitesimally

$$c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 =: \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu =: \mathrm{d}x_\nu\,\mathrm{d}x^\nu. \tag{1.16}$$

The Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.17}$$

and its inverse $\eta^{\mu\nu}$ are used to lower and raise indices. We note that, without quantum mechanics, only derivatives of the potential $A_\mu$ are measurable. They are called field strength or curvature,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial_\mu := \frac{\partial}{\partial x^\mu}\,. \tag{1.18}$$

The field strength is an antisymmetric matrix made up from the electric field $\vec{E}$ and the magnetic field $\vec{B}$,

$$F_{\mu\nu} = -\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{1.19}$$

The important property of the field strength is that it is invariant under the gauge or phase transformations,

$$\rho_V(\exp i\theta)A_\mu = A_\mu + i\tfrac{\hbar}{e}\exp i\theta\,\partial_\mu \exp -i\theta = A_\mu + \tfrac{\hbar}{e}\partial_\mu\theta. \tag{1.20}$$

The force law for test charges (1.14) is nothing but the Lorentz force describing 'the coupling between matter and the electromagnetic field'. In a second stroke we want to generalize the static differential equation (1.12) that tells us that charge is the source of the electric field. We simply replace the potential $V$ by the four potential $A$, the charge density $\rho$ by the four current $j$ and the Laplace operator $\Delta = \partial^2/\partial\vec{x}^2$ by its Lorentz invariant extension, the wave or d'Alembert operator

$$\Box = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta =: \eta^{\mu\nu}\partial_\mu\partial_\nu =: \partial^\nu\partial_\nu. \tag{1.21}$$

Then we obtain Maxwell's equations

$$\Box\, A_\nu = \frac{1}{\epsilon_0 c^2}\, j_\nu \tag{1.22}$$

in the Lorentz gauge $\partial_\mu A^\mu = 0$. The gauge invariant Maxwell equations read:

$$\partial^\mu F_{\mu\nu} = \partial^\mu\partial_\mu A_\nu - \partial_\nu\partial_\mu A^\mu = \frac{1}{\epsilon_0 c^2}\, j_\nu. \tag{1.23}$$

They make the electromagnetic field dynamical, it propagates with the speed of light and therefore it is often called radiation. In particular, Maxwell's equation contains Ampère's static law,

$$\mathrm{rot}\vec{B} = \mu_0\vec{j}, \tag{1.24}$$

and we can identify the static magnetic coupling constant $\mu_0 = 1/(\epsilon_0 c^2)$.

Maxwell's equations derive from an action,

$$S[A] = -\int_{\mathbb{R}^4}\left(\frac{\epsilon_0 c}{4}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{c} j_\nu A^\nu\right)\mathrm{d}^4x. \tag{1.25}$$

If we measure electric charge in units of the electron charge then we must replace $\epsilon_0$ by $\epsilon_0/e^2$. The first term is manifestly gauge invariant, the second, the minimal coupling to matter, is gauge invariant thanks to charge conservation,

$$\partial_\nu j^\nu = 0. \tag{1.26}$$

Let us summarize our first geometric dreisatz: the extension of Coulomb's static force law with its coupling $\epsilon_0$ to Minkowskian geometry characterized by the speed of light $c$ produces an additional force, the magnetic force with feeble coupling $\mu_0$. Maxwell's theory is celebrated today as Abelian or should we say, commutative Yang-Mills theory. Historically, the chronological order was different. Both the static electric and magnetic forces where known, Maxwell unified them by rendering them dynamical. Plane waves came out as particular dynamical solutions to his equations and he found that the velocity of the waves, the speed of light, was $c = (\epsilon_0\mu_0)^{-1/2}$. At his time physicists still believed that the speed of light, like any velocity, depended on the reference system. But nobody really dared to object to Maxwell's relating the speed of light to static constants, experimentally Maxwell was right. Lorentz timidly introduced his transformations to understand the puzzle. Only Einstein dared to take the Lorentz transformations serious. He operated a revolution on spacetime, e.g. abolishing absolute time. His revolution is accessible to experimental verification, without talking about forces.

# Chapter 2

# A technical interlude on differential forms

The two dreisätze discussed so far call for differential forms, which will also make the generalization of Maxwell's theory to curved spacetimes easy. Here is a crash course on the local theory [3].

## 2.1 Vector fields

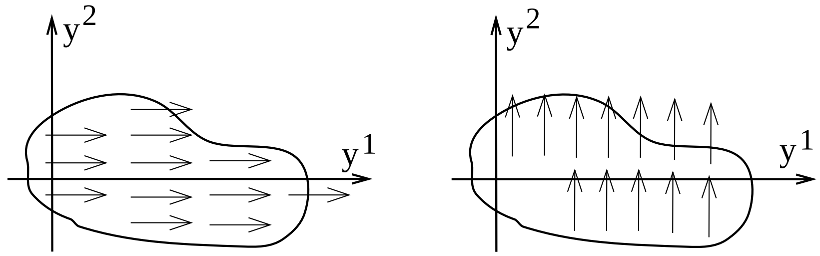


Figure 2.1: The Cartesian vector fields $\partial/\partial y^1$ and $\partial/\partial y^2$

Let $U$ be an open subset of $\mathbb{R}^n$. A vector field $v$ on $U$ is a differentiable family $v(x)$ of vectors in $\mathbb{R}^n$ indexed by the points in $U$. (For us, differentiable always means infinitely many times differentiable.) For example, $U$ is a lake and $v$ the wind. Note that the 'velocity' vectors $v(x)$ are not confined to lie in a subset of $\mathbb{R}^n$ as is the case for the points $x$. In Cartesian coordinates $y^\mu, \mu = 1, 2, \cdots, n$, any vector field may be decomposed:

$$v = \sum_{\mu=1}^{n} v^\mu(x) \frac{\partial}{\partial y^\mu}, \tag{2.1}$$

where $\partial/\partial y^\mu$ are the vector fields with Cartesian components $(0, \cdots, 0, 1, 0, \cdots, 0)$. The one is the $\mu$th entry. Figure 2.1 shows an example. Note that here $\partial/\partial y^\mu$ is not a differential operator, but just a symbol. Its mnemo-technical utility comes from the definition in arbitrary coordinates $x^\mu$:

$$\frac{\partial}{\partial x^\mu}(x) := \sum_\nu \frac{\partial y^\nu}{\partial x^\mu}(x) \frac{\partial}{\partial y^\nu} \tag{2.2}$$

where $\partial y^\nu / \partial x^\mu$is the Jacobian matrix of the (general) coordinate transformation. We shall consider explicitly the example of polar coordinates later

## 2.2 Differential forms

By definition a (differential) $p$-form $\varphi$ is a differentiable family of maps $\varphi_x$

$$\begin{array}{rcl} \varphi_x : \mathbb{R}^n \times \cdots \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (v_1(x), \cdots, v_p(x)) & \longmapsto & \varphi_x(v_1(x), \cdots, v_p(x)). \end{array} \tag{2.3}$$

Each map $\varphi_x$ is required to be multilinear (with respect to the real numbers) and alternating, i.e.

$$\varphi(\cdots, v_i, \cdots, v_j, \cdots) = -\varphi(\cdots, v_j, \cdots, v_i \cdots). \tag{2.4}$$

For convenience, we often suppress the point $x$. We denote by $\Omega^p U$ the set of all $p$-forms on $U$. Note that if $p > n$ this set only contains the zero element. For $p = 0$ we define $\Omega^0 U$ to be the set of all (differentiable) functions from $U$ into the real numbers.

## 2.3 Wedge product

The wedge product of a $p$-form with a $q$-form is the $(p+q)$-form defined by:

$$\begin{array}{rrcl} \wedge : & \Omega^p U \times \Omega^q U & \longrightarrow & \Omega^{p+q} U \\ & (\varphi, \psi) & \longmapsto & \varphi \wedge \psi \end{array}$$

$$(\varphi \wedge \psi)(v, ..., v_{p+q}) := \frac{1}{p!q!} \sum_{\pi \epsilon S_{p+q}} \mathrm{sig}\pi \quad \varphi(v_{\pi(1)}, ..., v_{\pi(p)}) \psi(v_{\pi(p+1)}, \cdots, v_{\pi(p+q)}), \tag{2.5}$$

where the sum is over all permutations of $p+q$ objects and sig$\pi$ is the sign of the permutation $\pi$.

The wedge product is bilinear, associative and graded commutative, i.e.

$$\varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi. \tag{2.6}$$

In any coordinate system $x^\mu$ a $p$-form may now be decomposed

$$\varphi = \sum_{\mu_1,\cdots,\mu_p} \varphi_{\mu_1,\cdots\mu_p}(x)\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}, \tag{2.7}$$

where for each $\mu = 1, 2, \cdots, n$, $\mathrm{d}x^\mu$ is the 1-form defined by

$$\mathrm{d}x^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_\nu. \tag{2.8}$$

In tensor language a vector field $v$ constitutes a contravariant tensor $v^\mu$ of degree (rank) one while a $p$-form constitutes a completely antisymmetric covariant tensor $\varphi_{\mu_1\cdots\mu_p}$ of degree $p$. The real number obtained by evaluating a $p$-form on $p$ vector fields corresponds to the complete contraction and the wedge product corresponds to the antisymmetrized tensor product of antisymmetric covariant tensors.

A collection of vector spaces $\Omega^p U, p = 0, 1, \cdots, n$, together with a bilinear, associative, graded commutative product $\wedge$ is also called exterior algebra or Grassmann algebra. Later, in order to alleviate notations, we shall suppress the wedge symbol.

## 2.4 Exterior derivative

We define the exterior derivative of a form using a coordinate system $x^\mu$:

$$\begin{array}{rcl} \mathrm{d}: \Omega^p U & \longrightarrow & \Omega^{p+1} U \\ \varphi & \longmapsto & \mathrm{d}\varphi \end{array}$$

$$\mathrm{d}\varphi := \sum_{\mu_1,\cdots,\mu_p,\nu} (\frac{\partial}{\partial x^\nu}\varphi_{\mu_1\cdots\mu_p})\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \tag{2.9}$$

This definition does not depend on the choice of the coordinate system $x^\mu$.

The exterior derivative is a linear first order differential operator. It obeys the Leibniz rule

$$\mathrm{d}(\varphi \wedge \psi) = (\mathrm{d}\varphi) \wedge \psi + (-1)^p \varphi \wedge \mathrm{d}\psi \tag{2.10}$$

and the so-called co-boundary condition

$$\mathrm{d}^2 = 0. \tag{2.11}$$

In tensor language the exterior derivative amounts to taking the gradient of an antisymmetric covariant tensor and then antisymmetrizing the covariant index of the gradient with the others. The co-boundary condition is just the statement that partial derivatives commute.

## 2.5 Integration

Let $\varphi$ be a $p$–form and $K$ a $p$–dimensional sufficiently regular piece of $U$ parameterized by $x^1, x^2, ..., x^p$, for example a cube. Then we define the integral of $\varphi$ over $K$:

$$\int_K \varphi := \int_K \varphi_{12...p}\, \mathrm{d}x^1 \cdots \mathrm{d}x^p, \tag{2.12}$$

where the rhs is just the multiple Riemannian integral of the coefficient function of $\varphi$. The increasing order of the indices in the coefficient function $\varphi_{12...p}$ means that we suppose a fixed numbering of the coordinates of $K$, i.e. an orientation. The definition of the integral of a form does not depend on the choice of the coordinate system. This is assured by the theorem that under a change of coordinates the integrand in the Riemannian integral changes with the absolute value of the determinant of the Jacobian matrix.

Let us mention **Stokes' theorem**: Let $\varphi$ be a $(p-1)$-form, $K$ a $p$–dimensional piece of $U$, $\partial K$ its properly oriented boundary. Then

$$\int_K d\varphi = \int_{\partial K} \varphi. \tag{2.13}$$

This theorem is useful to derive field equations from an action. Together with the Leibniz rule it allows to carry out partial integrations. Finally, we remark that the boundary of a boundary is empty,

$$\partial\partial K = \emptyset, \tag{2.14}$$

which explains the term co-boundary condition for $\mathrm{d}^2 = 0$.

## 2.6 Vector valued differential forms

Let $W$ be a finite dimensional real vector space. Since all operations introduced so far are linear, we can generalize the values of differential forms from the real numbers to vectors in $W$:

$$\Phi_x : \mathbb{R}^n \times ... \times \mathbb{R}^n \to W. \tag{2.15}$$

We denote by $\Omega^p(U, W)$ the set of $p$–forms on $U$ with values in $W$. In later applications $W$ will be a Lie algebra or a vector space carrying a linear representation of some symmetry group. With respect to a basis $T_a, a = 1, 2, ..., \dim W$, any element $w \in W$ can be written

$$w = \sum_{a=1}^{\dim W} w^a T_a, \tag{2.16}$$

where the $w^a$ are real numbers. Likewise any $p$–form $\Phi$ with values in $W$ can be written as

$$\Phi = \sum_a \varphi^a T_a, \tag{2.17}$$

where now the $\varphi^a$ are real valued differential forms on $U$. Of course, in order to define a wedge product in this more general setting, $W$ must have a multiplication law, i.e. $W$ must be an algebra. For example, if $W$ is a Lie algebra, we define the the commutator of a $p$– form and a $q$–form, both with values in the Lie algebra, by

$$[\Phi, \Psi](v_1, ..., v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \epsilon S_{p+q}} \operatorname{sig}\pi [\Phi(v_{\pi(1)}, ..., v_{\pi(p)}), \Psi(v_{\pi(p+1)}, ..., v_{\pi(p+q)})], \tag{2.18}$$

or with respect to a basis $T_a$:

$$\Phi = \sum_a \varphi_a T^a, \qquad \Psi = \sum_b \psi_b T^b,$$
$$[\Phi, \Psi] = \sum_{a,b} \varphi_a \wedge \psi_b [T^a, T^b]. \tag{2.19}$$

The commutator of forms is graded commutative:

$$[\Phi, \Psi] = -(-1)^{pq} [\Psi, \Phi], \tag{2.20}$$

where one minus sign comes from the anticommutativity of the commutator of two Lie algebra elements and the others from equation (2.6).

## 2.7 Frames

A frame on an open subset $U$ of $\mathbb{R}^n$ is a set of $n$ vector fields $b_1, b_2, ..., b_n$ such that in each point $x \in U$ the $n$ vectors $b_1(x), ..., b_n(x)$ are linearly independent. Other words used for frames are tetrads (for $n = 4$), vielbein or $n$–bein, repère (mobile). If $x^\mu$ is a coordinate system, then $\partial/\partial x^\mu, \mu = 1, 2, ..., n$, is a frame. However, not every frame $b_i$ can be derived from a coordinate system and we call a frame of the particular kind $\partial/\partial x^\mu$ holonomic. Later we shall learn a recipe how to decide whether a frame is holonomic.

Given two frames $b_i$ and $b'_i$ on $U$, we can always at a given point $x$ expand one in terms of the other:

$$b'_i(x) = \sum_j (\gamma^{-1}(x))^j{}_i b_j(x), \tag{2.21}$$

where $\gamma^{-1}(x)$ is an invertible $n \times n$ matrix:

$$\gamma^{-1}(x) \in GL_n. \tag{2.22}$$

Both frames depend differentiably on $x$ and so does $\gamma^{-1}(x)$, i.e. $\gamma^{-1}$ is a differentiable function from $U$ into $GL_n$. The set of all such functions forms a group where the multiplication is defined pointwise by the matrix product. We call this group the $GL_n$ gauge group

$$^U GL_n = \{\gamma : U \to GL_n\}. \tag{2.23}$$

A dual frame (or simply frame, when there is no risk of confusion) is a set of $n$ 1–forms $\beta^1, \beta^2, ..., \beta^n$ such that for every $x \in U$ $\beta^1(x), \beta^2(x), ..., \beta^n(x)$ are linearly independent. A frame is called holonomic if it is of the form $\mathrm{d}x^\mu$ where $x^\mu$ is a coordinate system.

**Theorem:** Let $U$ be simply connected. Then the frame $\beta^i$ is holonomic if and only if

$$\mathrm{d}\beta^i = 0 \tag{2.24}$$

for $i = 1, 2, ..., n$.

A dual frame $\beta^i$ is called dual to a frame $b_i$ if

$$\beta^i(b_j) = \delta^i_j \tag{2.25}$$

for $i, j = 1, 2, ..., n$ and a frame is holonomic if and only if its dual frame is holonomic. If two frames $b_i$ and $b'_i$ are related by the gauge transformation $\gamma^{-1}$, equation (2.21), their corresponding dual frames are related by the inverse transposed gauge transformation:

$$\beta'^i = \sum_j \gamma^i{}_j \beta^j, \tag{2.26}$$

transposed because of the 'wrong' order of the indices in equation (2.21). Our convention is that the first index of a matrix counts the rows, the second index the columns, irrespective of whether the indices are upper or lower.

As an example let us consider three-dimensional polar coordinates, $U$ is $\mathbb{R}^3$ without the $x - z$ half plane:

$$U = \mathbb{R}^3 - \{(x, y, z), x \geq 0, y = 0\}. \tag{2.27}$$

Let $b_i$ be the holonomic frame of Cartesian coordinates,

$$b_1 = \frac{\partial}{\partial x}, \quad b_2 = \frac{\partial}{\partial y}, \quad b_3 = \frac{\partial}{\partial z}, \tag{2.28}$$

and $b'_i$ the holonomic frame of polar coordinates,

$$b'_1 = \frac{\partial}{\partial r}, \quad b'_2 = \frac{\partial}{\partial \varphi}, \quad b'_3 = \frac{\partial}{\partial \theta}, \tag{2.29}$$

with

$$
\begin{aligned}
x &= r\cos\varphi\sin\theta && (2.30)\\
y &= r\sin\varphi\sin\theta && (2.31)\\
z &= r\cos\theta. && (2.32)
\end{aligned}
$$

In order to calculate the gauge transformation $\gamma$ relating the two frames we use the definition (2.2):

$$
\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}, \tag{2.33}
$$

and two similar identities; $\gamma^{-1}$ is just the Jacobian matrix of equations (2.30-2.32):

$$
\gamma^{-1} = \begin{pmatrix} \cos\varphi\ \sin\theta & -r\ \sin\varphi\ \sin\theta & r\ \cos\varphi\ \cos\theta \\ \sin\varphi\ \sin\theta & r\ \cos\varphi\ \sin\theta & r\ \sin\varphi\ \cos\theta \\ \cos\theta & 0 & -r\ \sin\theta \end{pmatrix}. \tag{2.34}
$$

The corresponding holonomic dual frames are given by

$$
\beta^1 = \mathrm{d}x, \quad \beta^2 = \mathrm{d}y, \quad \beta^3 = \mathrm{d}z, \tag{2.35}
$$

and

$$
\beta'^1 = \mathrm{d}r, \quad \beta'^2 = \mathrm{d}\varphi, \quad \beta'^3 = \mathrm{d}\theta. \tag{2.36}
$$

Using equation (2.26) we then find

$$
\mathrm{d}x = \frac{\partial x}{\partial r}\mathrm{d}r + \frac{\partial x}{\partial \varphi}\mathrm{d}\varphi + \frac{\partial x}{\partial \theta}\mathrm{d}\theta, \tag{2.37}
$$

and similar equations for $\mathrm{d}y$ and $\mathrm{d}z$.

## 2.8 Metrics on a vector space

Let $V$ be an $n$–dimensional real vector space. A (pseudo–)metric (or scalar product) on $V$ is a bilinear form

$$
\begin{aligned}
g : V\times V &\longrightarrow \mathbb{R}\\
(v,w) &\longmapsto g(v,w) && (2.38)
\end{aligned}
$$

which is symmetric:

$$
g(v,w) = g(w,v) \quad \text{for all} \quad v,w\in V \tag{2.39}
$$

and nondegenerate. The last requirement means that only the zero vector has vanishing scalar product with all vectors in $V$. If $b_1, b_2, ..., b_n$ is a basis of $V$, then due to the bilinearity the metric $g$ is uniquely specified by the $n \times n$ matrix of scalar products of the basis vectors:

$$g_{ij} := g(b_i, b_j). \tag{2.40}$$

The symmetry and nondegeneracy of $g$ imply that the matrix of $g$ with respect to the basis is symmetric and nondegenerate:

$$\begin{aligned} g_{ij} &= g_{ji}, \\ \det(g_{ij}) &\neq 0. \end{aligned} \tag{2.41}$$

The matrix ${g'}_{ij}$ of the metric $g$ with respect to a different basis ${b_i}'$,

$$b'_i = \sum_i (\gamma^{-1})^j{}_i b_j, \tag{2.42}$$

is given by

$$g'_{ij} := g(b'_i, b'_j) = (\gamma^{-1T} g \gamma^{-1})_{ij}. \tag{2.43}$$

Note here that we use $n \times n$ matrices to describe a change of coordinates as well as a metric, two quite different mathematical objects.

The following two theorems of linear algebra are of fundamental importance for us.

**Theorem (Gram & Schmidt):** Any metric has an orthonormal basis $e_i$, i.e. a basis such that

$$g(e_i, e_j) = \eta_{ij} := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}. \tag{2.44}$$

**Theorem (Sylvester):** The number $r$ of plus signs and the number $s$ of minus signs, $r+s=n$, does not depend on the choice of the orthonormal basis $e_i$.

From now on we shall reserve the letter $e$ for an orthonormal basis. Of course, an orthonormal basis is not unique, for instance

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.45}$$

and

$${e'}_1 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad {e'}_2 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ +1 \end{pmatrix} \tag{2.46}$$

are both orthonormal for the Euclidean metric of $\mathbb{R}^2$. In general, given an orthonormal basis $e_i$, any other basis $e'_i$ with

$$e'_i = \sum_j (\Lambda^{-1})^j{}_i e_j, \quad \Lambda \in GL_n, \tag{2.47}$$

is also orthonormal if and only if

$$\eta = \Lambda^{-1T} \eta \Lambda^{-1}. \tag{2.48}$$

The set of all $\Lambda$'s satisfying this condition forms a subgroup of $GL_n$, the Lorentz group denoted by $O(r, s)$. It is of dimension $\frac{1}{2}n(n-1)$.

There are two ways to parameterize all possible metrics with given signature $(r, s)$ on $V$.
**(i)** Choose a fixed basis $b_i$ of $V$. Then any metric is parameterized by the symmetric $n \times n$ matrix $g_{ij}$ of scalar products, that is $\frac{1}{2}n(n+1)$ real numbers.
**(ii)** Given any metric, choose an orthonormal basis $e_i$. This basis characterizes the metric as well. With respect to the fixed basis $b_i$, the $e_i$ are parameterized by the $n \times n$ matrix $\gamma^{-1}$ consisting of $n^2$ numbers. However, any other basis obtained from $e_i$ by a Lorentz rotation describes the same metric. Therefore we have to subtract from $n^2$ the number of dimensions of the Lorentz group $\frac{1}{2}n(n-1)$ yielding again

$$n^2 - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}n(n+1). \tag{2.49}$$

Being nondegenerate a metric $g$ on a vector space $V$ induces a canonical metric $g^*$ on the dual vector space $V^*$: Let $\beta^i$ be the basis of $V^*$ dual to the basis $b_i$:

$$\beta^i(b_j) = \delta^i_j. \tag{2.50}$$

Define a metric on $V^*$ by

$$g^*(\beta^i, \beta^j) = (g_{ij})^{-1}. \tag{2.51}$$

This metric is canonical, i.e. it does not depend on the choice of the basis $b_i$.

It follows that the dual basis of an orthonormal basis $e_i$ of $V$ is itself orthonormal with respect to $g^*$, because $\eta$ is its own inverse. Attention, in the following we denote an orthonormal basis of $V^*$ by $e^i$, only the position of the index distinguishes basis from dual basis.

## 2.9 Metrics on an open subset of $\mathbb{R}^n$

We defined a vector field on an open subset $U$ of $\mathbb{R}^n$ as a differentiable family of vectors indexed by the points $x$ of $U$. Likewise we now define a metric $g$ on $U$ to be a differentiable family $g_x$ of vector space metrics. With respect to a frame $b_i(x)$ this family is described by the symmetric $n \times n$ matrix

$$g_{ij}(x) := g_x(b_i(x), b_j(x)) \tag{2.52}$$

whose elements are real valued functions on $U$. For convenience we shall often suppress the $x$'s in the following.

Since the orthonormalization procedure by Gram and Schmidt only involves addition, multiplication and division, that is differentiable operations, it also immediately guarantees the existence of orthonormal frames $e_i(x)$,

$$g_x(e_i(x), e_j(x)) = \eta_{ij}, \tag{2.53}$$

with $x$-independent rhs.

A frame may now have two nice properties: being holonomic or being orthonormal. As often in life we can have both only in trivial situations.

**Theorem:** An open subset $U$ of $\mathbb{R}^n$ admits a holonomic and orthonormal frame if and only if it is flat.

We do not yet have a definition of flatness, but it is sufficient to take the naive sense of the word, for instance meaning that the angles of a triangle add up to 180º.

Let us return to our example of $\mathbb{R}^3$ minus a half plane and endow it with the Euclidean metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.54}$$

with respect to the Cartesian holonomic frame, which is therefore also orthonormal. On the other hand, the polar holonomic frame is not orthonormal:

$$g'_{ij} = (\gamma^{-1T}\, 1\, \gamma^{-1})_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \tag{2.55}$$

or in the dual frame

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.56}$$

and

$$g'^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} \sin^{-2}\theta & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}. \tag{2.57}$$

To have a non-flat example consider a piece of the unit sphere, $r = 1$. It is an open subset of $\mathbb{R}^2$ parameterized by $\varphi$ and $\theta$. Its metric is given by

$$g^{ij} = \begin{pmatrix} \sin^{-2}\theta & 0 \\ 0 & 1 \end{pmatrix} \tag{2.58}$$

with respect to the holonomic frame $\mathrm{d}\varphi, \mathrm{d}\theta$. An orthonormal frame is for instance

$$e^1 = \sin\theta \mathrm{d}\varphi \ , \ e^2 = \mathrm{d}\theta. \tag{2.59}$$

It is not holonomic:

$$\mathrm{d}e^1 = \mathrm{d}(\sin\theta d\mathrm{d}\varphi) = \cos\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi \neq 0. \tag{2.60}$$

We will show in section 6.5 that the sphere is not flat and the above theorem then implies that there is no holonomic and orhonormal frame on the sphere.

## 2.10 Hodge star

The Hodge star is a map turning a $p$–form into an $(n-p)$–form. We define it in terms of a holonomic frame:

$$\begin{aligned} * : \Omega^p U & \longrightarrow \Omega^{n-p} U \\ \varphi & \longmapsto *\varphi \end{aligned}$$

$$\begin{aligned} *\varphi \quad := \quad & \frac{1}{(n-p)!} \sum_{\mu_{p+1}\cdots\mu_n} \left[ \frac{1}{p!} \sum_{\mu_1..\mu_p} \epsilon_{\mu_1\cdots\mu_n} \sqrt{|\det g_{..}|} \right. \\ & \left. \times \sum_{\nu_1\cdots\nu_p} \varphi_{\nu_1\cdots\nu_p} g^{\mu_1\nu_1} \cdots g^{\mu_p\nu_p} \right] \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}, \end{aligned} \tag{2.61}$$

where $\epsilon_{\mu_1\cdots\mu_n}$ is the completely antisymmetric tensor with

$$\epsilon_{1\ldots n} = 1. \tag{2.62}$$

Note that this definition requires the choice of an orientation in $\mathbb{R}^n$, but does not depend on the particular coordinate system used. Just as the wedge product the Hodge star is a purely algebraic operation. It is linear and its square is plus or minus the identity:

$$**\varphi = (-1)^{p(n-1)+s}\varphi. \tag{2.63}$$

Recall that $s$ is the number of minus signs in the metric. Note that the Hodge star has a particularly simple expression in an orthonormal frame.

## 2.11 Coderivative and Laplace operator

Just as the exterior derivative, the coderivative is a linear first order differential operator which however lowers the degree of a differential form by one unit:

$$\begin{aligned} \delta : \Omega^p U &\longrightarrow \Omega^{p-1} U \\ \varphi &\longmapsto \delta\varphi := (-1)^{np+n+1+s} * \mathrm{d} * \varphi. \end{aligned} \tag{2.64}$$

It inherits nilpotency from the exterior derivative: $\delta^2 = 0$.

If $U$ is 'compact' and if the metric has Euclidean signature, then $\Omega U$ is a pre-Hilbert space with scalar product

$$(\kappa, \varphi) := \int_U \kappa \wedge *\varphi, \tag{2.65}$$

for two differential forms $\kappa$, $\varphi$ of equal degree. The scalar product vanishes if the degrees are not equal. In this situation, the coderivative is the formal adjoint of the exterior derivative.

In general, the Laplace operator is the linear second order differential operator defined by:

$$\Delta := -(d\delta + \delta d) \, : \, \Omega^p U \to \Omega^p U. \tag{2.66}$$

If the metric is Euclidean, the Laplace operator is Hermitean. If the metric is indefinite, the Laplace operator is usually called wave or d'Alembert operator and written as $\Box$.

## 2.12 Summary

Before returning to physics, let us summarize: We have recast a part of tensor analysis in a coordinate free language using differential forms. This serves two purposes:

- They carry less indices, making some calculations more transparent.
- Being coordinate independent they can easily be generalized to more general spaces like manifolds.

May be, the following dictionary can be useful:

| | |
|---|---|
| $v$ | $v^{\mu}$ |
| $\varphi \in \Omega^p U$ | $\varphi_{[\mu_1 \cdots \mu_p]}$ |
| $\varphi(v_1, v_2, \cdots, v_p)$ | $\sum_{\mu_1 \cdots \mu_p} \varphi_{[\mu_1 \cdots \mu_p]} v_1^{\mu_1} \cdots v_p^{\mu_p}$ |
| $\varphi \wedge \psi$ | $\varphi_{[\mu_1 \cdots \mu_p} \quad \psi_{\mu_{p+1} \cdots \mu_q]}$ |
| $d\varphi$ | $\partial_{[\mu_1} \varphi_{\mu_2 \cdots \mu_{p+1}]}$ |
| $\int_K \varphi$ | $\int_K \varphi_{1 \cdots p} \, dx^1 \cdots dx^p$ |
| $g$ | $g_{(ij)}$ |
| $g^*$ | $g^{ij} = (g^{-1})_{ij}$ |
| $* \varphi$ | $\sum_{\substack{\mu_1 \cdots \mu_p \\ \nu_1 \cdots \nu_p}} \sqrt{\lvert \det g_{\cdot\cdot} \rvert} \, \varphi_{\nu_1 \cdots \nu_p} g^{\nu_1 \mu_1} \cdots g^{\nu_p \mu_p} \epsilon_{\mu_1 \cdots \mu_p, \mu_{p+1} \cdots \mu_n}$ |
| $-d * d * - * d * d$ | $\Delta$ |

## 2.13 Maxwell's equations

Consider Minkowski space $U = \mathbb{R}^4$ equipped with the Minkowski metric of signature $+ - --$. We subscribe again to Einstein's summation convention, (summing over indices that appear twice). The sources, electric charge and current densities, are combined into a real valued 3-form:

$$j = \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} j^{\mu} \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\lambda} \wedge \mathrm{d}x^{\rho} \quad \in \ \Omega^3(\mathbb{R}^4). \tag{2.67}$$

Integrating $j$ over a 3-dimensional space-like volume yields the total charge inside that volume as a function of time. Charge conservation reads

$$\mathrm{d}j = 0. \tag{2.68}$$

The field strength is a real valued 2-form

$$F = \tfrac{1}{2} F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{2.69}$$

Then Maxwell's equations read:

$$\mathrm{d}F = 0, \tag{2.70}$$

$$\delta F = \tfrac{1}{\epsilon_0 c^2} * j. \tag{2.71}$$

equation (2.71) implies charge conservation. Therefore only conserved currents, $\mathrm{d}j = 0$, may be coupled to the electromagnetic field. Our spacetime being simply connected, equation (2.70) implies the existence of a potential, a real valued 1-form $A$ such that

$$F = \mathrm{d}A. \tag{2.72}$$

Expressed in terms of the potential, equation (2.71) can be obtained from the action

$$S[A] = -\int_{\mathbb{R}^4} \left( \tfrac{\epsilon_0 c}{2}\, F \wedge *F + \tfrac{1}{c}\, j \wedge A \right) \tag{2.73}$$

upon variation of the potential. This means we replace $A$ by $A + a$ in the action, expand it and put the term linear in $a$ equal to zero. Note that if spacetime was Euclidean, Maxwell's action in the vacuum would be simply $S = \frac{\epsilon_0}{2}(F, F)$. This will be Connes' starting point.

Writing Maxwell's theory with differential forms has four advantages:

- Lorentz invariance is immediate; $SO(1,3)$, the group of linear transformations preserving the metric and the orientation of $\mathbb{R}^4$, also leaves the Hodge star and consequently the Maxwell action (2.73) invariant.

- In Maxwell's equations or in the action the flat Minkowski metric may be replaced by any curved metric. This tells us how electromagnetism couples to gravity.

- Gauge invariance now reads

  $$\rho_V(g)A = A + i\tfrac{\hbar}{e} g \mathrm{d}g^{-1} = A + \tfrac{\hbar}{e}\mathrm{d}\theta, \quad g = \exp i\theta \ \in \ {}^{\mathbb{R}^4}U(1). \tag{2.74}$$

  Its abelian group $U(1)$ may easily be generalized to a non-Abelian, compact Lie group. One then gets the celebrated Yang-Mills theories.

- The invariance of the action under diffeomorphisms is manifest. They form a semidirect product with the gauge group, here:

  $$\mathrm{Diff}(M) \circledS {}^{\mathbb{R}^4}U(1). \tag{2.75}$$

# Chapter 3

# Yang-Mills-Higgs theories

To get started we describe Yang-Mills-Higgs theories as a black box or better as a slot machine. There are four slots for four bills. Once you have decided which bills you choose and entered them, a certain number of small slots will open for coins. Their number depends on the choice of bills. You make your choice of coins, feed them in, and the machine starts working. It produces as output a complete particle phenomenology: the particle spectrum with their quantum numbers, cross sections, life times, branching ratios. You compare the phenomenology to experiment to find out whether your input wins or loses.

## 3.1 The bills

The first bill is a finite dimensional, real, compact Lie group $G$. The gauge bosons, spin 1, will live in its adjoint representation whose Hilbert space is the complexified of the Lie algebra $\mathfrak{g}$.

The remaining bills are three unitary representations of $G$, $\rho_L,\ \rho_R,\ \rho_S$ defined on the complex Hilbert spaces, $\mathcal{H}_L,\ \mathcal{H}_R,\ \mathcal{H}_S$. They classify the left- and right-handed fermions, spin $\frac{1}{2}$, and the scalars, spin 0. The group $G$ is chosen compact to ensure that the unitary representations are finite dimensional, we want a finite number of different Lego bricks.

## 3.2 The coins

The coins are numbers, coupling constants, more precisely coefficients of invariant polynomials. We need an invariant scalar product on $\mathfrak{g}$. The set of all these scalar products is a cone and the gauge couplings are particular coordinates of this cone. If the group is simple, say $G = SU(n)$, then the most general, invariant scalar product is

$$(X, X') = \tfrac{2}{g_n^2}\mathrm{tr}[X^* X'], \quad X, X' \in su(n). \tag{3.1}$$

If $G = U(1)$ we have

$$(Y, Y') = \tfrac{1}{g_1^2}\bar{Y}Y', \quad Y, Y' \in u(1). \tag{3.2}$$

Mind the different normalizations, they are conventional. The $g_n$ are positive numbers, *the gauge couplings.* For every simple factor of $G$ there is one gauge coupling.

Then we need the Higgs potential $V(\varphi)$. It is an invariant, fourth order, stable polynomial on $\mathcal{H}_S \ni \varphi$. Stable means bounded from below. For $G = SU(2)$ and the Higgs scalar in the fundamental or defining representation, $\varphi \in \mathcal{H}_S = \mathbb{C}^2$, $\rho_S(g) = g$, we have

$$V(\varphi) = \lambda\,(\varphi^*\varphi)^2 - \tfrac{1}{2}\mu^2\,\varphi^*\varphi. \tag{3.3}$$

The coefficients of the Higgs potential are the Higgs couplings, $\lambda$ must be positive for stability. We say that the potential breaks $G$ spontaneously if its minimum is not a trivial orbit under $G$. In our example, if $\mu$ is positive the minimum of $V(\varphi)$ is the 3-sphere $|\varphi| = v := \frac{1}{2}\mu/\sqrt{\lambda}$. $v$ is called vacuum expectation value and $SU(2)$ is said to break down spontaneously to its little group $U(1)$. The little group leaves invariant any given point of the minimum, e.g. $\varphi = {}^t(v, 0)$. On the other hand if $\mu$ is purely imaginary, then the minimum of the potential is the origin, no spontaneous symmetry breaking.

Finally, we need the Yukawa couplings $g_Y$. They are the coefficients of the most general trilinear invariant on $\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes (\mathcal{H}_S \oplus \mathcal{H}_S^*)$. For every 1-dimensional invariant subspace in the reduction of this tensor representation, we have one complex Yukawa coupling.

We will see that, if the symmetry is broken spontaneously, gauge and Higgs bosons acquire masses related to the Higgs couplings, fermions acquire masses related to the Yukawa couplings.

## 3.3 The winner

Physicists have spent some thirty years and billions of Swiss Francs playing on the slot machine by Yang-Mills & Higgs. There is a winner, the standard model of electro-weak and strong interactions. Its bills are

$$G = SU(3) \times SU(2) \times U(1)$$

$$\mathcal{H}_L = \bigoplus_1^3 \left[(3, 2, \tfrac{1}{6}) \oplus (1, 2, -\tfrac{1}{2})\right], \tag{3.4}$$

$$\mathcal{H}_R = \bigoplus_1^3 \left[(3, 1, \tfrac{2}{3}) \oplus (3, 1, -\tfrac{1}{3}) \oplus (1, 1, -1)\right], \tag{3.5}$$

$$\mathcal{H}_S = (1, 2, -\tfrac{1}{2}), \tag{3.6}$$

where $(n_3, n_2, y)$ denotes the tensor product of an $n_3$ dimensional representation of $SU(3)$, an $n_2$ dimensional representation of $SU(2)$ and the one dimensional representation of $U(1)$ with hypercharge $y$: $\rho(\exp(i\theta)) = \exp(iy\theta)$. For historical reasons the hypercharge is an integer multiple of $\frac{1}{6}$. This is irrelevant: only the product of the hypercharge by its gauge coupling is measurable. In the direct sum, we recognize the three generations of fermions, the quarks are $SU(3)$ colour triplets, the leptons colour singlets. The basis of the fermion representation is the periodic table,

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

$$\begin{matrix} u_R, & c_R, & t_R, & & & \\ d_R, & s_R, & b_R, & e_R, & \mu_R, & \tau_R \end{matrix}$$

The parentheses indicate isospin doublets.

We recognize the eight gluons in $su(3)$. Attention, the $U(1)$ is not the one of electric charge, it is called hypercharge, the electric charge is a linear combination of hypercharge and weak isospin, parameterized by the weak mixing angle $\theta_w$ to be introduced below. This mixing is necessary to give electric charges to the $W$ bosons. The $W^+$ and $W^-$ are pure isospin states, while the $Z^0$ and the photon are (orthogonal) mixtures of the third isospin generator and hypercharge.

Because of the high degree of reducibility in the bills, there are many coins, among them 27 complex Yukawa couplings. Not all of them have a physical meaning. They can be converted into 18 physically significant, positive numbers [4], three gauge couplings,

$$g_3 = 1.218 \pm 0.026, \quad g_2 = 0.6567 \pm 0.0007, \quad g_1 = 0.3575 \pm 0.0001, \tag{3.7}$$

two Higgs couplings, $\lambda$ and $\mu$, and 13 positive parameters from the Yukawa couplings. The Higgs couplings are related to the boson masses:

$$m_W = \tfrac{1}{2} g_2 v = 80.33 \pm .15 \text{ GeV}, \tag{3.8}$$

$$m_Z = \tfrac{1}{2}\sqrt{g_1^2 + g_2^2}\, v = m_W / \cos\theta_w = 91.187 \pm .007 \text{ GeV}, \tag{3.9}$$

$$m_H = 2\sqrt{2}\sqrt{\lambda}\, v > 65 \text{ GeV}, \tag{3.10}$$

with the vacuum expectation value $v := \frac{1}{2}\mu/\sqrt{\lambda}$ and the weak mixing angle $\theta_w$ defined by

$$\sin^2\theta_w := g_2^{-2}/(g_2^{-2} + g_1^{-2}) = 0.2315 \pm 0.0005. \tag{3.11}$$

For the standard model, there is a one–to–one correspondence between the physically relevant part of the Yukawa couplings and the fermion masses and mixings,

$$\begin{aligned}
m_e &= 0.51099906 \pm 0.00000015 \text{ MeV}, & m_u &= 5 \pm 3 \text{ MeV}, & m_d &= 10 \pm 5 \text{ MeV},\\
m_\mu &= 0.105658389 \pm 0.000000034 \text{ GeV}, & m_c &= 1.3 \pm 0.3 \text{ GeV}, & m_s &= 0.2 \pm 0.1 \text{ GeV},\\
m_\tau &= 1.7771 \pm 0.0005 \text{ GeV}, & m_t &= 175 \pm 6 \text{ GeV}, & m_b &= 4.3 \pm 0.2 \text{ GeV},
\end{aligned}$$

Since the neutrinos are massless, the mixing only occurs for quarks and is given by a unitary matrix, the Cabibbo-Kobayashi-Maskawa matrix

$$C_{KM} := \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{3.12}$$

For physical purposes it can be parameterized by three angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and one $CP$ violating phase $\delta$:

$$C_{KM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

with $c_{kl} := \cos\theta_{kl}$, $s_{kl} := \sin\theta_{kl}$. The absolute values of the matrix elements are:

$$\begin{pmatrix} 0.9753 \pm 0.0006 & 0.221 \pm 0.003 & 0.004 \pm 0.002 \\ 0.221 \pm 0.003 & 0.9745 \pm 0.0007 & 0.040 \pm 0.008 \\ 0.010 \pm 0.006 & 0.039 \pm 0.009 & 0.9991 \pm 0.0004 \end{pmatrix}.$$

The physical meaning of the quark mixings is the following: when a sufficiently energetic $W^+$ decays into a $u$ quark, this $u$ quark is produced together with a $\bar{d}$ quark with probability $|V_{ud}|^2$, together with a $\bar{s}$ quark with probability $|V_{us}|^2$, together with a $\bar{b}$ quark with probability $|V_{ub}|^2$. The fermion masses and mixings together are an entity, the fermionic mass matrix or the matrix of Yukawa couplings multiplied by the vacuum expectation value.

Let us note four important properties of the standard model.

- The gluons couple in the same way to left- and right-handed fermions, the gluon coupling is vectorial, strong interaction do not break parity.
- The scalar is a colour singlet, the $SU(3)$ part of $G$ does not suffer spontaneous break down, the gluons remain massless.
- The $SU(2)$ couples only to left-handed fermions, its coupling is chiral, weak interaction break parity maximally.
- The scalar is an isospin doublet, the $SU(2)$ part suffers spontaneous break down, the $W^\pm$ and the $Z^0$ are massive.

## 3.4 The rules

It is time to open the slot machine and to see how it works. Its mechanism falls into five pieces:

**The Yang-Mills action:**

- Maxwell + non-Abelian gauge = Yang-Mills.

The actor in this piece is $A$ called a connection, gauge potential, gauge bosons or Yang-Mills field. It is a 1-form on spacetime $M$ with values in the Lie algebra $\mathfrak{g}$,

$$A \in \Omega^1(M, \mathfrak{g}). \tag{3.13}$$

We define its curvature or field strength,

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A] \ \in \Omega^2(M, \mathfrak{g}), \tag{3.14}$$

and the Yang-Mills action,

$$S_{YM}[A] = -\tfrac{1}{2} \int_M (F, *F). \tag{3.15}$$

The space of all connections carries an affine representation $\rho_V$ of the gauge group ${}^M G \ni g$:

$$\rho_V(g)A = gAg^{-1} + g\mathrm{d}g^{-1}. \tag{3.16}$$

Restricted to $x$-independent gauge transformation, the representation is linear, the adjoint one. The field strength transforms homogeneously under any gauge transformation,

$$\rho_V(g)F = gFg^{-1}, \tag{3.17}$$

and, as the scalar product $(\cdot, \cdot)$ is invariant, the Yang-Mills action is gauge invariant,

$$S_{YM}[\rho_V(g)A] = S_{YM}[A] \quad \text{for all } g \in^M G. \tag{3.18}$$

Note that a mass term for the gauge bosons,

$$\tfrac{1}{2} \int_M m_A^2 (A, *A), \tag{3.19}$$

is not gauge invariant because of the inhomogeneous term in the transformation law of a connection (3.16). Gauge invariance forces the gauge bosons to be massless.

In the Abelian case $G = U(1)$, the Yang-Mills action is nothing but Maxwell's action, quantum electro-dynamics ($QED$). Note however, that now the vector potential is purely imaginary, while conventionally, in Maxwell's theory it is chosen real. For $G = SU(3)$ and

$\mathcal{H}_L = \mathcal{H}_R = \mathbb{C}^3$ we have today's theory of strong interaction, quantum chromo-dynamics ($QCD$).

**The Dirac action:** Schrödinger's action is non-relativistic. Dirac generalized it to be Lorentz invariant, e.g. [5]. The price to be paid is twofold. His generalization only works for spin $\frac{1}{2}$ particles and requires that for every such particle there must be an antiparticle with same mass and opposite charges. Therefore Dirac's wave function $\psi(x)$ takes values in $\mathbb{C}^4$, spin up, spin down, particle, antiparticle. Antiparticles have been discovered and Dirac's theory was celebrated. Here it is in short for (flat) Minkowski space of signature $+---$. Define the four Dirac matrices,

$$\gamma^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.20}$$

$$\gamma^2 := \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

They satisfy the anticommutation relations,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} 1_4. \tag{3.22}$$

In even spacetime dimensions, the chirality,

$$\gamma_5 := \tfrac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.23}$$

is a natural operator and it paves the way to an understanding of the chirality in weak interactions. The chirality is a unitary matrix of unit square that anticommutes with all four Dirac matrices. $(1-\gamma_5)/2$ projects on the left-handed part, $(1+\gamma_5)/2$ projects on the right-handed part. The chirality applied to a left-handed spinor produces its right-handed part. Similarly, there is the charge conjugation, an anti-unitary operator of unit square, that applied on a particle $\psi$ produces its antiparticle

$$\psi^c := i\gamma^2 \psi^*. \tag{3.24}$$

Here $\cdot^*$ denotes complex conjugation. The charge conjugation commutes with all four Dirac matrices. In flat spacetime, the free Dirac operator is simply defined by,

$$\partial\!\!\!/ := i\hbar \gamma^\mu \partial_\mu. \tag{3.25}$$

It is sometimes referred to as square root of the wave operator because $\partial\!\!\!/^2 = -\,\square\,$. The coupling of the Dirac spinor to the gauge potential $A = A_\mu \mathrm{d}x^\mu$ is done via the covariant derivative, and called minimal coupling. In order to break parity we write left- and right-handed part independently:

$$\begin{aligned} S_D[A,\psi_L,\psi_R] &= \int_M \bar\psi_L\,[\partial\!\!\!/ + i\hbar\gamma^\mu\tilde\rho_L(A_\mu)]\,\frac{1-\gamma_5}{2}\,\psi_L\,\mathrm{d}^4x \\ &\quad + \int_M \bar\psi_R\,[\partial\!\!\!/ + i\hbar\gamma^\mu\tilde\rho_R(A_\mu)]\,\frac{1+\gamma_5}{2}\,\psi_R\,\mathrm{d}^4x. \end{aligned} \tag{3.26}$$

The new actors in this piece are $\psi_L$ and $\psi_R$, two multiplets of Dirac spinors or fermions, that is vectors in the Hilbert spaces $\mathbb{C}^4\otimes\mathcal{H}_L$ and $\mathbb{C}^4\otimes\mathcal{H}_R$. We use the notations, $\bar\psi := \psi^*\gamma^0$, where $\cdot^*$ denotes the dual with respect to the scalar product in the (internal) Hilbert space $\mathcal{H}_L$ or $\mathcal{H}_R$. The $\gamma^0$ is needed for energy reasons and for invariance of the pseudo–scalar product of spinors under (covered) Lorentz transformations. The $\gamma^0$ is absent if spacetime is Euclidean. Then we have a genuine scalar product and the square integrable spinors form a Hilbert space $\mathcal{L}^2(\mathcal{S})$, the infinite dimensional brother of the internal one. The Dirac operator is then self-adjoint in this Hilbert space. We denote by $\tilde\rho_L$ the Lie algebra representation in $\mathcal{H}_L$. The covariant derivative, $\mathrm{D}_\mu := \partial_\mu + \tilde\rho_L(A_\mu)$, deserves its name,

$$[\partial_\mu + \tilde\rho_L(\rho_V(g)A_\mu)]\,(\rho_L(g)\psi_L) = \rho_L(g)\,[\partial_\mu + \tilde\rho_L(A_\mu)]\,\psi_L, \tag{3.27}$$

for all gauge transformations $g \in^M G$. This ensures that the Dirac action is gauge invariant.

If $\mathcal{H}_L = \mathcal{H}_R$ we may add a mass term

$$-\,c\int_M \bar\psi_R\,m_\psi\,\frac{1-\gamma_5}{2}\,\psi_L\,\mathrm{d}^4x \;-\; c\int_M \bar\psi_L\,m_\psi\,\frac{1+\gamma_5}{2}\,\psi_R\,\mathrm{d}^4x \;=\; -c\int_M \bar\psi\,m_\psi\,\psi\,\mathrm{d}^4x \tag{3.28}$$

to the Dirac action. It gives identical masses to all members of the multiplet. The fermion masses are gauge invariant if all fermions in $\mathcal{H}_L = \mathcal{H}_R$ have the same mass. Remember that gauge invariance forces gauge bosons to be massless. Here it is parity *non*-invariance that forces fermions to be massless.

Let us conclude by reviewing briefly why the Dirac equation is the Lorentz invariant generalization of the Schrödinger equation. Take the free Schrödinger equation on (flat) $\mathbb{R}^4$ it is a linear differential equation with constant coefficients,

$$\left(\frac{2m}{i\hbar}\,\frac{\partial}{\partial t} - \Delta\right)\psi = 0. \tag{3.29}$$

We compute its polynomial following Fourier and de Broglie,

$$-\,\frac{2m}{\hbar}\,\omega + k^2 = -\,\frac{2m}{\hbar^2}\,\left[E - \frac{p^2}{2m}\right]. \tag{3.30}$$

Energy conservation in Newtonian mechanics is equivalent to the vanishing of the polynomial. Likewise, the polynomial of the free, massive Dirac equation $(\partial\!\!\!/ - cm_\psi)\psi = 0$ is

$$\tfrac{\hbar}{c}\,\omega\gamma^0 + \hbar\,k_j\gamma^j - c\,m1. \tag{3.31}$$

Putting it to zero implies energy conservation in special relativity,

$$(\tfrac{\hbar}{c})^2\,\omega^2 - \hbar^2\,\vec{k}^2 - c^2\,m^2 = 0. \tag{3.32}$$

In short

- Schrödinger + Minkowskian geometry = Dirac.

So far we have seen the two noble pieces, Yang-Mills and Dirac. Their noblesse has even convinced mathematicians, Donaldson has used a non-Abelian Yang-Mills theory to discover exotic differential structures on $\mathbb{R}^4$ and the Dirac operator has been elected differential operator of the decade by Atiyah & Singer. I feel that these two actions deserve the comparison with the circles of planetary motion and we are ready for the epicycles, the other three pieces are indeed cheap copies of the circles with the gauge boson $A$ replaced by a scalar $\varphi$. We need these three epicycles to cure only one problem, give masses to some gauge bosons and to some fermions. These masses are forbidden by gauge invariance and parity violation. To simplify the notation we will work from now on in units with $c = \hbar = 1$.

**The Klein-Gordon action:** The Yang-Mills action contains the kinetic term for the gauge boson. This is simply the quadratic term, $(\mathrm{d}A, \mathrm{d}A)$ that by Euler-Lagrange produces linear field equations. We copy this for our new actor, a multiplet of scalar fields or Higgs bosons,

$$\varphi \in \Omega^0(M, \mathcal{H}_S), \tag{3.33}$$

by writing the Klein-Gordon action,

$$S_{KG}[A, \varphi] = \tfrac{1}{2}\int_M (\mathrm{D}\varphi)^* * \mathrm{D}\varphi, \tag{3.34}$$

with the covariant derivative here defined with respect to the scalar representation,

$$\mathrm{D}\varphi := \mathrm{d}\varphi + \tilde{\rho}_S(A)\varphi. \tag{3.35}$$

Again we need this minimal coupling $\varphi^* A \varphi$ for gauge invariance.

**The Higgs potential:** The non-Abelian Yang-Mills action contains interaction terms for the gauge bosons, a bounded, invariant, fourth order polynomial, $2(\mathrm{d}A, [A, A]) + ([A, A], [A, A])$. We mimic these interactions for scalar bosons by adding the integrated Higgs potential $\int_M *V(\varphi)$ to the action.

**The Yukawa terms:** We also mimic the (minimal) coupling of the gauge boson to the fermions $\psi^* A\psi$ by writing all possible trilinear invariants,

$$S_Y[\psi_L,\psi_R,\varphi] := \mathrm{Re}\int_M * \left(\sum_{j=1}^{n} g_{Yj}\,(\psi_L^*,\psi_R,\varphi)_j + \sum_{j=n+1}^{m} g_{Yj}\,(\psi_L^*,\psi_R,\varphi^*)_j\right). \tag{3.36}$$

In the standard model, there are 27 complex Yukawa couplings, $m = 27$.

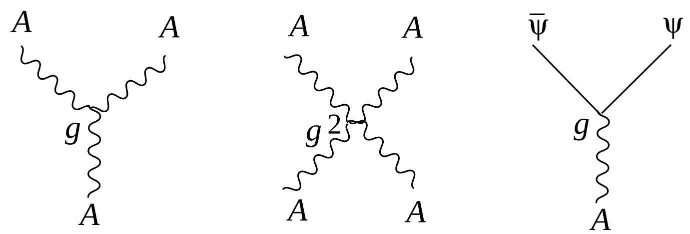


Figure 3.1: The tri- and quadrilinear gauge couplings and the minimal gauge coupling to fermions

The two circles, Yang-Mills and Dirac, contain three types of couplings, a trilinear self coupling $AAA$, a quadrilinear self coupling $AAAA$ and a the trilinear minimal coupling $\psi^* A\psi$. The gauge self couplings are absent if the group $G$ is Abelian, the photon has no electric charge, Maxwell's equations are linear.

The beauty of gauge invariance is that if $G$ is simple, all these couplings are fixed in terms of one positive number, the gauge coupling $g$. To see this, take an orthonormal basis $T_b,\ b = 1,2,\dots \dim G$ of the complexified $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra with respect to the invariant scalar product and an orthonormal basis $F_k,\ k = 1,2,\dots \dim \mathcal{H}_L$ of the fermionic Hilbert space, say $\mathcal{H}_L$, and develop the actors,

$$A =: A^b_\mu T_b \mathrm{d}x^\mu, \quad \psi =: \psi^k F_k. \tag{3.37}$$

Insert these expressions into the Yang-Mills and Dirac actions, then you get the following interaction terms, figure 3.1,

$$g\,\partial_\rho A^a_\mu A^b_\nu A^c_\sigma\, f_{abc}\,\epsilon^{\rho\mu\nu\sigma}, \quad g^2\, A^a_\mu A^b_\nu A^c_\rho A^d_\sigma\, f_{ab}{}^e f_{ecd}\,\epsilon^{\rho\mu\nu\sigma}, \quad g\,\psi^{k*} A^b_\mu \gamma^\mu \psi_\ell\, t_{bk}{}^\ell, \tag{3.38}$$

with the structure constants $f_{ab}{}^e$,

$$[T_a, T_b] =: f_{ab}{}^e T_e. \tag{3.39}$$

The indices of the structure constants are raised and lowered with the matrix of the invariant scalar product in the basis $T_b$, that is the identity matrix. The ${t_{bk}}^\ell$ is the matrix of the operator $\tilde\rho_L(T_b)$ with respect to the basis $F_k$. The difference between the circles and the epicycles is that the Higgs couplings, $\lambda$ and $\mu$ in the standard model, and the Yukawa couplings $g_{Yj}$ are arbitrary, are neither connected among themselves nor connected to the gauge couplings $g_i$.

The standard model is the most painful humiliation of physics today. The humiliation has four levels:

- The rules of the Yang-Mills-Higgs model building kit contain three epicycles.
- The winning bills are unmotivated except for the $U(1)$ coming from quantum mechanics.
- The winning coins are numerous, 18, and beg for an understanding.
- The theory of gravity is completely different from the Yang-Mills description of the electro-weak and strong forces. The underlying group of gravity is the group of diffeomorphisms of spacetime, $\mathrm{Diff}(M)$, that formalizes the coordinate transformations. This group is not a Lie group. Any attempt to unify all four forces has failed so far.

Nevertheless, and this makes the humiliation painful, the standard model reproduces correctly millions of experimental numbers that cost billions of Swiss Francs. Every anomaly free Yang-Mills-Higgs model, in particular the standard model, is renormalizable. Renormalizable theories are rare and therefore precious. Connes has shown that noncommutative geometry eases the humiliation on all four levels.

## 3.5 An example

We illustrate this chapter with the current model of electro-weak interactions for one generation of leptons. This is the Glashow-Salam-Weinberg model, a submodel of the standard model. There are simpler examples on the market, in particular models not containing a $U(1)$ factor. Mathematically the $U(1)$ is so degenerate that it makes some computations perfidious.

$$G = SU(2)\times U(1) \ni (a,b), \quad \mathfrak{g} = su(2)\oplus u(1) \ni (X,Y), \tag{3.40}$$

$$((X,Y),(X',Y')) = \tfrac{2}{g_2^2}\,\mathrm{tr}(X^*X') + \tfrac{1}{g_1^2}\,\bar{Y}Y', \tag{3.41}$$

$$\mathcal{H}_L = \mathbb{C}^2 \ni \psi_L, \quad \rho_L(a,b) = ab^{y_L}, \quad y_L = -\tfrac{1}{2}, \tag{3.42}$$

$$\mathcal{H}_R = \mathbb{C} \ni \psi_R, \quad \rho_R(a,b) = b^{y_R}, \quad y_R = -1, \tag{3.43}$$

$$\begin{aligned} \mathcal{H}_S &= \mathbb{C}^2 \ni \varphi, \quad \rho_S(a,b) = ab^{y_S}, \quad y_S = -\tfrac{1}{2}, & (3.44) \\ V(\varphi) &= \lambda\,(\varphi^*\varphi)^2 - \tfrac{1}{2}\mu^2\,\varphi^*\varphi, & (3.45) \\ \mathcal{L}_Y &= \mathrm{Re}[g_e(-\bar\psi_{1L}\bar\varphi_2 + \bar\psi_{2L}\bar\varphi_1)\psi_R]. & (3.46) \end{aligned}$$

To see the physical content of the theory, we need orthonormal bases of the Hilbert spaces $\mathfrak{g}^{\mathbb{C}}$, $\mathcal{H}_L$, $\mathcal{H}_R$ and $\mathcal{H}_S$.

A Cartan subalgebra of $\mathfrak{g}$ is spanned by the two orthonormal vectors, 'third isospin' and 'hypercharge',

$$I_3 := i\left(g_2\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, 0\right), \quad Y := i\,(0, g_1)\,. \tag{3.47}$$

The uncommitted choice for the electric charge generator $Q$ is:

$$iQ := i\left(g_2\sin\theta_w\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, g_1\cos\theta_w\right), \tag{3.48}$$

where $\theta_w$ is the weak mixing angle. We complete $iQ$ to an orthonormal basis of $\mathfrak{g}^{\mathbb{C}}$ of eigenvectors of $[Q,\cdot]$

$$\begin{aligned} \tilde Z &:= i\left(g_2\cos\theta_w\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, -g_1\sin\theta_w\right), \\ I^+ &:= i\left(\frac{g_2}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right), \\ I^- &:= i\left(\frac{g_2}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right). \end{aligned}$$

The eigenvalues are 0 and $\pm g_2\sin\theta_w =: \pm e$. The multiplet of gauge bosons is now written as

$$A_\mu(x) := \gamma_\mu(x)\,iQ + Z_\mu(x)\,\tilde Z + \frac{1}{\sqrt{2}}\left(W_\mu(x)\,I^+ + W^*_\mu(x)\,I^-\right),$$

where the photon $\gamma_\mu(x)$ and the $Z_\mu(x)$ are real fields while the $W$ is complex. The kinetic term in the Yang-Mills Lagrangian now has its standard form, a sum of three pieces each of the form

$$-\tfrac{1}{2}\partial_\mu W^*_\nu\partial^\mu W^\nu + \tfrac{1}{2}\partial_\mu W^{*\mu}\partial_\nu W^\nu + \tfrac{1}{2}m_W^2 W^*_\mu W^\mu. \tag{3.49}$$

The mass term is absent from the Yang-Mills Lagrangian because of gauge invariance. We will now get it from the Klein-Gordon action by spontaneous symmetry breaking.

Our group $SU(2)\times U(1)$ is broken spontaneously down to $U(1)$. The former $U(1)$ defines the hypercharge. We will identify the latter $U(1)$ with the electric charge. The minimum of the Higgs potential is located at scalars $\varphi_0$ of norm $|\varphi_0| = v$ where $v = \frac{1}{2}\mu/\sqrt{\lambda}$ is the vacuum expectation value. Any such minimal $\varphi_0$ is left invariant by a residual subgroup, the little group. Without loss of generality let us choose $y_S = -\frac{1}{2}$ and $\varphi_0 = {}^t(v, 0)$ and let us compute the little group. We are looking for elements $(a, b) \in G$ such that

$$\rho_S(a, b)\varphi_0 = ab^{-1/2}\varphi_0 = \varphi_0. \tag{3.50}$$

The solution is,

$$a = \begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix}, \quad b = \exp(i\theta), \tag{3.51}$$

the little group is $U(1)$ generated by $iQ$ if and only if

$$g_1 \cos\theta_w = g_2 \sin\theta_w = e. \tag{3.52}$$

Then

$$\frac{1}{i}\tilde{\rho}_S(iQ) = \begin{pmatrix} 0 & 0 \\ 0 & -e \end{pmatrix}. \tag{3.53}$$

Next we compute the boson masses. We have to develop the scalar field around a minimum of the action, $\varphi = \varphi_0$ and not around $\varphi = 0$ which is not a minimum. To this end we define

$$\varphi(x) =: \varphi_0 + h(x). \tag{3.54}$$

Then the mass matrix of the gauge bosons is the term quadratic in $A$ contained in the Klein-Gordon Lagrangian,

$$\tfrac{1}{2}\left|\tilde{\rho}_S(A_\mu)\varphi_0\right|^2 = \tfrac{1}{2}m_Z^2 Z_\mu Z^\mu + \tfrac{1}{2}m_W^2 W_\mu^* W^\mu \tag{3.55}$$

with

$$m_W = \tfrac{1}{2}g_2 v \quad \text{and} \quad m_Z = \tfrac{1}{2}\sqrt{g_1^2 + g_2^2}\, v = m_W/\cos\theta_w. \tag{3.56}$$

The spontaneous symmetry breaking has given masses to the $W$ and $Z$ bosons. Massless spin 1 particles have two degrees of freedom, 'the transverse modes', the spin is orthogonal to the direction of motion. A massive spin 1 particle has one more degree of freedom, 'the longitudinal mode', the spin is parallel to the direction of motion. To become massive the massless gauge boson takes this additional degree of freedom from the Higgs field. In our example, we parameterize the scalar as

$$\varphi(x) = \varphi_0 + \begin{pmatrix} H(x) + ih_Z(x) \\ h_W(x) \end{pmatrix}, \tag{3.57}$$

corresponding to an orthonormal basis of $\mathcal{H}_S$. The neutral $Z$ boson eats the neutral scalar field $h_Z$ to become massive and the charged $W$ eats the charged scalar $h_W$. There remains only one physical scalar field $H$ which is neutral. Let us compute its mass. To this end we must develop the Higgs potential in terms of the fields $H$, $h_Z$ and $h_W$,

$$V(\varphi(x)) = V(\varphi_0) + \tfrac{1}{2} m_H^2 H^2(x) + \text{terms of order 3 and 4}\ , \tag{3.58}$$

with

$$m_H = 2\sqrt{2}\sqrt{\lambda}\, v. \tag{3.59}$$

The constant term $V(\varphi_0)$ is the energy of the vacuum or cosmological constant.

One defines the $\rho$-factor by

$$\rho := \frac{m_W^2}{\cos^2\theta_w\ m_Z^2}. \tag{3.60}$$

It is unit if the scalar sits in a doublet and it can take any other real value with more complicated scalar representations. Experimentally we have today $\rho = 1.0012 \pm 0.0031$.

Finally let us turn to the fermionic action. The spontaneous symmetry breaking also produces the electron mass from the Yukawa term with $\varphi = \varphi_0$. With respect to orthonormal bases of $\mathcal{H}_L$ and $\mathcal{H}_R$, we have

$$\psi_L = \nu_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e_L \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \psi_R = e_R, \tag{3.61}$$

and the fermionic Lagrangian reads to second order:

$$\bar{e}\, \partial\!\!\!/ e + \bar{\nu}\, \partial\!\!\!/ \, \frac{1-\gamma_5}{2}\, \nu + m_e \bar{e} e, \tag{3.62}$$

with

$$m_e = \mathrm{Re}\, g_e\, v. \tag{3.63}$$

The remaining terms are of order three, the minimal couplings fermion-fermion-gauge boson and the Yukawa couplings fermion-fermion-Higgs. They describe interactions, terms giving rise to non-linear field equations via Euler-Lagrange. For instance the coupling of the photon to the neutrino $\bar{\nu}_L \gamma \nu_L$ is,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \tfrac{1}{i} \tilde{\rho}_L(iQ) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e \begin{pmatrix} 1 & 0 \end{pmatrix} \left[ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \tfrac{1}{2} 1_2 \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{3.64}$$

The couplings of the photon to the left-handed electron and of the photon to the right-handed electron are both $-e$. The photon coupling is vectorial, electromagnetism preserves parity. On the other hand, the coupling of the $W$ to the left-handed electron is $g_2$, to the right-handed electron it vanishes, the $W$ coupling is axial.

# Chapter 4

# Connes' first dreisatz

Noncommutative geometry explains the Higgs field as a magnetic field accompanying certain Yang-Mills fields, among them the ones of the standard model.

- Yang-Mills + noncommutative geometry = Yang-Mills-Higgs.

The geometric noblesse of the two circles allows their promotion to noncommutative geometries. The promotion of the two circles to one of these, an almost commutative geometry, produces the three epicycles from the two promoted circles.

To construct a Yang-Mills action $\int(F, *F)$, we need four ingredients, differential forms on spacetime $M$, a Lie group $G$, 'the internal space', a scalar product on the space of differential forms $\Omega M$ and an invariant scalar product on the Lie algebra $\mathfrak{g}$ of the group $G$. To construct the action which is a real number, we take the scalar products of the field strength with itself. The first scalar product involves the spacetime metric $g$ hidden in the Hodge star $*$, $(\kappa, \varphi) := \int_M \kappa^* * \varphi$, $\kappa$ and $\varphi$ differential forms of same degree. The second scalar product is on the Lie algebra, e.g. for $G = SU(n)$, the general invariant scalar product is $(a, b) = \frac{2}{g_n^2}\mathrm{tr}(a^*b)$, $a, b \in su(n)$ and the coupling constant $g_n$ is a positive number. Noncommutative geometry in its almost commutative version unifies spacetime and internal space and the two scalar products are derived from one common scalar product. At the same time coordinate transformations on spacetime are unified with gauge transformations. They are nothing but the automorphisms of the almost commutative geometry. This last point will be the starting point of the fourth geometric dreisatz unifying Yang-Mills with gravity.

## 4.1 Spectral triples

Noncommutative geometry does to spacetime $M$, what quantum mechanics did to phase space $\mathcal{P}$

- Hamilton + noncommutative geometry = Schrödinger.

An uncertainty relation is introduced by allowing the commutative algebra of functions $\mathcal{C}^\infty(\mathcal{P})$ to become noncommutative. Let us call $\mathcal{A}$ this new algebra that we suppose defined over the real numbers, associative and equipped with a unit and an involution. $\mathcal{A}$ is the algebra of quantum observables. Now on spacetime $M$ we have a metric. But how define a distance on a space that has lost its points? Following Connes [6], we need a faithful representation $\rho$ of $\mathcal{A}$ via bounded operators on a complex Hilbert space $\mathcal{H}$, the space of fermions, and a selfadjoint 'Dirac' operator $\mathcal{D}$ on $\mathcal{H}$. Connes calls these three ingredients a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. They satisfy axioms. These axioms are simply taken from the properties of the commutative case, $\mathcal{A} = \mathcal{C}^\infty(M)$, where from now on we must suppose that spacetime $M$ is Euclidean and compact. The Hilbert space $\mathcal{H}$ is the space of ordinary, square integrable Dirac spinors. An element $f$ of $\mathcal{A}$ is a differentiable function on spacetime, $f(x)$, and it acts on a spinor $\psi(x)$ by multiplication $(\rho(f)\psi)(x) := f(x)\psi(x)$. $\mathcal{D} = \partial\!\!\!/$ is the ordinary Dirac operator. Only recently Connes has completed the list of axioms [7] as to have a one-to-one correspondence between commutative spectral triples and Riemannian spin manifolds. To this end, he needed two other old friends from particle physics, a chirality operator $\chi$ and a real structure $J$. The chirality is a unitary operator of square one that commutes with the representation. Therefore $\chi$ decomposes the representation space into a left-handed piece $(1-\chi)/2\,\mathcal{H}$ and a right-handed piece $(1+\chi)/2\,\mathcal{H}$. In the commutative case, of course $\chi = \gamma_5$. The real structure is an anti-unitary operator that in the commutative case reduces to the charge conjugation operator $C$. $J$ is of square plus or minus one, depending on spacetime dimension and signature. Also depending on spacetime dimension and signature, $J$ commutes or anticommutes with $\chi$. The charge conjugation as well decomposes the representation space into two pieces, particles and anti-particles, all together

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c. \tag{4.1}$$

Here are a few more properties from the commutative case that become axioms

- $\rho(a)$ commutes with $J\rho(\tilde{a})J^{-1}$, for all $a, \tilde{a} \in \mathcal{A}$,
- $\mathcal{D}\chi = -\chi\mathcal{D}$,
- $\mathcal{D}J = +J\mathcal{D}$,
- $[\mathcal{D}, \rho(a)]$ is bounded for all $a$ in $\mathcal{A}$,
- $[\mathcal{D}, \rho(a)]$ commutes with $J\rho(\tilde{a})J^{-1}$, for all $a, \tilde{a}$ in $\mathcal{A}$.

The last axiom is called first order, because in the commutative case, it just says that the Dirac operator is a first order differential operator. The dimensionality of $M$ can be recovered from the spectrum of the Dirac operator. Indeed for compact manifolds, the spectrum is discrete and the eigenvalues $\lambda_n$ grow like $n^{1/\dim M}$. This motivates the name spectral triple. Let us mention two more axioms. The orientability axiom relates the chirality to the volume form, a differential form of maximal degree. The Poincaré duality on manifolds is promoted to an axiom in quite an abstract form. We anticipate that, in the case of the standard model, this Poincaré duality will prohibit right-handed neutrinos [8].

**Warning:** My presentation of noncommutative geometry is that of a modest physicist. For a precise account the reader is referred to Joe Várilly's beautiful lectures at this School [9].

Since we are now in Euclidean signature let us spell out again the case of a four dimensional spacetime. A spinor has four square integrable components,

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \in \mathcal{L}^2(\mathcal{S}). \tag{4.2}$$

The (flat) Dirac operator is

$$\partial\!\!\!/ \psi := i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi. \tag{4.3}$$

We choose the gamma matrices self adjoint,

$$\gamma^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{4.4}$$

$$\gamma^2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{4.5}$$

They satisfy the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1 \tag{4.6}$$

with the flat Euclidean metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.7}$$

The chirality operator is by definition

$$\gamma_5 := \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

It is unitary and of unit square as postulated. Since it anticommutes with all other gamma matrices

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0, \tag{4.9}$$

the Dirac operator is odd

$$\partial\!\!\!/\gamma_5 + \gamma_5\,\partial\!\!\!/ = 0. \tag{4.10}$$

The charge conjugation is

$$\psi^c = \gamma^0\gamma^2\psi^*. \tag{4.11}$$

Let us note that in four dimensional Euclidean spacetime, the chirality commutes with charge conjugation,

$$(\psi_L)^c = (\psi^c)_L =: \psi_L^c. \tag{4.12}$$

In the following we will take advantage of this notational simplification. Attention, in Minkowskian signature, the notation $\psi_L^c$ is ambiguous, because there the two operators anti-commute. Finally, we abbreviate the representation of a function $f$ on a spinor $\psi$ by $\rho(f) =: \underline{f}$, $(\underline{f}\psi)(x) = f(x)\psi(x)$.

## 4.2 Differential forms

Our next aim is to construct differential forms starting from a spectral triple. In the commutative case, we want this construction to reproduce de Rham's differential forms, $\Omega M$.

We start with an auxiliary differential algebra $\Omega\mathcal{A}$, the universal differential envelope of $\mathcal{A}$: $\Omega^0\mathcal{A} := \mathcal{A}$. $\Omega^1\mathcal{A}$ is generated by symbols $\delta a$, $a \in \mathcal{A}$ with relations $\delta 1 = 0$, $\delta(aa') = (\delta a)a' + a\delta a'$. $\Omega^1\mathcal{A}$ consists of finite sums of terms of the form $a_0\delta a_1$, and likewise for higher degree $p$,

$$\Omega^p\mathcal{A} = \left\{ \sum_j a_0^j\delta a_1^j ... \delta a_p^j, \quad a_q^j \in \mathcal{A} \right\}. \tag{4.13}$$

The differential $\delta$ is defined by

$$\delta(a_0\delta a_1 ... \delta a_p) := \delta a_0\delta a_1 ... \delta a_p. \tag{4.14}$$

The involution $^*$ is extended from the algebra $\mathcal{A}$ to $\Omega^1\mathcal{A}$ by putting $(\delta a)^* := \delta(a^*) =: \delta a^*$ and to the entire differential envelope by $(\kappa\varphi)^* = \varphi^*\kappa^*$. The next step is to extend the representation $\rho$ from the algebra $\mathcal{A}$ to its envelope $\Omega\mathcal{A}$. This extension deserves a new name:

$$\pi : \Omega\mathcal{A} \longrightarrow \bigoplus_p \mathrm{End}(\mathcal{H})$$

$$\pi(a_0\delta a_1...\delta a_p) := (-i)^p\rho(a_0)[\mathcal{D},\rho(a_1)]...[\mathcal{D},\rho(a_p)]. \tag{4.15}$$

$\pi$ is a representation of $\Omega\mathcal{A}$ as graded involution algebra. Note the $(-i)^p$ on the rhs which is not uniform in the literature. We are tempted to define also a differential, again denoted by $\delta$, on $\pi(\Omega\mathcal{A})$ by $\delta\pi(\hat\varphi) := \pi(\delta\hat\varphi)$. However, this definition does not make sense because there are forms $\hat\varphi \in \Omega\mathcal{A}$ with $\pi(\hat\varphi) = 0$ and $\pi(\delta\hat\varphi) \neq 0$. By dividing out these unpleasant forms, we arrive at the desired differential algebra $\Omega_\mathcal{D}\mathcal{A}$,

$$\Omega_\mathcal{D}\mathcal{A} := \frac{\pi\left(\Omega\mathcal{A}\right)}{\mathcal{J}}, \quad \text{with} \quad \mathcal{J} := \pi\left(\delta\ker\pi\right) =: \bigoplus_p \mathcal{J}^p, \tag{4.16}$$

($\mathcal{J}$ for junk). On the quotient, the differential is now well defined. Degree by degree we have:

$$\Omega^0_\mathcal{D}\mathcal{A} = \rho(\mathcal{A}) \tag{4.17}$$

because $\mathcal{J}^0 = 0$ ,

$$\Omega^1_\mathcal{D}\mathcal{A} = \pi(\Omega^1\mathcal{A}) \tag{4.18}$$

because $\rho$ is faithful, and in degree $p \geq 2$

$$\Omega^p_\mathcal{D}\mathcal{A} = \frac{\pi(\Omega^p\mathcal{A})}{\pi(\delta(\ker\pi)^{p-1})}. \tag{4.19}$$

In the commutative case, $\delta = \mathrm{d}$, $\Omega_{\partial\!\!\!/}\mathcal{C}^\infty(M)$ is isomorphic to de Rham's differential algebra $\Omega M$ with

$$\pi(f_0\mathrm{d}f_1\mathrm{d}f_2...\mathrm{d}f_p) \cong f_0\gamma^{\mu_1}\left(\frac{\partial}{\partial x^{\mu_1}}f_1\right)\gamma^{\mu_2}\left(\frac{\partial}{\partial x^{\mu_2}}f_2\right)...\gamma^{\mu_p}\left(\frac{\partial}{\partial x^{\mu_p}}f_p\right). \tag{4.20}$$

Dividing out the junk renders the lhs graded commutative.

Let us illustrate this isomorphism for 1- and 2-forms on a four dimensional spacetime $M$. We need the commutator

$$[\partial\!\!\!/, \underline{f}]\psi \;=\; i\gamma^\mu\frac{\partial}{\partial x^\mu}(f\psi) - if\gamma^\mu\frac{\partial}{\partial x^\mu}\psi$$

$$= i\left[\gamma^\mu \frac{\partial}{\partial x^\mu} f\right]\psi. \tag{4.21}$$

Therefore

$$[\not\partial, \underline{f}] = i\gamma^\mu \frac{\partial}{\partial x^\mu} f =: i\gamma(\mathrm{d}f) \tag{4.22}$$

with

$$\mathrm{d}f = \left[\frac{\partial}{\partial x^\mu} f\right] \mathrm{d}x^\mu. \tag{4.23}$$

At this point, we see that the restriction to flat spacetime can be dropped. Let us anticipate the Dirac operator on curved manifolds

$$i\gamma^\mu(x)\left[\frac{\partial}{\partial x^\mu} + \omega_\mu\right]. \tag{4.24}$$

It differs from the flat one in two respects, the gamma matrices are $x$ dependent, no problem in the above commutator, and an additional algebraic term, a spin connection $\omega = \omega_\mu \mathrm{d}x^\mu$ valued in $so(4)$ appears but drops out from the commutator. Since the Dirac operator only shows up in commutators, Connes' algorithm works on any Riemannian spin manifold.

The representation of functions by multiplication on spinors is faithful, of course, and

$$\Omega^1_{\not\partial}\mathcal{A} \cong \pi(\Omega^1\mathcal{A}). \tag{4.25}$$

A general element of the rhs is a finite sum of terms

$$\pi(f_0 \mathrm{d}f_1), \quad f_0, f_1 \in \mathcal{A}. \tag{4.26}$$

It is identified with the differential 1-form on $M$

$$f_0 \mathrm{d}f_1 \quad \in \Omega^1 M. \tag{4.27}$$

For 2-forms the situation is less trivial, we must compute the junk $\mathcal{J}^2 = \pi(\mathrm{d}(\ker\pi)^1)$. Consider

$$h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1} \tag{4.28}$$

an element in $\Omega^1\mathcal{A}$ where $h \in \mathcal{A}$ is a non-vanishing function, $h^{-1}(x) = 1/h(x)$. As $\Omega\mathcal{A}$ is not graded commutative this element does not vanish!

$$h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1} \neq h^{-1}\mathrm{d}h + (\mathrm{d}h^{-1})h = \mathrm{d}(h^{-1}h) = \mathrm{d}1 = 0. \tag{4.29}$$

Its image under $\pi$ however does vanish

$$\pi(h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1}) \quad = \quad \gamma(h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1})$$

$$= \quad \gamma(h^{-1}\mathrm{d}h + (\mathrm{d}h^{-1})h) = 0. \tag{4.30}$$

Therefore the considered element is in $(\ker\pi)^1$ and the corresponding element in $\pi(\mathrm{d}(\ker\pi)^1)$ is

$$\begin{aligned}
\pi(\mathrm{d}h^{-1}\mathrm{d}h + \mathrm{d}h\mathrm{d}h^{-1}) &= \gamma(\mathrm{d}h^{-1})\gamma(\mathrm{d}h) + \gamma(\mathrm{d}h)\gamma(\mathrm{d}h^{-1}) \\
&= \gamma^\mu \left(\frac{\partial}{\partial x^\mu}h^{-1}\right)\gamma^\nu \frac{\partial}{\partial x^\nu}h + \gamma^\nu \left(\frac{\partial}{\partial x^\nu}h\right)\gamma^\mu \frac{\partial}{\partial x^\mu}h^{-1} \\
&= [\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]\left(\frac{\partial}{\partial x^\mu}h^{-1}\right)\frac{\partial}{\partial x^\nu}h \\
&= -\left(\frac{2}{h^2}g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}h\right)\frac{\partial}{\partial x^\nu}h\right)1.
\end{aligned} \tag{4.31}$$

By linear combination we get the junk,

$$\pi(\mathrm{d}(\ker\pi)^1) = \{f1, \quad f \in \mathcal{A}\}. \tag{4.32}$$

On the other hand

$$\pi(\Omega^2\mathcal{A}) = \{f_{\mu\nu}\gamma^\mu\gamma^\nu, \quad f_{\mu\nu} \in \mathcal{A}\} \tag{4.33}$$

and

$$\pi(\mathrm{d}f_1\mathrm{d}f_2 + \mathrm{d}f_2\mathrm{d}f_1) = \left(2g^{\mu\nu}\frac{\partial}{\partial x^\mu}f_1\frac{\partial}{\partial x^\nu}f_2\right)1 \tag{4.34}$$

After dividing out the junk, $\pi(\mathrm{d}f_1)$ and $\pi(\mathrm{d}f_2)$ anticommute whereas they did not anticommute in $\pi(\Omega^2\mathcal{A})$. We may now identify a general element

$$\pi(f_0\mathrm{d}f_1\mathrm{d}f_2) \quad \in \Omega^2_{\partial\!\!\!/}\mathcal{A} \tag{4.35}$$

with the differential 2-form on $M$

$$f_0\mathrm{d}f_1\mathrm{d}f_2 \quad \in \Omega^2 M. \tag{4.36}$$

Note that we have treated the quotient space like a subspace which is legitimate only in presence of an appropriate scalar product. This scalar product will be defined in terms of the involution and a trace in the next section.

The involution that $\Omega M$ inherits from $\Omega_{\partial\!\!\!/}\mathcal{A}$ via the sketched isomorphism is with our conventions

$$(f_0\mathrm{d}f_1\mathrm{d}f_2...\mathrm{d}f_p)^* = (-1)^{(1/2)p(p-1)}\bar{f}_0\mathrm{d}\bar{f}_1\mathrm{d}\bar{f}_2...\mathrm{d}\bar{f}_p. \tag{4.37}$$

The orientability axiom alluded to above is motivated from this isomorphism, $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}x^4 \cong (\det g_{..})^{1/2}\gamma^1\gamma^2\gamma^3\gamma^4 = (\det g_{..})^{1/2}\gamma_5$.

## 4.3 The scalar products in noncommutative geometry

To play the Yang-Mills game, we need a scalar product for differential forms. In the noncommutative context, the scalar product has another utility. It allows us to interpret the differential forms in $\Omega_\mathcal{D}\mathcal{A}$ not as classes but as concrete operators on the Hilbert space $\mathcal{H}$: degree by degree, we embed $\Omega^p_\mathcal{D}$ in $\pi(\Omega^p\mathcal{A})$ as orthogonal complement of $\mathcal{J}^p$. If $\mathcal{H}$ was finite dimensional, we would naturally take as scalar product of two operators $\kappa$ and $\varphi$, $<\kappa,\varphi>= \mathrm{Re}\,\mathrm{tr}(\kappa^*\varphi)$. For infinite dimensional Hilbert spaces $\mathcal{H}$, like the space of spinors $\mathcal{L}^2(\mathcal{S})$, we have to regularize and we use the Dirac operator to do so. Thanks to the asymptotic behavior of its spectrum, $\mathrm{Re}\,\mathrm{tr}[\kappa^*\varphi\,|\mathcal{D}|^{-\dim M}]$ only diverges logarithmically. The Dixmier trace $\mathrm{tr}_\omega$ gets rid of this divergence [10]: For any bounded, positive operator $Q$ on $\mathcal{H}$ we define the *Dixmier trace* $\mathrm{tr}_\omega$ by

$$\mathrm{tr}_\omega\,(Q|\mathcal{D}|^{-\dim}) := \lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\lambda_n, \tag{4.38}$$

where the $\lambda_n$ are the eigenvalues of $Q|\mathcal{D}|^{-\dim}$ arranged in a decreasing sequence discarding the zero modes of the Dirac operator. Now we proceed as in the finite dimensional case ($\dim M = 0$) and define a scalar product on $\pi(\Omega\mathcal{A})$ by

$$<\kappa,\varphi>:= \mathrm{Re}\ \mathrm{tr}_\omega\,(\kappa^*\varphi|\mathcal{D}|^{-\dim}),\quad \kappa,\varphi\in\pi(\Omega^p\mathcal{A}). \tag{4.39}$$

Note that $\kappa$ and $\varphi$ are bounded because $[\mathcal{D},\rho(a)]$ are by axiom. In the commutative case, for a four dimensional spacetime $M$, this scalar product can be computed to be

$$<\kappa,\varphi>= \tfrac{1}{32\pi^2}\,\mathrm{Re}\int_M \mathrm{tr}_4\,[\kappa^*\varphi]\,\mathrm{d}^4x. \tag{4.40}$$

It is independent of $M$. $\mathrm{tr}_4$ denotes the trace over the gamma matrices. With this scalar product $\Omega_\mathcal{D}\mathcal{A}$ is a subspace of $\pi(\Omega\mathcal{A})$, by definition orthogonal to the junk. As subspace $\Omega_\mathcal{D}\mathcal{A}$ inherits a scalar product, that we denote by

$$(\kappa,\varphi) = <\kappa,\varphi>,\quad \kappa,\varphi\in\Omega^p_\mathcal{D}\mathcal{A}. \tag{4.41}$$

In the commutative case in four dimensions, thanks to well known results for $\mathrm{tr}_4\left[\gamma^{\mu_1}v_{\mu_1}...\gamma^{\mu_q}v_{\mu_q}\right]$ this scalar product vanishes for forms with different degree. By the isomorphism (4.20) between $\Omega_{\partial\!\!\!/}\mathcal{A}$ and $\Omega M$ the corresponding scalar product on differential forms, still denoted by $(\cdot,\cdot)$, is

$$(\kappa,\varphi) = \tfrac{1}{8\pi^2}\,\mathrm{Re}\int_M \kappa^* * \varphi,\quad \kappa,\varphi\in\Omega^p M. \tag{4.42}$$

Let us illustrate this by a simple example on the flat four torus with all circumferences measuring $2\pi$. Denote by $\psi_B(x)$, $B=1,2,3,4$, the four components of the spinor. The Dirac operator is

$$\partial\!\!\!/ = \begin{pmatrix} i\partial/\partial x^0 & 0 & -\partial/\partial x^3 & -\partial/\partial x^1 + i\partial/\partial x^2 \\ 0 & i\partial/\partial x^0 & -\partial/\partial x^1 - i\partial/\partial x^2 & \partial/\partial x^3 \\ \partial/\partial x^3 & \partial/\partial x^1 - i\partial/\partial x^2 & -i\partial/\partial x^0 & 0 \\ \partial/\partial x^1 + i\partial/\partial x^2 & -\partial/\partial x^3 & 0 & -i\partial/\partial x^0 \end{pmatrix}. \quad (4.43)$$

After a Fourier transform

$$\psi_B(x) \; =: \sum_{j_0,\dots,j_3\in\mathbb{Z}} \hat\psi_B(j_0,\dots,j_3)\exp(-ij_\mu x^\mu), \quad B=1,2,3,4 \quad (4.44)$$

the eigenvalue equation $\partial\!\!\!/\psi = \lambda\psi$ reads

$$\begin{pmatrix} j_0 & 0 & ij_3 & ij_1+j_2 \\ 0 & j_0 & ij_1-j_2 & -ij_3 \\ -ij_3 & -ij_1-j_2 & -j_0 & 0 \\ -ij_1+j_2 & ij_3 & 0 & -j_0 \end{pmatrix} \begin{pmatrix} \hat\psi_1 \\ \hat\psi_2 \\ \hat\psi_3 \\ \hat\psi_4 \end{pmatrix} = \lambda \begin{pmatrix} \hat\psi_1 \\ \hat\psi_2 \\ \hat\psi_3 \\ \hat\psi_4 \end{pmatrix}. \quad (4.45)$$

Its characteristic equation is $\left[\lambda^2 - (j_0^2+j_1^2+j_2^2+j_3^2)^2\right]^2 = 0$ and for fixed $j_\mu$, each eigenvalue $\lambda = \pm\sqrt{j_0^2+j_1^2+j_2^2+j_3^2}$ has multiplicity two. Therefore asymptotically for large $\Lambda$ there are $4B_4\Lambda^4$ eigenvalues (counted with their multiplicity) whose absolute values are smaller than $\Lambda$. $B_4=\pi^2/2$ denotes the volume of the unit ball in $\mathbb{R}^4$. Let us arrange the absolute values of the eigenvalues in an increasing sequence. Taking due account of their multiplicities we have for large $n$

$$|\lambda_n| \approx \left(\frac{n}{2\pi^2}\right)^{1/4} \quad (4.46)$$

and we can check the Dixmier trace in equation (4.40) for instance with $\kappa=\varphi=1 \;\in \pi(\Omega^0\mathcal{A})$

$$\begin{aligned} <1,1> &= \mathrm{tr}_\omega(|\partial\!\!\!/|^{-4}) = \lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}|\lambda_n|^{-4} \\ &= \lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{2\pi^2}{n} = \lim_{N\to\infty}\frac{1}{\log N}\int_1^N \frac{2\pi^2}{n}\,\mathrm{d}n \\ &= 2\pi^2 = \frac{1}{32\pi^2}\int_M \mathrm{tr}_4[1]\mathrm{d}^4x. \end{aligned} \quad (4.47)$$

In the commutative case the following two scalar products

$$<\kappa,\varphi> \;:=\; \mathrm{Re}\ \mathrm{tr}_\omega\left(\kappa^*\varphi|\mathcal{D}|^{-\dim}\right), \quad \kappa,\varphi\in\pi(\Omega^p\mathcal{A}), \quad (4.48)$$

$$<\kappa,\varphi> \;:=\; \tfrac{1}{2}\mathrm{Re}\ \mathrm{tr}_\omega\left([\kappa+J\kappa J^{-1}]^*[\varphi+J\varphi J^{-1}]\,|\mathcal{D}|^{-\dim}\right), \quad (4.49)$$

are identical. This is not true in general. We anticipate that the generalization of the principle of general relativity to noncommutative geometry, Connes' second dreisatz, will exclude the first scalar product.

## 4.4 The commutative Yang-Mills action

The message of this section is that the commutative spectral triple of spacetime $M$ is a natural tool to reconstruct Maxwell's theory: this reconstruction unifies spacetime with internal space, $G = U(1)$. The first sign for this unification comes from the group of unitaries of $\mathcal{A}$. Remember that $\mathcal{A}$ is the algebra of complex valued functions on $M$ with involution just complex conjugation. The group of unitaries $U(\mathcal{A}) := \{u \in \mathcal{A},\ uu^* = u^*u = 1\}$ for this algebra is the group of functions from spacetime into $U(1)$ and this is Maxwell's gauge group, $U(\mathcal{A}) = {}^M U(1)$. Maxwell's four potential $A \in \Omega^1_{\partial\!\!\!/}\mathcal{A}$ is a 1-form that we take *anti*-Hermitean now in order to harmonize the abelian and non-Abelian case. A gauge transformation or unitary $u = \exp i\theta$ acts affinely on the gauge potential by

$$\rho_V(u)A := \rho(u)A\rho(u^{-1}) + \rho(u)\mathrm{d}\rho(u^{-1}) = A - i\mathrm{d}\theta. \tag{4.50}$$

The field strength

$$F := \mathrm{d}A + A^2 = \mathrm{d}A \quad \in \Omega^2_{\partial\!\!\!/}\mathcal{A} \tag{4.51}$$

transforms homogeneously under unitaries and is even gauge invariant in the commutative case,

$$\rho_V(u)F = \rho(u)F\rho(u^{-1}) = F. \tag{4.52}$$

The obviously gauge invariant Maxwell action can be written,

$$\begin{aligned} S_{\mathrm{Maxwell}}[A] &= z(F,F) = z\mathrm{Re}\ \mathrm{tr}_\omega\,(F^*F|\partial\!\!\!/|^{-4}) = \tfrac{z}{8\pi^2}\int_M F^* * F \\ &= \tfrac{z}{16\pi^2}\int_M F^*_{\mu\nu}F^{\mu\nu}(\det g_{\cdot\cdot})^{1/2}\mathrm{d}^4x =: \tfrac{\epsilon_0}{4e^2}\int_M F^*_{\mu\nu}F^{\mu\nu}(\det g_{\cdot\cdot})^{1/2}\mathrm{d}^4x, \end{aligned} \tag{4.53}$$

where $z = \pi/\alpha_{\mathrm{em}}$ is the fine-structure constant or gauge coupling $\alpha_{\mathrm{em}} := e^2/(4\pi\epsilon_0\hbar c)$. The commutative pure Yang-Mills theory is linear and to justify the word coupling, we have to add matter, say an electron $\psi$. The Dirac operator acts on it defining its kinetic energy, unitaries act on it by

$$\rho_{\mathrm{spinor}}(u)\psi = \rho(u)\psi, \quad u \in U(\mathcal{A}), \quad \psi \in \mathcal{H}, \tag{4.54}$$

and we define the minimal coupling by the covariant Dirac operator $D\!\!\!\!/ := \partial\!\!\!/ - \pi(A)$. We have already noted that the gravitational field $\omega$ drops out when we construct the differential forms. The same is true for the electromagnetic field, $\Omega_{\partial\!\!\!/}\mathcal{A} = \Omega_{D\!\!\!\!/}\mathcal{A}$. The Dirac action then reads

$$S_D[\psi, A] = \int_M \psi^*\, D\!\!\!\!/\psi\, |\det g_{\cdot\cdot}|^{1/2}\mathrm{d}^4x, \tag{4.55}$$

A mass term $m_\psi \psi^* \psi$ may be added.

Let us stress again that in Connes' formulation, the gauge coupling, that is the invariant scalar product in internal space, is induced from the scalar product of differential forms over spacetime.

## 4.5 Almost commutative geometries

One way to see the above commutative example is to say that the associative algebra of the spectral triple is $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}_f$, a tensor product of the commutative, infinite dimensionsal algebra of *real* valued functions $\mathcal{C}^\infty(M)$ on spacetime and the commutative, finite dimensional, *real* algebra $\mathcal{A}_f = \mathbb{C}$. The gauge group then is Abelian, $G = U(1) \subset \mathcal{A}_f$. It is natural to try noncommutative algebras for $\mathcal{A}_f$ to get non-Abelian gauge groups [11]. In this spirit we consider tensor products of entire spectral triples, and the message of this section is that if the fermionic representation breaks parity, the Higgs scalar and the symmetry breaking potential come free of charge. We call almost commutative geometry this cheap tensor product of the commutative, infinite dimensional spectral triple of a spacetime with a noncommutative finite dimensional spectral triple of a matrix algebra [12]. Remember that the spinning particle in quantum mechanics is also such a cheap tensor product, of an ordinary wave function with a vector in a representation space of $SU(2)$.

Let us denote by $(\mathcal{F}, \mathcal{L}^2(\mathcal{S}), \partial\!\!\!/, \gamma_5, C)$ the commutative spectral triple of a four dimensional spacetime and by $(\mathcal{A}_f, \mathcal{H}_f, \mathcal{D}_f, \chi_f, J_f)$, $\cdot_f$ for finite, the one of a (zero dimensional) internal space. Note that our $C$ is anti-unitary. According to the rules of noncommutative geometry the tensor product of these two spectral triples $(\mathcal{A}_t, \mathcal{H}_t, \mathcal{D}_t, \chi_t, J_t)$, $\cdot_t$ for tensor, is:

$$\begin{aligned} \mathcal{A}_t &= \mathcal{F} \otimes \mathcal{A}_f, \qquad \mathcal{H}_t = \mathcal{L}^2(\mathcal{S}) \otimes \mathcal{H}_f, \qquad \mathcal{D}_t = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D}_f, \\ &\qquad\qquad \chi_t = \gamma_5 \otimes \chi_f, \qquad J_t = C \otimes J_f. \end{aligned} \tag{4.56}$$

Before turning the crank, we must talk about the internal Dirac operator $\mathcal{D}_f$. From the axioms, we infer that with respect to the decomposition (4.1) of the fermionic Hilbert space $\mathcal{H}_f$ the internal Dirac operator has the form:

$$\mathcal{D}_f = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathcal{M}} \\ 0 & 0 & \bar{\mathcal{M}}^* & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{D}_f = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.57}$$

where $\mathcal{M}$ is the fermionic mass matrix. This is another manifestation of the unification of spacetime and internal space, the naked Dirac operator $\partial\!\!\!/$ and its mass matrix obey the same axioms.

As in the commutative case, we start by identifying the gauge group, the functions from spacetime into the finite dimensional Lie group $G = U(\mathcal{A}_f)$. It is represented affinely on the bosonic fields. They are anti-Hermitean 1-forms. But now,

$$\begin{aligned}\Omega^1_{\mathcal{D}_t}\mathcal{A}_t &= \Omega^1_{\not\partial}\mathcal{F}\otimes\Omega^0_{\mathcal{D}_f}\mathcal{A}_f \,\oplus\, \Omega^0_{\not\partial}\mathcal{F}\otimes\Omega^1_{\mathcal{D}_f}\mathcal{A}_f \\ &\cong \Omega^1(M,\mathcal{A}_f)\,\oplus\,\mathcal{F}\otimes\Omega^1_{\mathcal{D}_f}\mathcal{A}_f \;\ni A_t =: (A,H). \end{aligned} \tag{4.58}$$

From the anti-Hermiticity of $A_t$, it follows that $A$ is in fact a Lie algebra valued 1-form on spacetime, $A \in \Omega^1(M,\mathfrak{g})$, i.e. a Yang-Mills potential. $\mathfrak{g} := u(\mathcal{A}_f) := \{X \in \mathcal{A}_f,\ X + X^* = 0\}$ is the Lie algebra of the group of unitaries $G = U(\mathcal{A}_f)$. On the other hand, the Higgs scalar $H$ is a 0-form on spacetime, valued in a representation of the Lie group $G$. The inhomogeneous transformation law,

$$\rho_{tV}(u)A_t := \rho_t(u)A_t\rho_t(u^{-1}) + \rho_t(u)\delta_t\rho_t(u^{-1}) = (\rho_V(u)A, \rho_S(u)H), \tag{4.59}$$

$$\rho_V(u)A = \rho_f(u)A\rho_f(u)^{-1} + \rho_f(u)\mathrm{d}\rho_f(u)^{-1}, \tag{4.60}$$

$$\rho_S(u)H = \rho_f(u)H\rho_f(u^{-1}) + \rho_f(u)\delta_f\rho_f(u^{-1}), \tag{4.61}$$

determines according to which *group* representation $\rho_S$ the Higgs scalar transforms and this depends on the details of the internal spectral triple. We denote by $\rho_t$ the representation of $\mathcal{A}_t$ on $\mathcal{H}_t$, by $\rho_f$ the representation of $\mathcal{A}_f$ on $\mathcal{H}_f$, by $\delta_t$ the differential of $\Omega_{\mathcal{D}_t}\mathcal{A}_t$ and so forth. Next we define the field strengh,

$$F_t = \delta_t A_t + A_t^2 \quad \in \Omega^2_{\mathcal{D}_t}\mathcal{A}. \tag{4.62}$$

To decompose the field strength, it is comfortable to change scalar variables,

$$\Phi(x) := H(x) - i\mathcal{D}_f = -\Phi^* \in \Omega^0(M,\Omega^1_{\mathcal{D}_f}\mathcal{A}_f). \tag{4.63}$$

This change of variables is well defined within $\Omega^0(M,\Omega^1_{\mathcal{D}_f}\mathcal{A}_f)$ thanks to the orientability axiom [13]. $\Phi$ has the good taste to transform homogeneously under a gauge transformation $u$ and we can define its covariant exterior derivative,

$$\mathrm{D}\Phi := \mathrm{d}\Phi + [\rho_f(A),\Phi] \in \Omega^1(M,\Omega^1_{\mathcal{D}_f}\mathcal{A}_f). \tag{4.64}$$

The field strength decomposes as

$$F_t = (F, C - \alpha C, -\mathrm{D}\Phi\gamma_5), \tag{4.65}$$

with

$$F = \mathrm{d}A + A^2 \in \Omega^2(M,\mathfrak{g}), \tag{4.66}$$

$$C = \delta_f H + H^2 \in \Omega^0(M,\Omega^2_{\mathcal{D}_f}\mathcal{A}_f). \tag{4.67}$$

The internal field strength $C$, $C$ for curvature, should not be confused with the $C$ of charge conjugation. $\alpha C \in \Omega^0(M, \Omega^2_{\mathcal{D}_f}\mathcal{A}_f + \mathcal{J}_f^2)$ is the tricky piece of the computation, it comes from the interference in degree two of spacetime junk and internal junk. The former is isomorphic to $\Omega^0 M$, a happy circumstance that allows us to compute $\alpha C$ pointwise [14]. For fixed $x$, $C \in \Omega_{\mathcal{D}_f}\mathcal{A}_f \subset \mathrm{End}\mathcal{H}_f$ and $\alpha C \in \pi(\Omega\mathcal{A}_f) \subset \mathrm{End}\mathcal{H}_f$ are finite dimensional operators, i.e. matrices. Let us denote by $<\kappa,\varphi> = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}[(\kappa + J_f\kappa J_f^{-1})^*(\varphi + J_f\varphi J_f^{-1})]$ the finite dimensional scalar product. Then $\alpha C$ is uniquely determined by the linear equations

$$< r, C - \alpha C > = 0 \qquad \text{for all} \quad r \in \rho_f(\mathcal{A}_f), \tag{4.68}$$

$$< j, C - \alpha C > = 0 \qquad \text{for all} \quad j \in \mathcal{J}_f^2, \tag{4.69}$$

where the trace is over the finite dimensional Hilbert space $\mathcal{H}_f$. Under a gauge transformation $u(x)$, the field strength transforms homogeneously and we can define, as before, the Yang-Mills action,

$$S_{\mathrm{YM}}[A_t] = z(F_t, F_t) = z\mathrm{Re}\ \mathrm{tr}_\omega\,(F_t^* F_t |\mathcal{D}_t|^{-4}). \tag{4.70}$$

The differential algebra contains the Lie algebra as 0-forms and the scalar product $(\cdot,\cdot)$ restricted to the Lie algebra is an invariant scalar product. Therefore this action is gauge invariant. Let us decompose it, $S_{\mathrm{YM}}[A_t] = S_{\mathrm{YM}}[A, H]$:

$$S_{\mathrm{YM}}[A,H] = \tfrac{z}{8\pi^2}\int_M (F, *F) + \tfrac{z}{8\pi^2}\int_M (\mathrm{D}\Phi, *\mathrm{D}\Phi) + \tfrac{z}{8\pi^2}\int_M *V(H), \tag{4.71}$$

with

$$V(H) = < C - \alpha C, C - \alpha C > = (C, C) - < \alpha C, \alpha C > . \tag{4.72}$$

The first term, a non-Abelian Yang-Mills action, is no surprise. The second, a Klein-Gordon action, propagates the Higgs scalar. The Higgs potential $V(H)$ breaks the gauge group spontaneously, if the fermions break parity. As we shall see, the computation of the Higgs sector, representation and potential, will be intricate even though it follows from a simple geometric definition, $S_{\mathrm{YM}}[A_t] = z(F_t, F_t)$.

To end this section, we mention the Dirac Lagrangian, $\mathcal{L}_{\mathrm{Dirac}} = \psi^*\mathcal{D}_{t,\mathrm{cov}}\psi$. The total, covariant Dirac operator is

$$\mathcal{D}_{t,\mathrm{cov}} = \mathcal{D}_t - \pi_t(A_t) + J_t(\mathcal{D}_t - \pi_t(A_t))J_t^{-1}. \tag{4.73}$$

It is covariant with respect to the *group* representation,

$$\rho_{\mathrm{spinor}}(u)\,\psi \;=\; \rho_t(u)\,J_t\rho_t(u)J_t^{-1}\;\psi, \quad u \in U(\mathcal{A}_t) = {}^M U(\mathcal{A}_f). \tag{4.74}$$

Note the appearance of charge conjugation that will be crucial. The decomposition of this Lagrangian is:

$$\mathcal{L}_{\rm Dirac} = \psi^*(2\,\partial\!\!\!/ + i\rho_f(\,A\!\!\!/\,) + J_t i\rho_f(\,A\!\!\!/\,)J_t^{-1})\psi + \psi^*(i\Phi\gamma_5 + J_t i\Phi\gamma_5 J_t^{-1})\psi. \tag{4.75}$$

In words: almost commutative geometry promotes the Higgs scalar to a connection and thereby unifies the gauge couplings hidden in $\rho_f(\,A\!\!\!/\,)$ with the Yukawa couplings hidden in $\Phi$.

The general Poincaré duality of noncommutative geometry is beyond the scope of this introduction. In the almost commutative case the Poincaré duality can be stated easily. Since it holds in commutative geometry we only have to worry about the finite dimensional internal space. Let $p_j$ be a set of minimal projectors of $\mathcal{A}_f$ and define the intersection form $\cap$ to be the matrix

$$\cap_{ij} := \mathrm{tr}\left[\chi_f\,\rho_f(p_i)\,J_f\rho_f(p_j)J_f^{-1}\right]. \tag{4.76}$$

Poincaré duality holds if and only if the intersection form is non-degenerate, $\det\cap \neq 0$. Note that in the finite dimensional case, the Poincaré duality does not involve the Dirac operator.

## 4.6 A minimax example

It is time for an example. To the best of my knowledge, the simplest, nontrivial example – a maximum of pleasure with a minimum of effort – is quite complicated. Strange enough, it resembles the standard model of electro-weak forces, the example section 3.5.

We just learned that all computations can be done in the finite dimensional, internal space. Therefore we drop the subscript $\cdot_f$. Consider the internal spectral triple,

$$\mathcal{A} \;=\; \mathbb{H}\oplus\mathbb{C} \;\ni\; (a,b), \tag{4.77}$$

$$\mathcal{H} \;=\; \mathcal{H}_L\oplus\mathcal{H}_R\oplus\mathcal{H}_L^c\oplus\mathcal{H}_R^c \;=\; (\mathbb{C}^2\oplus\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C})\otimes\mathbb{C}^N, \tag{4.78}$$

$$\begin{aligned}\rho(a,b) \;&=\; \begin{pmatrix}\rho_L(a)&0&0&0\\0&\rho_R(b)&0&0\\0&0&\bar\rho_L^c(b)&0\\0&0&0&\bar\rho_R^c(b)\end{pmatrix}\\ &=\; \begin{pmatrix}a\otimes 1_N&0&0&0\\0&\bar b 1_N&0&0\\0&0&b1_2\otimes 1_N&0\\0&0&0&b1_N\end{pmatrix},\end{aligned} \tag{4.79}$$

$$\mathcal{D} \;=\; \begin{pmatrix}0&\mathcal{M}&0&0\\\mathcal{M}^*&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{M} = \begin{pmatrix}0\\1\end{pmatrix}\otimes M_e,\quad M_e = \begin{pmatrix}m_e&0\\0&m_\mu\end{pmatrix}, \tag{4.80}$$

$$\chi = \begin{pmatrix} -1_{2N} & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & -1_{2N} & 0 \\ 0 & 0 & 0 & 1_N \end{pmatrix}, \tag{4.81}$$

$$J = \begin{pmatrix} 0 & 1_{3N} \\ 1_{3N} & 0 \end{pmatrix} \circ \text{complex conjugation.} \tag{4.82}$$

We denote by $\mathbb{H}$ the *real*, four dimensional algebra of quaternions. We write its elements as complex $2 \times 2$ matrices,

$$a = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \quad x, y \in \mathbb{C}. \tag{4.83}$$

The involution in $\mathbb{H}$ is Hermitian conjugation and the group of unitaries of $\mathbb{H}$ is $SU(2)$. The algebra $\mathbb{C} \ni b$ is also taken as real, two dimensional algebra. The physical basis of the *complex* fermionic Hilbert space consists of an electron and its left-handed neutrino in the first generation and a muon and its left-handed neutrino in the second generation. Of course there are also the anti-particles,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad e_R, \quad \mu_R, \qquad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \quad e_R^c, \quad \mu_R^c. \tag{4.84}$$

$N$ counts the number of generations, $N = 2$.

We are ready to turn the crank and start with the commutator

$$[\mathcal{D}, \rho(a,b)] = \begin{pmatrix} 0 & \mathcal{M}\rho_R(b) - \rho_L(a)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(a) - \rho_R(b)\mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.85}$$

We take advantage of the following simplification in our model,

$$\mathcal{M}\rho_R(b) = \rho_L \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \mathcal{M} =: \rho_L(B)\mathcal{M} \tag{4.86}$$

to compute a general 1-form. It is a sum of terms

$$\pi((a_0, b_0)\delta(a_1, b_1)) = -i \begin{pmatrix} 0 & \rho_L(a_0(B_1 - a_1))\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(B_0(a_1 - B_1)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.87}$$

and as vector space

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \left\{ i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(\tilde{h}^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ h, \tilde{h} \in \mathbb{H} \right\}. \tag{4.88}$$

The Higgs being an anti-Hermitian 1-form

$$H = i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} \in \mathbb{H} \tag{4.89}$$

is parameterized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}. \tag{4.90}$$

Likewise a general element in $\pi(\Omega^2\mathcal{A})$ is

$$\pi((a_0, b_0)\delta(a_1, b_1)\delta(a_2, b_2)) = \tag{4.91}$$

$$\begin{pmatrix} \rho_L(a_0)\rho_L(a_1 - B_1)\mathcal{M}\mathcal{M}^*\rho_L(a_2 - B_2) & 0 & 0 & 0 \\ 0 & \mathcal{M}^*\rho_L(B_0)\rho_L(a_1 - B_1)\rho_L(a_2 - B_2)\mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We rewrite the (1,1) matrix element,

$$(1_2 \otimes \Sigma)\rho_L(a_0)\rho_L(a_1 - B_1)\rho_L(a_2 - B_2) + (1_2 \otimes \Delta)\rho_L(a_0)\rho_L(a_1 - B_1)(\sigma_3 \otimes 1_N)\rho_L(a_2 - B_2),$$

where we have used the decomposition

$$\mathcal{M}\mathcal{M}^* = \begin{pmatrix} 0 & 0 \\ 0 & M_e M_e^* \end{pmatrix} = 1_2 \otimes \Sigma + \sigma_3 \otimes \Delta \tag{4.92}$$

with

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma = \tfrac{1}{2} M_e M_e^*, \quad \Delta = -\tfrac{1}{2} M_e M_e^*. \tag{4.93}$$

A general element in $(\ker \pi)^1$ is a finite sum of the form

$$\sum_j (a_0^j, b_0^j)\delta(a_1^j, b_1^j) \tag{4.94}$$

with the two conditions

$$\left[\sum_j \rho_L(a_0^j)\rho_L(a_1^j - B_1^j)\right]\mathcal{M} = 0, \quad \mathcal{M}^*\left[\sum_j \rho_L(B_0^j)\rho_L(a_1^j - B_1^j)\right] = 0. \tag{4.95}$$

Therefore the corresponding general element in $\pi(\delta(\ker \pi)^1)$ has only one nonvanishing matrix element in position (1,1):

$$(1_2 \otimes \Sigma)\sum_j \rho_L(a_0^j - B_0^j)\rho_L(a_1^j - B_1^j) + (1_2 \otimes \Delta)\sum_j \rho_L(a_0^j - B_0^j)(\sigma_3 \otimes 1_N)\rho_L(a_1^j - B_1^j) \tag{4.96}$$

still subject to the two conditions and we have the following inclusion

$$\pi(\delta(\ker\pi)^1)\supset\left\{i\begin{pmatrix}(1_2\otimes\Delta)\sum_j\rho_L(a_0^j(i\sigma_3)a_1^j)&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad\sum_j a_0^j a_1^j=0\right\}.\tag{4.97}$$

Note that $\rho_L$ is faithful and that

$$\left\{\sum_j a_0^j(i\sigma_3)a_1^j,\quad\sum_j a_0^j a_1^j=0\right\}$$

is an ideal in $\mathbb{H}$. This ideal is not zero, take for example

$$a_0^1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad a_1^1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad a_0^2=\begin{pmatrix}0&i\\i&0\end{pmatrix},\quad a_1^2=\begin{pmatrix}i&0\\0&-i\end{pmatrix},$$

with

$$\sum_j a_0^j(i\sigma_3)a_1^j=-2\begin{pmatrix}0&i\\i&0\end{pmatrix}.\tag{4.98}$$

The quaternions being a simple algebra, the ideal is the entire algebra and the junk is

$$\mathcal{J}^2=\pi(\delta(\ker\pi)^1)=\left\{i\begin{pmatrix}j\otimes\Delta&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad j\in\mathbb{H}\right\}.\tag{4.99}$$

Next we have to project out the junk using the scalar product,

$$\Omega^2_{\mathcal{D}}\mathcal{A}=\left\{\begin{pmatrix}\tilde c\otimes\Sigma&0&0&0\\0&\mathcal{M}^*\rho_L(c)\mathcal{M}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad\tilde c,c\in\mathbb{H}\right\}.\tag{4.100}$$

Since $\pi$ is a homomorphism of involution algebras, the product in $\Omega_{\mathcal{D}}\mathcal{A}$ is given by matrix multiplication followed by the orthogonal projection $P$ and the involution is given by transposition complex conjugation. In order to calculate the differential $\delta$, we go back to the universal differential envelope. The result is

$$\delta:\Omega^1_{\mathcal{D}}\mathcal{A}\longrightarrow\Omega^2_{\mathcal{D}}\mathcal{A}$$

$$i\begin{pmatrix}0&\rho_L(h)\mathcal{M}&0&0\\\mathcal{M}^*\rho_L(\tilde h^*)&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\longmapsto\begin{pmatrix}\tilde c\otimes\Sigma&0&0&0\\0&\mathcal{M}^*\rho_L(c)\mathcal{M}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\tag{4.101}$$

with $\tilde{c} = c = h + \tilde{h}^*$.

We are now in position to compute the curvature:

$$C := \delta H + H^2 = \left(1 - |\varphi|^2\right) \begin{pmatrix} 1_2 \otimes \Sigma & 0 & 0 & 0 \\ 0 & \mathcal{M}^*\mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.102}$$

with the homogeneous scalar variable

$$\Phi := H - i\mathcal{D} =: i \begin{pmatrix} 0 & \rho_L(\phi)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(\phi^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H}, \tag{4.103}$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad |\varphi|^2 = |\varphi_1|^2 + |\varphi_2|^2. \tag{4.104}$$

In the example of section 3.5 we also had two useful parameterizations of the scalar field, $\varphi$ and $h$. They coincide precisely with the two parameterizations here, only they appear in opposite chronology. The computation of $\alpha C$ is long but presents no difficulty. In this example there is no junk component:

$$\alpha C = (1 - |\varphi|^2)\, \rho(\alpha 1_2, \beta). \tag{4.105}$$

The real numbers $\alpha$ and $\beta$ are determined by the two linear equations

$$\begin{aligned} N\alpha \ + \ N\beta &= \tfrac{1}{2}(m_e^2 + m_\mu^2), \\ N\alpha \ + \ 3N\beta &= \tfrac{3}{2}(m_e^2 + m_\mu^2). \end{aligned} \tag{4.106}$$

Their solution is

$$\alpha = 0, \quad \beta = \tfrac{1}{2N}(m_e^2 + m_\mu^2), \tag{4.107}$$

and the Higgs potential is,

$$V(H) = (C, C) - < \alpha C, \alpha C > = \tfrac{z}{8\pi^2} \tfrac{3}{2} (1 - |\varphi|^2)^2 [m_e^4 + m_\mu^4 - \tfrac{1}{N}(m_e^2 + m_\mu^2)^2]. \tag{4.108}$$

Now we can explain why our minimax model must contain at least $N = 2$ generations of leptons with distinct masses. Otherwise the Higgs potential vanishes.

Next we compute the Klein-Gordon action,

$$\tfrac{z}{8\pi^2} \int_M (\mathrm{D}\Phi, *\mathrm{D}\Phi) = \tfrac{z}{8\pi^2} 2(m_e^2 + m_\mu^2) \int_M \mathrm{D}\varphi^* * \mathrm{D}\varphi. \tag{4.109}$$

The covariant derivative with respect to the gauge potential $A = ({}^{(2)}A,\ \frac{1}{2}\,{}^{(1)}A) \in \Omega^1(M, su(2) \oplus u(1))$ follows from the group representation carried by the scalar doublet, equations (4.64) and (4.103),

$$\mathrm{D}\varphi = \mathrm{d}\varphi \,+\, {}^{(2)}A\varphi \,+\, \tfrac{1}{2}\,{}^{(1)}A\varphi. \tag{4.110}$$

The factor $\frac{1}{2}$ in front of ${}^{(1)}A$ is conventional: we want the hypercharge of the Higgs scalar to be one half. To put the scalar Lagrangian into conventional form,

$$\tfrac{1}{2}\mathrm{D}_\mu\varphi_{\mathrm{ph}}^*\mathrm{D}^\mu\varphi_{\mathrm{ph}} \,+\, \lambda|\varphi_{\mathrm{ph}}|^4 \,-\, \tfrac{1}{2}\mu^2|\varphi_{\mathrm{ph}}|^2, \tag{4.111}$$

we renormalize the scalar field,

$$|\varphi_{\mathrm{ph}}|^2 := \left[\tfrac{z}{8\pi^2}\, 4\,(m_e^2 + m_\mu^2)\right]\ |\varphi|^2. \tag{4.112}$$

The physical scalar $\varphi_{\mathrm{ph}}$ now has the correct dimensions of a mass and we will drop the subscript $\cdot_{\mathrm{ph}}$. For the scalar couplings we get,

$$\lambda \,=\, \frac{3\pi^2}{4z}\left[\frac{m_e^4 + m_\mu^4}{(m_e^2+m_\mu^2)^2} - \frac{1}{N}\right], \quad \mu^2 \,=\, \tfrac{3}{2}\left[\frac{m_e^4 + m_\mu^4}{(m_e^2+m_\mu^2)} \,-\, \frac{1}{N}\,(m_e^2+m_\mu^2)\right]. \tag{4.113}$$

The energy of the vacuum or cosmological constant $V(\varphi_0)$ vanishes automatically. The vacuum expectation is,

$$|\varphi_0|^2 = v^2 = \tfrac{z}{2\pi^2}(m_e^2 + m_\mu^2). \tag{4.114}$$

and the group of unitaries $SU(2) \times U(1)$ is broken spontaneously down to $U(1)$. To avoid any misunderstanding, the miracle is not the symmetry breaking. This symmetry breaking is introduced by hand with the masses for chiral fermions. The miracle is that this explicit symmetry breaking produces a Higgs field and that this Higgs field promotes the symmetry breaking from explicit to spontaneous. The spontaneous symmetry breaking in turn produces the gauge boson masses. In other words, in almost commutative geometry the invariance group of the fermionic mass matrix is necessarily equal to the invariance group of the mass matrix of the gauge bosons, the little group. This is not true in a general Yang-Mills-Higgs model, but it is true in the standard model.

We compute the Yang-Mills action,

$$\tfrac{z}{8\pi^2}\int_M (F, *F) = \tfrac{z}{8\pi^2}\int_M \left[N\mathrm{tr}\,{}^{(2)}F^* * {}^{(2)}F \,+\, \tfrac{3}{2}N\,{}^{(1)}F^* * {}^{(1)}F\right]. \tag{4.115}$$

Comparing with the action in conventional form,

$$\tfrac{1}{2}\int_M \left[\tfrac{2}{g_2^2}\mathrm{tr}\,{}^{(2)}F^* * {}^{(2)}F \,+\, \tfrac{1}{g_1^2}\,{}^{(1)}F^* * {}^{(1)}F,\right] \tag{4.116}$$

we get the gauge couplings,

$$g_2^2 = \frac{8\pi^2}{Nz}\,, \quad g_1^2 = \frac{8\pi^2}{3Nz}\,. \tag{4.117}$$

The weak mixing angle $\theta_w$ is therefore fixed,

$$\sin^2\theta_w := \frac{g_1^2}{g_1^2+g_2^2} = \tfrac{1}{4}. \tag{4.118}$$

Also fixed is the $\rho$-factor,

$$\rho := \frac{m_W^2}{\cos^2(\theta_w)\ m_Z^2} = 1. \tag{4.119}$$

It is unit because the scalar sits in a doublet.

Noncommutative geometry unifies the gauge, Higgs and Yukawa couplings, in the same way that gauge invariance unifies the tri- and quadri-linear self couplings of the gauge bosons and the minimal couplings of the gauge bosons to fermions and scalars:

$$\lambda = \tfrac{3}{32}(N-1)g_2^2 + O(m_e^2/m_\mu^2)\,g_2^2, \tag{4.120}$$

$$g_e^2 = \frac{m_e^2}{v^2} = \tfrac{2\pi^2}{z}\,\frac{m_e^2}{m_e^2+m_\mu^2} = O(m_e^2/m_\mu^2)\,g_2^2, \tag{4.121}$$

$$g_\mu^2 = \frac{m_\mu^2}{v^2} = \tfrac{2\pi^2}{z}\,\frac{m_\mu^2}{m_e^2+m_\mu^2} = \tfrac{N}{4}g_2^2 + O(m_e^2/m_\mu^2)\,g_2^2. \tag{4.122}$$

A general lesson that we learn from our minimax example is the link between parity break down and spontaneous gauge symmetry break down. They go together in almost commutative models. Indeed take any vectorial model, that means $\rho_L = \rho_R$ and a mass matrix $\mathcal{M}$ commuting with this representation $\rho_L$. Examples of vectorial theories are the parity preserving electromagnetic and strong forces. For these models, the internal differential forms vanish identically except in degree zero. Consequently there is no Higgs scalar, no spontaneous symmetry break down and the gauge bosons, e.g. the photon and the gluons remain massless. The Yang-Mills-Higgs model building kit on the other hand allows for spontaneous symmetry break down of any model, parity violating or vectorial.

The last item we have to discuss is Poincaré duality. There are two minimal projectors, $p_1 = (1_2, 0)$ and $p_2 = (0,1)$. The intersection form is computed easily,

$$\cap_{ij} := \mathrm{tr}\left[\chi\,\rho(p_i)\,J\rho(p_j)J^{-1}\right] = -2N\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}. \tag{4.123}$$

It is non-degenerate and Poincaré duality holds.

Before we leave our minimax model we must talk about its short coming, the quarks with their electric charges are difficult to fit in. This problem will be cured by the inclusion of strong interactions.

## 4.7 The standard model from Connes' first dreisatz

The strong interactions being vectorial their addition to the minimax example is not difficult and we go quickly over the calculations [6][11][8][15]. The finite dimensional algebra is chosen to reproduce $SU(2) \times U(1) \times SU(3)$,

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c). \tag{4.124}$$

The fermionic Hilbert spaces are copied from the standard model,

$$\begin{aligned} \mathcal{H}_L &= \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}\right), & (4.125) \\ \mathcal{H}_R &= \left((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}\right). & (4.126) \end{aligned}$$

In each summand, the first factor denotes weak isospin doublets or singlets, the second denotes $N$ generations, $N = 3$, and the third denotes color triplets or singlets. Let us choose the following basis of $\mathcal{H} = \mathbb{C}^{90}$:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L;$$

$$\begin{matrix} u_R, & c_R, & t_R, \\ d_R, & s_R, & b_R, \end{matrix} \quad e_R, \quad \mu_R, \quad \tau_R;$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L^c, \begin{pmatrix} c \\ s \end{pmatrix}_L^c, \begin{pmatrix} t \\ b \end{pmatrix}_L^c, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^c;$$

$$\begin{matrix} u_R^c, & c_R^c, & t_R^c, \\ d_R^c, & s_R^c, & b_R^c, \end{matrix} \quad e_R^c, \quad \mu_R^c, \quad \tau_R^c.$$

The representation $\rho$ acts on $\mathcal{H}$ by

$$\rho(a, b, c) := \begin{pmatrix} \rho_w(a, b) & 0 \\ 0 & \bar{\rho}_s(b, c) \end{pmatrix} := \begin{pmatrix} \rho_{wL}(a) & 0 & 0 & 0 \\ 0 & \rho_{wR}(b) & 0 & 0 \\ 0 & 0 & \bar{\rho}_{sL}(b, c) & 0 \\ 0 & 0 & 0 & \bar{\rho}_{sR}(b, c) \end{pmatrix} \tag{4.127}$$

with

$$\rho_{wL}(a) := \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 \\ 0 & a \otimes 1_N \end{pmatrix}, \quad \rho_{wR}(b) := \begin{pmatrix} B \otimes 1_N \otimes 1_3 & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}, \tag{4.128}$$

$$B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \tag{4.129}$$

$$\rho_{sL}(b, c) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_2 \otimes 1_N \end{pmatrix}, \quad \rho_{sR}(b, c) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}. \tag{4.130}$$

The chosen representation $\rho$ takes into account weak interactions $\rho_w(a,b),\ a \in \mathbb{H},\ b \in \mathbb{C}$, and strong interactions $\rho_s(b,c),\ c \in M_3(\mathbb{C})$, $c$ for color. This choice discriminates between leptons (color singlets) and quarks (color triplets). The role of $b \in \mathbb{C}$ appearing in both weak interactions $\rho_w(a,b)$ and strong interactions $\rho_s(b,c)$ is crucial to make $\rho(a,b,c)$ a representation of $\mathcal{A}$ and is crucial for weak hypercharge computations. There is an apparent asymmetry between particles and anti-particles, the former are subject to weak, the latter to strong interactions. However, since particles and anti-particles are permuted in the covariant Dirac operator (4.73) by

$$J = \begin{pmatrix} 0 & 1_{15N} \\ 1_{15N} & 0 \end{pmatrix} \circ \text{ complex conjugation}, \tag{4.131}$$

the theory is invariant under charge conjugation. For completeness, we record the chirality as matrix

$$\chi = \begin{pmatrix} -1_{8N} & 0 & 0 & 0 \\ 0 & 1_{7N} & 0 & 0 \\ 0 & 0 & -1_{8N} & 0 \\ 0 & 0 & 0 & 1_{7N} \end{pmatrix}. \tag{4.132}$$

The third item in the spectral triple is the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.133}$$

The fermionic mass matrix of the standard model is

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes M_u \otimes 1_3 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes M_d \otimes 1_3 & 0 \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes M_e \end{pmatrix}, \tag{4.134}$$

with

$$M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \qquad M_d := C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \tag{4.135}$$

$$M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{4.136}$$

All indicated fermion masses are supposed positive and different. The Cabibbo-Kobayashi-Maskawa matrix $C_{KM}$ is supposed non-degenerate in the sense that there is no simultaneous mass and weak interaction eigenstate.

Let us turn the crank and record a few intermediate steps:

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \left\{ i \begin{pmatrix} 0 & \rho_{wL}(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_{wL}(\tilde{h}^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ h, \tilde{h} \in \mathbb{H} \right\}. \tag{4.137}$$

The Higgs being an anti-Hermitian 1-form

$$H = i \begin{pmatrix} 0 & \rho_{wL}(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} \in \mathbb{H}$$

is parameterized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}. \tag{4.138}$$

The internal junk in degree two is again

$$\mathcal{J}^2 = \left\{ i \begin{pmatrix} j \otimes \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j \in \mathbb{H} \right\} \tag{4.139}$$

with

$$\Delta := \tfrac{1}{2} \begin{pmatrix} (M_u M_u^* - M_d M_d^*) \otimes 1_3 & 0 \\ 0 & -M_e M_e^* \end{pmatrix}. \tag{4.140}$$

also containing the quark masses. The homogeneous scalar variable is:

$$\Phi := H - i\mathcal{D} =: i \begin{pmatrix} 0 & \rho_{wL}(\phi)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_{wL}(\phi^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H}, \tag{4.141}$$

and with $\varphi := {}^t(\varphi_1, \varphi_2)$, the internal field strength is:

$$C := \delta H + H^2 = \left(1 - |\varphi|^2\right) \begin{pmatrix} 1_2 \otimes \Sigma & 0 & 0 & 0 \\ 0 & \mathcal{M}^*\mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.142}$$

$$\Sigma := \tfrac{1}{2} \begin{pmatrix} (M_u M_u^* + M_d M_d^*) \otimes 1_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix}. \tag{4.143}$$

Again $\alpha C$ has no junk component,

$$\alpha C = (1 - |\varphi|^2)\, \rho(\alpha 1_2, \beta, \gamma 1_3). \tag{4.144}$$

To compute the real numbers $\alpha,\ \beta,\ \gamma$, we neglect all fermion masses with respect to the top mass. This approximation is good to $m_b^2/m_t^2 = 0.0006$ and we have the three linear equations:

$$\begin{array}{ccccccc} 4N\,\alpha & + & N\,\beta & + & 3N\,\gamma & = & \frac{3}{2}m_t^2 \\ 2N\,\alpha & + & 12N\,\beta & + & 6N\,\gamma & = & 3m_t^2 \\ 3N\,\alpha & + & 3N\,\beta & + & 6N\,\gamma & = & 3m_t^2, \end{array} \tag{4.145}$$

with solution

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \frac{1}{2N}\, m_t^2. \tag{4.146}$$

The Higgs and Yukawa couplings follow:

$$\mu^2 = \left(\frac{3}{2} - \frac{1}{N}\right) m_t^2, \tag{4.147}$$

$$\lambda = \frac{\pi^2}{6z}\left(\frac{3}{2} - \frac{1}{N}\right), \tag{4.148}$$

$$g_t^2 = \frac{m_t^2}{v^2} = \frac{2\pi^2}{3z}. \tag{4.149}$$

Before computing the gauge couplings, we face a problem. The group of unitaries $SU(2) \times U(1) \times U(3)$ is too big by one $U(1)$ factor. Indeed there is no associative algebra with $SU(3)$ as unitary group. Howerver there is an encouraging miracle, the representation of a linear combination of the two $u(1)$s coincides with the representation of the hypercharge $Y$ in the standard model. This miracle needs three colours and vectorial couplings of the $U(3)$. These vectorial couplings, in turn, are an immediate consequence of the first order condition in spectral triples together with the maximal parity violation of weak interactions [8][16][17]. All four *ad hoc* features of the standard model,

- gluons couple vectorially,
- gluons are massless,
- the $W$ couples axially,
- the $W$ is massive,

are rigidly tied together by the axioms of the spectral triple. To obtain the standard model we can project out the other, unwanted linear combination of the two $u(1)$s in $\mathfrak{g}$ by imposing the so-called unimodularity condition,

$$\mathrm{tr}\left[P\left(\rho(a,b,c) + J\rho(a,b,c)J^{-1}\right)\right] = 0, \tag{4.150}$$

where $P$ is the projection on $\mathcal{H}_L \oplus \mathcal{H}_R$, the space of particles. We note that this condition is equivalent to the condition of vanishing gauge anomalies [18]. Nevertheless the unimodularity condition is at this stage an artefact. Connes second dreisatz will improve this situation [7].

The computation of the gauge couplings is now straightforward,

$$g_3^2 = \frac{2\pi^2}{Nz}, \quad g_2^2 = \frac{2\pi^2}{Nz}, \quad g_1^2 = \frac{6\pi^2}{5Nz}. \tag{4.151}$$

In particular we have

$$\sin^2\theta_w = \tfrac{3}{8}, \quad g_3 = g_2, \tag{4.152}$$

as in grand unified theories and from the geometric unification of gauge and Higgs bosons,

$$\lambda = \tfrac{3N-2}{24} g_2^2, \quad g_t^2 = \tfrac{N}{3} g_2^2. \tag{4.153}$$

The confrontation of these four constraints with experiment calls for the renormalization group flow to be discussed in the next chapter.

Before leaving the standard model, we must verify its Poincaré duality. We have now three minimal projectors,

$$p_1 = (1_2, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = \left(0, 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right). \tag{4.154}$$

Note that $1_3$ is not minimal in $M_3(\mathbb{C})$ because it is a sum of three projectors of rank one,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.155}$$

All three are unitarily equivalent. The intersection form,

$$\cap = -2N \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{4.156}$$

is non-degenerate. However if we add right-handed neutrinos to the standard model, massive or not, then the intersection form,

$$\cap = -2N \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{4.157}$$

is degenerate and Poincaré duality fails.

## 4.8 Necessary conditions

We have become accustomed to see supersymmetric versions of any theory or model already on the market, supersymmetric quantum mecanics, supersymmetric Yang-Mills theories, supersymmetric $\sigma$-models, super gravities, super strings,... You should not believe that you can put noncommutative in front of any theory, not even in front of any Yang-Mills theory. It remains a miracle that the standard model is in the tiny, priviledged class of Yang-Mills theories allowing a noncommutative generalization and that putting almost commutative in front of the standard model produces its correct Higgs sector. The purpose of the present section is to assess this miracle. Needless to say that we call it a miracle because we do not have the slightest explanation for it today.

Recall the input bills of a Yang-Mills theory, a finite dimensional, real, compact Lie group $G$ and two unitary representations $\rho_L$ and $\rho_R$. The classification of all such groups teaches us that its Lie algebras $\mathfrak{g}$ are direct sums of simple Lie algebras from the three classical series $so(n)$, $su(n)$, $sp(n)$, and of the five exceptional Lie algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. Each of the simple Lie algebras has an infinite number of irreducible representations, for example $u(1) = so(2)$ has one irreducible representation for any charge $y \in \mathbb{R}$ and $su(2) = sp(1)$ has an irreducible representation of any dimension $d \in \mathbb{N}$ corresponding to spin $j = (d-1)/2$.

The input bills of an almost commutative Yang-Mills theory are a finite dimensional, real, associative involution algebra with unit, $\mathcal{A}$ and two faithful representations $\rho_L$ and $\rho_R$. The classification of these algebras is easier than the one for groups. Any such algebra is a direct sum of finite algebras from the three series, $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$, the $n \times n$ matrices with real, complex and quaternionic entries. The corresponding groups of unitaries have Lie algebras $so(n)$, $su(n)$, $sp(n)$. Therefore the exceptional Lie groups are unsuitable for Connes' dreisatz, not a great loss. Things are more exciting concerning the representations. Any associative algebra representation induces a Lie algebra representation but only very few Lie algebra representations can be extended to a representation of the ambient associative algebra. The tensor product of two $\mathfrak{g}$ representations is a $\mathfrak{g}$ representation. The tensor product of two $\mathcal{A}$ representations is not an $\mathcal{A}$ representation. The only irreducible representations of $M_1(\mathbb{C})$ have charge 1 and $-1$, the only irreducible representation of $M_1(\mathbb{H})$ is on $\mathbb{C}^2$. In general $M_n(\mathbb{R})$ has only one irreducible representation, the fundamental one, on $\mathbb{R}^n$, $M_n(\mathbb{C})$ has two, the fundamental one, on $\mathbb{C}^n$, and its conjugate, and $M_n(\mathbb{H})$ has one, the fundamental one, on $\mathbb{C}^{2n}$. Note that the fermions of the standard model only contain colour triplets and singlets and isospin doublets and singlets. The singlets are admitted thanks to the real structure $J$. The hypercharges may deviate from $\pm 1$ thanks to the unimodularity condition. The above general conditions on the group and its fermionic representations exclude already all popular grand unified models from almost commutative geometry. The axioms of the spectral triple contain further restrictions on the fermionic representations, the first order axiom and Poincaré duality. The complete classification of almost commutative geometries is given in [17]. The standard model is in this classification, the first order axiom implies that strong interactions are vectorial, Poincaré duality excludes right-handed neutrinos.

Concerning the coins, the Yang-Mills input is any invariant scalar product on the Lie algebra. In almost commutative geometry, this scalar product is the restriction of a scalar product on the entire space of differential forms. However, anticipating on Connes' second dreisatz we have not taken the most general such scalar product on forms, but we have picked one of the two simplest, (4.49). It involves only one positive constant, $z$, and consequently the three gauge couplings of the standard model are related by two constraints, equations (4.152). Finally, in almost commutative geometry all parameters in the fermionic mass matrix are input coins. The scalar representation is only a group representation and is computed to be a subrepresentation of the tensor product $\mathcal{H}_L^* \otimes \mathcal{H}_R$ and its conjugate [19]. This subrepresentation depends on the details of the fermionic mass matrix. The inclusion is however sufficient to exclude all left-right symmetric models from almost commutative geometry [20]. In left-right symmetric models parity violation is spontaneous, induced from the mass matrix of the gauge bosons. Finally the Higgs couplings are also computed as a function of the fermionic mass matrix, equations (4.153) for the standard model. The induced mass matrix of the gauge bosons has the same invariance as the fermionic mass matrix. As the minimax example shows, the computation of the Higgs representation and couplings is involved. The most modest Yang-Mills-Higgs model beyond the standard model has the group $SU(3)\times SU(2)\times U(1)\times U(1)$. Any model in almost commutative geometry yielding this group, like for instance the standard model without the unimodularity condition, is incompatible with experiment [21]. At present we have no complete classification of all Yang-Mills-Higgs models accessible to almost commutative geometry. Figure 4.1 tries to give an impression of the situation.

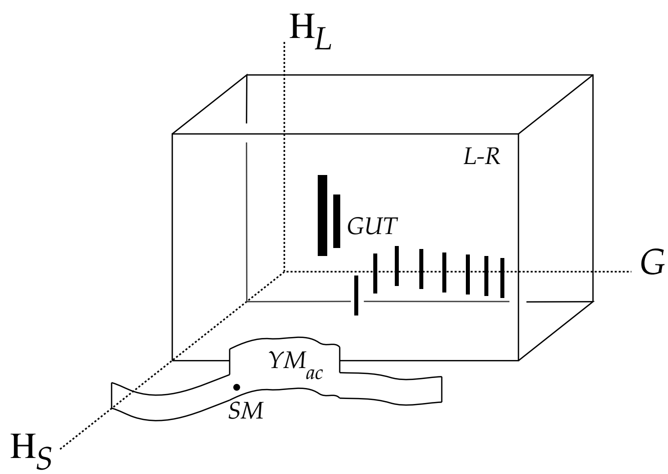


Figure 4.1: An artist's partial view of the space of bills of all Yang-Mills-Higgs models and some of its subsets. $GUT$ stands for 'Grand Unified Theories', $L-R$ stands for left-right symmetric models, $SM$ stands for standard model and $YM_{ac}$ for almost commutative Yang-Mills models.

# Chapter 5

# Running coupling constants

Quantum field theory teaches us that coupling constants are functions of the energy used to measure them. Today this energy dependence is accessible to accelerator experiments. Physically it can be understood in analogy with the screening effect from condensed matter physics. The computational origin of this energy dependence lies in divergent Feynman diagrams.

Consider an electric charge $Q$ placed in a dielectric medium, like water. The water molecules carry an electric dipole moment. These dipoles orient themselves around the charge such that the effective charge seen from far away is smaller than $Q$: the cloud of dipoles surrounding the charge partially screens $Q$. By convention we keep the charge constant and say that the effective coupling constant $(e^2/\epsilon)^{1/2}$ has decreased. In the vacuum $\epsilon_0 = 8.85 \cdot 10^{-12} \mathrm{s}^2\, \mathrm{C}^2/(\mathrm{m}^3\, \mathrm{kg})$ is also called vacuum permittivity, otherwise $\epsilon$ is the permittivity of the dielectric medium, $\epsilon = 699 \cdot 10^{-12} \mathrm{s}^2\, \mathrm{C}^2/(\mathrm{m}^3\, \mathrm{kg})$ for water.

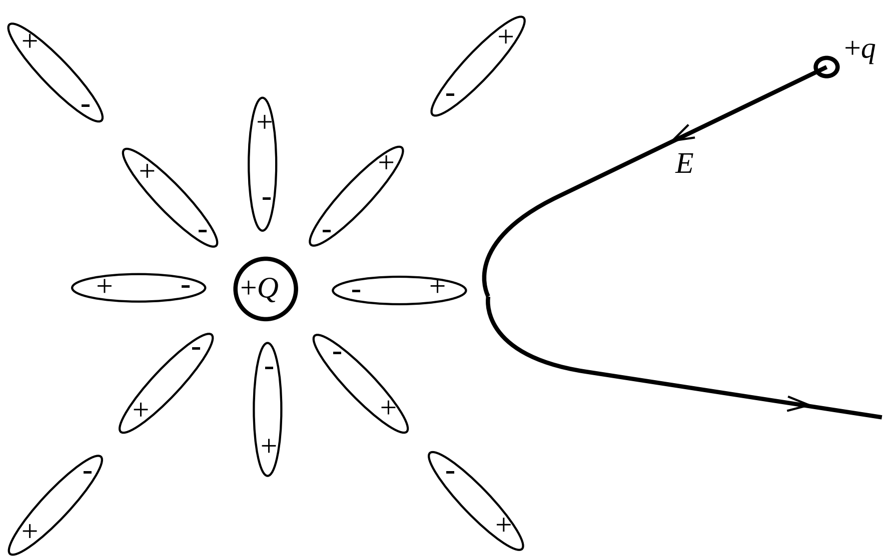


Figure 5.1: Vacuum polarization

Let us now place the central charge $Q$ in the vacuum and let us measure the effective coupling dynamically by scattering a test charge $q$ off the central charge $Q$ with an energy $E$, figure 5.1. Dirac tells us that electron positron pairs are created, dipoles that will screen the central charge like the dipole moments of the water molecules before. With increasing energy the test charge penetrates deeper into the dipole cloud and we measure an increasing effective charge or equivalently an increasing effective coupling constant squared $e^2/\epsilon_0(E)$. The vacuum permittivity is a decreasing function of energy, for instance $\epsilon_0 = 8.27 \cdot 10^{-12} \mathrm{s}^2\, \mathrm{C}^2/(\mathrm{m}^3\, \mathrm{kg})$ at $E = m_Z$. The effect is now called vacuum polarization or running coupling constant.

The quantitative treatment of the running coupling is cumbersome. So far we only have perturbative calculations. The cross section is computed as a power series in the fine structure constant $e^2/(4\pi\epsilon_0\, \hbar c)$ at a fixed energy. Even for small couplings this power series diverges and physicists take a pragmatic point of view. As the computation of the higher order terms is exceedingly complicated, the power series is truncated at first (or second) order. One talks about 1-loop contributions, this means that the photon exchanged between the central charge and the test charge produces one particle antiparticle pair only, figure 5.2. In 2-loop one admits the possibility that one of the particles of the pair may in turn produce new particles e.g. via bremsstrahlung, figure 5.3. Even at 1-loop, one has to live with divergent integrals, essentially short distance or ultra-violet divergences. For example consider figure 5.2, call $x_1$ the point of pair creation and $x_2$ the point of pair annihilation. The integral over $x_1$ and $x_2$ diverges for short distances between the two points. This divergence has to be regularized to get a finite cross section, a delicate manœuvre trusted only in renormalizable models. Even in renormalizable models, there are different regularization schemes leading to different cross sections. Fortunately the scheme dependence is weak and again physicists take a pragmatic point of view. This point of view is backed by an impressive agreement between the computed and measured numbers and adds to the humiliation of the standard model. Let us note that 95 % of physics is described without the use of loops. Renormalization is needed to fit the experimental numbers with higher precision.

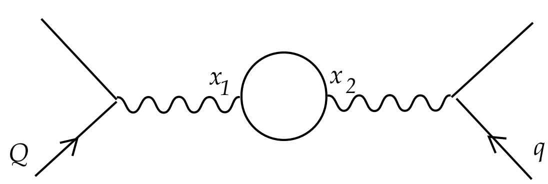


Figure 5.2: A 1-loop graph

One major motivation for noncommutative geometry in particle physics is that a spacetime uncertainty naturally cures short distance divergences. In fact most of these divergences are

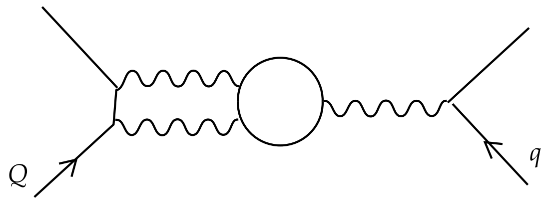


Figure 5.3: A 2-loop graph

logarithmic and resemble the divergences encountered under the Dixmier trace (4.38).

To cut a long story short, the running of the couplings is governed order by order by differential equations, the renormalization group equations,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \beta_g, \quad t := \log E/\Lambda. \tag{5.1}$$

$\Lambda$ is the energy cut off from the regularization. The rhs of the differential equation is called the $\beta$ function of the coupling $g$. For the standard model with $N = 3$ generations, in 1-loop 'approximation', neglecting threshold effects and neglecting all fermion masses with respect to the top mass, the $\beta$ functions are [22]

$$\beta_1 = \tfrac{1}{16\pi^2}\,\tfrac{41}{6}\,g_1^3 \tag{5.2}$$

$$\beta_2 = -\tfrac{1}{16\pi^2}\,\tfrac{19}{6}\,g_2^3 \tag{5.3}$$

$$\beta_3 = -\tfrac{1}{16\pi^2}\,7\,g_3^3 \tag{5.4}$$

$$\beta_t = \tfrac{1}{16\pi^2}\,(9g_t^3 - 8g_3^2 g_t - \tfrac{9}{4}g_2^2 g_t - \tfrac{17}{12}g_1^2 g_t), \tag{5.5}$$

$$\beta_\lambda = \tfrac{1}{16\pi^2}\,(96\lambda^2 + 24\lambda g_t^2 - 6g_t^4 - 9\lambda g_2^2 - 3\lambda g_1^2 + \tfrac{9}{32}g_2^4 + \tfrac{3}{32}g_1^4 + \tfrac{3}{16}g_2^2 g_1^2), \tag{5.6}$$

$$\beta_{\mu^2} = \tfrac{1}{16\pi^2}\,\mu^2(48\lambda + 12g_t^2 - \tfrac{9}{2}g_2^2 - \tfrac{3}{2}g_1^2). \tag{5.7}$$

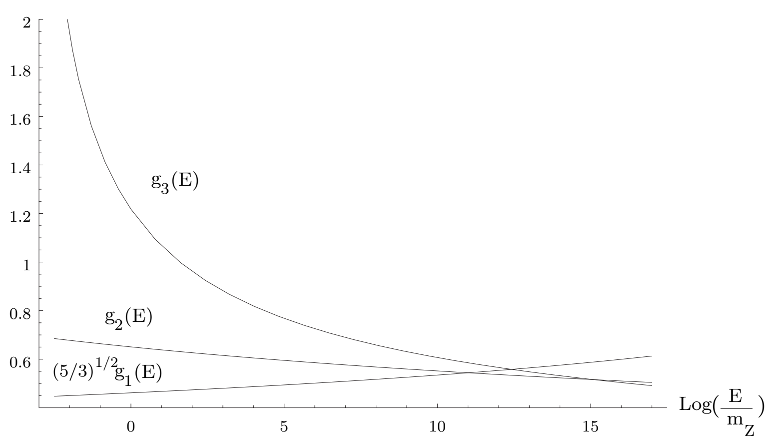


Figure 5.4: The evolution of the three gauge couplings

With the cited approximations, the three gauge couplings $g_i$ decouple from the Yukawa and Higgs couplings and can be solved immediately,

$$g_i^{-2}(t) = g_i^{-2}(0) - \tfrac{1}{8\pi^2} c_i t, \quad \beta_i =: \tfrac{1}{16\pi^2}\, c_i\, g_i^3. \tag{5.8}$$

Figure 5.4 shows the logarithmic running of the three gauge couplings with experimental initial values, $g_3 = 1.207,\ g_2 = 0.6507,\ g_1 = 0.3575$ at $E = m_Z$. In agreement with our hand waving argument, the abelian coupling $g_1$ increases with energy. The non-Abelian ones, the weak and strong couplings decrease with energy. This is called asymptotic freedom and has rendered non-Abelian Yang-Mills theories popular. At energies below 1 GeV the curve of the strong coupling constant loses all meaning because it leaves the perturbative regime. This is taken as evidence for confinement. On the other side, the curves have been extrapolated to science fiction energies of $10^{19}$ GeV with the insolent hypothesis of the big desert. I.e. we pretend that from presently accessible energies of $10^2$ GeV all the way up to $10^{19}$ GeV, energies that will never be accessible to man, no new forces, no new particles exist. This hypothesis was invented in the seventies together with grand unified theories. To ease somewhat the humiliation of the standard model, some physicists were looking for a simple Lie group like $SU(5)$ that contains $SU(3) \times SU(2) \times U(1)$. As a simple Lie group only has one coupling constant, this idea constrains the three gauge couplings:

$$g_3 = g_2, \quad g_1 = \sqrt{\tfrac{3}{5}}\, g_2. \tag{5.9}$$

The picture was that at the unification energy $\Lambda$ of around $10^{15}$ GeV $SU(5)$ breaks spontaneously down to $SU(3) \times SU(2) \times U(1)$. The gauge bosons that acquire a mass of the order of $\Lambda$ are called lepto-quarks because they mediate transitions between leptons and quarks rendering the proton unstable with a life time of the order of $\hbar\Lambda^4/m_p^5$, some $10^{29}$ years. $m_p c^2 = 0.938$ GeV is the proton mass. At energies $E$ below $\Lambda$ the lepto-quarks decouple leaving the standard model with its three couplings $g_i$ running as in figure 5.4. In the seventies the experimental initial conditions and uncertainties were different, such that the three curves could cross in one single point. Furthermore the experimental lower limit on the life time of the proton was $10^{28}$ years, compatible with the theoretical value. It was also clear that the lower limit could be improved by several orders of magnitude within a few years falsifying grand unification or discovering new physics. The former happened, today the proton life time is longer than $10^{32}$ years.

But grand unification also implied constraints on the Yukawa and Higgs couplings and therefore on the top and Higgs masses,

$$m_t = 2\, g_t/g_2\ m_W, \quad m_H = 4\sqrt{2}\,\sqrt{\lambda}/g_2\ m_W. \tag{5.10}$$

The $\beta$ functions (5.2-5.7) have been computed with dimensional regularization and the modified minimal subtraction scheme where only logarithmic divergences are kept. With Wilson's lattice regularization, $\mu$ has in addition to its logarithmic divergence a quadratic one that modifies its $\beta$ function. To avoid this ambiguity we note that, thanks to its dimensionality, $\mu$ decouples from the other couplings which are dimensionless. If we identify the pole masses $m_p = m(m_p)$ with their running masses at the $Z$ mass $m(m_Z)$, and only compute mass ratios we never need the ambiguous renormalization behaviour of $\mu$.

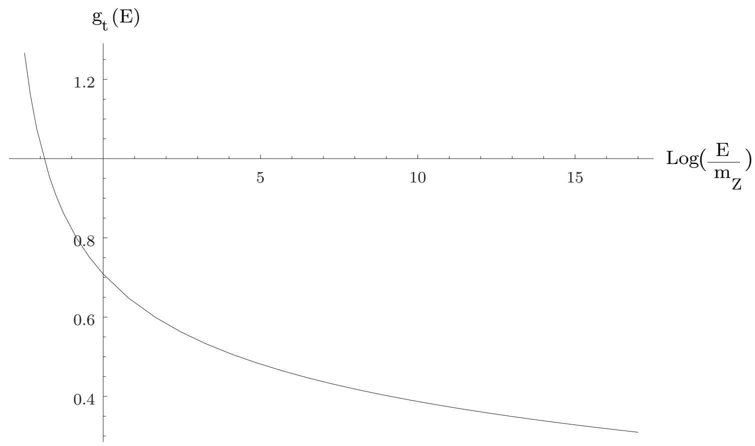


Figure 5.5: The top coupling

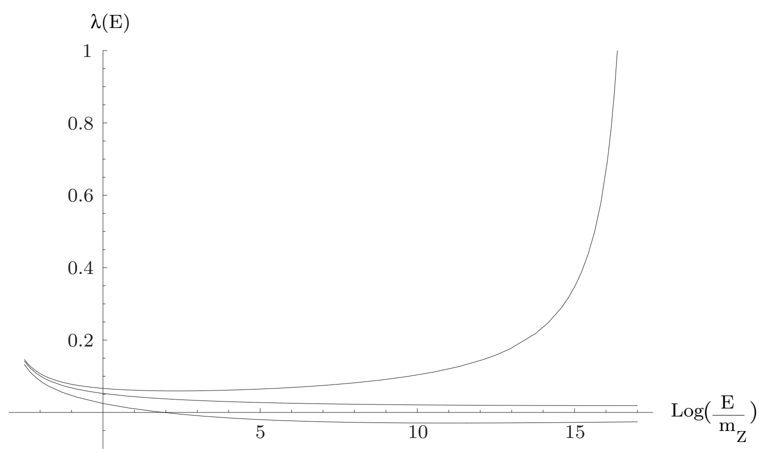


Figure 5.6: The Higgs selfcoupling for $m_H(m_Z) = 120$ (lower graph), 160 and 180 Gev (upper graph) for $m_t(m_Z) = 175$ GeV

In 1-loop the Yukawa coupling decouples from the Higgs couplings. Figure 5.5 shows its energy dependence with an initial value of $m_t(m_Z) = 175$ GeV. All initial conditions not mentioned are set to their central experimental values. Finally figure 5.6 shows the Higgs coupling $\lambda$ with three initial values $m_H(m_Z) = 120,\ 160$ and 180 GeV (upper curve) and

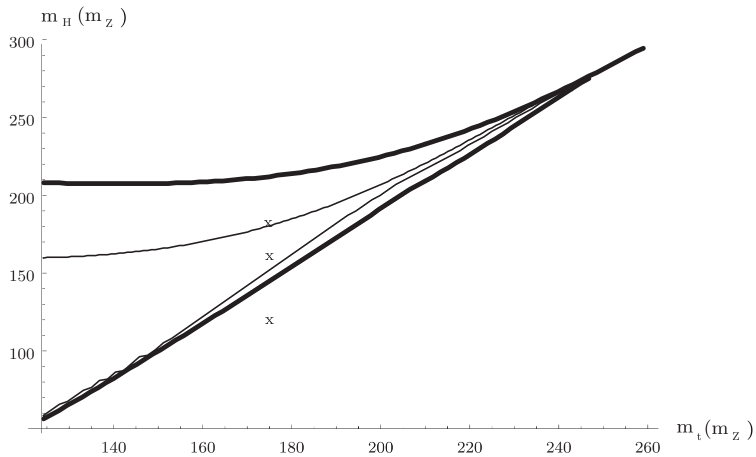


Figure 5.7: Two allowed domains of initial values for $\Lambda = 10^{10}$ GeV (fat lines) and $\Lambda = 10^{19}$ GeV (thin lines)

with $m_t(m_Z) = 175$ GeV. We see that two catastrophes may happen while traversing the big desert from $m_Z$ to $\Lambda$ [23]: the Higgs coupling may become negative, rendering the Higgs potential unstable or the Higgs coupling may become too large and ruin the perturbative computation. Figure 5.7 shows the allowed domain of initial values $m_t(m_Z)$, $m_H(m_Z)$ that avoid both catastrophes up to $\Lambda = 10^{10}$ GeV, thin lines, and up to $\Lambda = 10^{19}$ GeV, fat lines. The upper curves limit perturbation, the lower curves limit stability. The three points indicate the initial conditions of figure 5.6.

Let us now discuss the constraints from almost commutative Yang-Mills for the standard model,

$$g_3 = g_2, \qquad g_1 = \sqrt{\tfrac{3}{5}}\, g_2$$

$$g_t = g_2, \qquad \lambda = \tfrac{7}{24}\, g_2^2. \tag{5.11}$$

Again we suppose that they hold at some energy scale $\Lambda$ which immediately implies that we must swallow the big desert. In grand unification $\Lambda$ characterizes new gauge interaction, here it characterizes a new spacetime geometry. $\Lambda$ measures the spacetime uncertainty like $\hbar$ measures the phase space uncertainty. Today the experimental values of the three gauge couplings do not allow to fit the constraints at one energy, figure 5.4, $\Lambda = 10^{13} - 10^{17}$ GeV. The corresponding mismatch in the gauge couplings is on the 10 % level. We expect that the new uncertainty will explain this mismatch. Indeed at energies close to the cut off $\Lambda$ the $\beta$ functions computed from the ultra-violet divergences cannot be trusted together with noncommutative geometry. But so far we do not have a quantum field theory on noncommutative spacetimes. Nevertheless we cannot refrain from computing the numbers produced by the other two constraints:

$$m_t = 187 \pm 14 \text{ GeV}, \quad m_H = 197 \pm 9 \text{ GeV}. \tag{5.12}$$

The low value of $\Lambda$ produces the low top mass and the low Higgs mass. All masses are automatically compatible with a stable and perturbative Higgs coupling.

We believe that almost commutative geometry is just a low energy mirage of a truly noncommutative geometry on the high energy side of the big dessert. In grand unification, the direct product of groups was replaced by one group at the scale $\Lambda$. In the new picture, the tensor product of algebras should be replaced by one algebra at the scale $\Lambda$. We find it encouraging that this scale is close to, but lower than the Planck mass,

$$m_P = \sqrt{\frac{\hbar c}{G}} \cong 10^{19} \text{ GeV}. \tag{5.13}$$

Indeed there is an old hand waving argument combining Heisenberg's uncertainty relation of phase space with the Schwarzschild horizon to find an uncertainty relation in spacetime with a scale $\Lambda$ smaller than the Planck mass: To measure a position with a precision $\Delta x$ we need, following Heisenberg, at least a momentum $\hbar/\Delta x$ or, by special relativity, an energy $\hbar c/\Delta x$. According to general relativity, such an energy creates an horizon of size $G\hbar c^{-3}/\Delta x$. If this horizon exceeds $\Delta x$ all information on the position is lost. The best we can do is resolve positions with $\Delta x$ such that $\Delta x = G\hbar c^{-3}/\Delta x$, that is $\Delta x = \hbar/(m_P c)$, the Planck length. The problem with this argument is that we do not have a consistent quantum theory in curved spaces. Despite many efforts no renormalizable quantum field theory of gravity is known. Even the pragmatic physicists cannot agree on the energy dependence of the gravitational coupling $G$. The numerical static value, used in the hand waving argument does not seem reasonable.

It is time to talk about gravity.

# Chapter 6

# The Riemannian dreisatz

- Newton + Riemannian geometry = Einstein-Hilbert.

Einstein was a passionate sailor. We speculate that this was no accident. The subtle harmony between geometries and forces becomes palpable to the sailor, he sees the curvature of the sail and feels the force that it produces. Before Einstein, it was generally admitted that forces are vector fields in an Euclidean space, $\mathbb{R}^3$, the scalar product being necessary to define work and energy. Einstein generalized Euclidean to Minkowskian and Riemannian geometry and found special and general relativity with invariance groups, the Lorentz group $SO(1,3)$ and the diffeomorphism group $\mathrm{Diff}(M)$. These groups define the principles of special and general relativity.

## 6.1 First stroke

Newton's universal, static law of gravity,

$$F = G\,\frac{mM}{r^2}\,, \tag{6.1}$$

the proportionality constant being Newton's constant $G = 6.670 \cdot 10^{-11}\,\mathrm{m}^3/(\mathrm{s}^2\mathrm{kg})$, resembles Coulomb's law. However there is a subtle difference, the electric charge is Lorentz invariant, the mass is not. Minkowskian geometry is the geometry of a flat spacetime, with the flat Minkowski metric $\eta$. Riemannian geometry is the geometry of curved spacetimes, with an arbitrary metric $g$. Riemannian geometry also suggests the principle of general relativity, invariance under general coordinate transformations whereas in Minkowskian geometry or special relativity we only had invariance under the Lorentz group, under those special transformations that map inertial coordinates (holonomic, orthonormal frames) into inertial coordinates.

As for Maxwell, the extension of Newton's law is done in two strokes and starts with the trajectory of a test mass $m$,

$$m\,\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}=0, \tag{6.2}$$

in inertial coordinates, a straight line. In arbitrary coordinates, still in flat spacetime this becomes the geodesic equation,

$$m\,\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}p^2} \;=\; -m\Gamma^\lambda{}_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}p}\,, \tag{6.3}$$

$$\Gamma^\lambda{}_{\mu\nu}:=\tfrac{1}{2}\eta^{\lambda\kappa}\left[\partial_\mu\eta_{\kappa\nu}+\partial_\nu\eta_{\kappa\mu}-\partial_\kappa\eta_{\mu\nu}\right]. \tag{6.4}$$

The Christoffel symbols $\Gamma$ are first derivatives of the matrix $\eta_{\mu\nu}$ of the flat metric in the coordinates $x^\mu$, defined by equation (2.43). The geodesic equation and the definition of the Christoffel symbols are to be compared to their Maxwell brothers,

$$m\,\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} \;=\; qF^\mu{}_\nu\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,, \tag{6.5}$$

$$F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{6.6}$$

The geodesic equation simply describes the straight line in non-Cartesian coordinates. Nevertheless it already contains a lot of physics. If the $x^\mu$ are the coordinates of the rotating disk the geodesic equation is nothing but centrifugal and Coriolis forces. We can also repeat the above argument replacing the free particle of Newton's mechanics with the free particle of Schrödingers quantum mechanics, i.e. a plane wave. Then choosing for $x^\mu$ oscillating coordinates, we understand some observed interference patterns of neutrons [24].

The equivalence principle says that in absence of friction with air, a down falls as fast as a marble. In other words, inertial and gravitational masses are equal. This suggests to use a non-flat metric $g$ to describe the trajectory of the marble in a non-vanishing gravitational field. The mass on the lhs of the geodesic equation is inertial, on the rhs the mass is gravitational and the two masses still cancel by virtue of the equivalence principle. The Christoffel symbols,

$$\Gamma^\lambda{}_{\mu\nu}:=\tfrac{1}{2}g^{\lambda\kappa}\left[\partial_\mu g_{\kappa\nu}+\partial_\nu g_{\kappa\mu}-\partial_\kappa g_{\mu\nu}\right], \tag{6.7}$$

are the gravitational field, the underlying metric $g$ is the gravitational potential. The electromagnetic potential $A$ can only be measured partially as integral over a closed curve and this only via quantum effects, the Aharonov-Bohm effect. The metric can be measured classically, but again only as integral over a curve, the proper time,

$$c^2\mathrm{d}\tau^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{6.8}$$

Let us emphazise that the geodesic equation and the proper time are invariant under general coordinate transformations.

## 6.2 Second stroke

In the second stroke, Einstein used the full power of the principle of general relativity to derive the dynamics of the gravitational field. The source of electromagnetism is charge,

$$(\mathcal{D}_{\rm Maxwell}A)_\mu = -\tfrac{1}{\epsilon_0 c^2} j_\mu. \tag{6.9}$$

We know the coupling constant from Coulomb's law and we know that the differential operator $\mathcal{D}_{\rm Maxwell}$ must reduce to the Laplace operator in the static case. The source of gravity is mass or – with special relativity – energy,

$$(\mathcal{D}_{\rm Einstein}g)_{\mu\nu} = \tfrac{8\pi G}{c^4}\tau_{\mu\nu}. \tag{6.10}$$

The energy-momentum tensor has the good taste to be symmetric, $\tau_{00}$ is the energy density, $\tau_{0i}$ are the energy currents, $\tau_{i0}$ are the momentum densities and $\tau_{ij}$ their currents. Newton's law fixes the coupling constant $G$ and from the $1/r^2$ fall-off we know that $\mathcal{D}_{\rm Einstein}$ is second order. Covariance under general coordinate transformations and energy-momentum conservation then determine the differential operator uniquely:

$$(\mathcal{D}_{\rm Einstein}g)_{\mu\nu} = G_{\mu\nu} - \Lambda_C g_{\mu\nu}, \tag{6.11}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

$$R^\lambda{}_{\mu\nu\kappa} := \partial_\nu \Gamma^\lambda{}_{\mu\kappa} - \partial_\kappa \Gamma^\lambda{}_{\mu\nu} + \Gamma^\eta{}_{\mu\kappa}\Gamma^\lambda{}_{\nu\eta} - \Gamma^\eta{}_{\mu\nu}\Gamma^\lambda{}_{\kappa\eta} \tag{6.12}$$

is the Riemann tensor, $R_{\mu\kappa} := R^\lambda{}_{\mu\lambda\kappa}$ is the Ricci tensor and $R := R_{\mu\nu}g^{\mu\nu}$ is the curvature scalar. $\Lambda_C$ is the cosmological constant that we discard for phenomenological reasons. Maxwell's differential operator, equation (1.23), is linear and has eight terms. Einstein's operator is non-linear and has roughly 80 000 terms. Otherwise the two theories are very similar. As light from Maxwell's equation, Einstein's equation has plane wave solutions, 'gravitational waves'. They too travel at the speed of light. 'Gravito–magnetic' forces with feeble couplings are contained in Einstein's equations and have been measured, as the advance of perihelia, the curvature of light in a gravitational field, radar delay, or spin precession.

Einstein's equation derives from an action, the Einstein-Hilbert action

$$S[g] = \tfrac{-c^3}{16\pi G}\int_{\mathbb{R}^4} R\,|\det g_{\cdot\cdot}|^{1/2}\mathrm{d}^4x \ + \ \text{matter}, \tag{6.13}$$

where $R$ is the curvature scalar. The energy momentum tensor $\tau_{\mu\nu}$ is the variation of the matter action with respect to $g^{\mu\nu}$.

## 6.3 The principle of general relativity

Connes' second dreisatz will unify Yang-Mills theories with general relativity. To understand Yang-Mills theories in terms of noncommutative geometry, it was very useful to formulate them with differential forms. The same is true for general relativity. The remaining sections of this chapter continue the technical interlude of chapter 2. We will use the local concepts of chapter 2 to construct general relativity in presence of spinors. Spacetime is an open subset $U$ of $\mathbb{R}^4$ with signature $+---$ for concreteness. The generalization to any dimension and signature is immediate. The outcome of this construction will be a gauge theory based on the Lorentz group $SO(1,3)$ or its spin cover and the coupling of the gravitational field to matter will be minimal, i.e. a covariant derivative.

General relativity promotes the spacetime metric to a dynamical field describing gravity. Therefore we look for differential equations determining the metric. By definition the metric is a differentiable family of bilinear symmetric forms, and we do not know what differential equations for such objects are. We have seen that any metric can be described using a frame of 1-forms. For 1-forms we know differential operators. Einstein has used holonomic frames. The principle of general relativity requires that the metric and only the metric generates gravitational interaction. Therefore we want field equations that do not depend on the particular coordinate system used to define the holonomic frame. In the following, we use orthonormal frames of 1-forms to parameterize all metrics. The principle of general relativity now requires that the particular orthonormal frame chosen to describe a given metric is irrelevant. Our task therefore is to find differential equations for the orthonormal frames $e^i$ which are covariant under gauge transformations $\Lambda$:

$$e'^i = \Lambda^i_j e^i, \quad \Lambda \in {}^U SO(1,3). \tag{6.14}$$

We restrict ourselves to orientation preserving Lorentz transformations because we use the Hodge star. It is sometimes convenient to consider the orthonormal frame $e^i$ as a 1-form $e$ with values in the fundamental representation of $SO(1,3)$. To be more precise, we must add the restriction that the $e^i$ be linearly independent which is compatible with the gauge transformation $e' = \Lambda e$. To get gauge covariant field equations for $e$ we use the Yang-Mills trick: We introduce a connection, write down an invariant action and obtain the desired field equations by variation. In Yang-Mills theories the connection actually represents new physical fields like the photon or the weak bosons $W^\pm, Z$. Here we just signed the principle prohibiting the introduction of new fields. A natural solution of this dilemma will show up automatically, and for the moment we allow for a new field, the connection $\omega$, a 1-form with values in the Lie algebra of $SO(1,3)$

$$\omega \in \Omega^1(U, so(1,3)), \tag{6.15}$$

also called spin connection. As a connection it is supposed to transform under gauge transformations according to

$$\omega' = \Lambda \omega \Lambda^{-1} + \Lambda \mathrm{d}\Lambda^{-1}. \tag{6.16}$$

As before we define the curvature

$$R := \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] \ \in \ \Omega^2(U, so(1,3)). \tag{6.17}$$

This definition is known as Cartan's second structure equation. Again we have immediately the homogeneous transformation property of the curvature:

$$R' = \Lambda R \Lambda^{-1}. \tag{6.18}$$

We define torsion by Cartan's first structure equation

$$T := \mathrm{d}e + \omega e = \mathrm{D}e \in \Omega^1(U, \mathbb{R}^4). \tag{6.19}$$

As a covariant derivative, also the torsion transforms homogeneously under gauge transformations:

$$T' = \Lambda T. \tag{6.20}$$

From $\mathrm{d}^2 = 0$ and the Jacobi identity, we obtain the Bianchi identities:

$$\mathrm{D}R \;=\; \mathrm{d}R + [\omega, R] = 0, \tag{6.21}$$

$$\mathrm{D}T \;=\; \mathrm{d}T + \omega T = Re. \tag{6.22}$$

## 6.4 The Einstein-Cartan equations

For a Yang-Mills theory without matter the cheapest gauge invariant action is quadratic in the curvature:

$$S_{YM}[A] = -\frac{1}{g^2} \int F^{*a}{}_b * F^b{}_a. \tag{6.23}$$

For the moment the pure gravitational field is coded into two fields $e$ and $\omega$. Consequently, we have an invariant action already linear in the curvature,

$$S_{EH}[e, \omega] = \frac{1}{32\pi G} \int R^a{}_b * (e^b e_a), \tag{6.24}$$

where indices are raised and lowered with $\eta^{ab}$ and $\eta_{ab}$, and $e$ always denotes orthonormal frames of 1-forms. Equation (6.24) is the Einstein-Hilbert action. Using the definition of the Hodge star in four dimensions the Einstein-Hilbert action can also be written,

$$S_{EH}[e,\omega] = \frac{-1}{32\pi G}\int R^{ab}e^c e^d \epsilon_{abcd}. \tag{6.25}$$

We introduce matter by adding a functional $\int \mathcal{L}_M$ depending on the matter fields and on $e$ and $\omega$,

$$S[e,\omega] = \frac{-1}{32\pi G}\int R^{ab}e^c e^d \epsilon_{abcd} + \int \mathcal{L}_M[e,\omega,...]. \tag{6.26}$$

For example, the matter could be a Yang-Mills action (6.23) with $A$ now considered as matter field. This particular matter action depends only on $e$ (through the Hodge star) and not on $\omega$.

Let us derive the field equations following from (6.26).

**Variation of $e$:** We call $\tau$ the variation of the matter Lagrangian with respect to $e$:

$$\mathcal{L}_M[e+f] - \mathcal{L}_M[e] =: -f^c\tau_c + O(f^2), \tag{6.27}$$

where $\tau$ is a 3-form with values in $\mathbb{R}^4$,

$$\tau \in \Omega^3(U,\mathbb{R}^4), \tag{6.28}$$

the 'energy momentum tensor'. Integrating $\tau$ over a 3-dimensional volume yields the energy momentum contained in that volume. E.g. for pure electromagnetic radiation,

$$\mathcal{L}_M = -\tfrac{\epsilon_0}{2}F^* * F, \tag{6.29}$$

we obtain after a lengthy calculation

$$\tau_{00} = \tfrac{\epsilon_0}{2}(\vec{E}^2 + \vec{B}^2) \tag{6.30}$$

with

$$* \tau_c =: \tau_{ca}e^a. \tag{6.31}$$

Variation of the total action (6.26) with respect to $e$ gives immediately the Einstein equations:

$$R^{ab}e^d\ \epsilon_{abcd} = -16\pi G\tau_c. \tag{6.32}$$

For given energy momentum $\tau$, they are non-linear first order differential equations for the connection. They are also linear equations for the curvature, 'energy is the source of curvature'.

Despite the algebraic nature of the equations curvature propagates in four dimensions: Vanishing $\tau$ does not imply vanishing curvature as is illustrated, for example, by Schwarzschild's solution. This comes from the fact that the curvature has $6 \times 6 = 36$ independent coefficients $R^{ab}{}_{\mu\nu}$ (antisymmetric in $\mu$ and $\nu$ because $R$ is a 2-form, antisymmetric in $a$ and $b$ because $R$ takes values in the Lorentz algebra) while Einstein's equation, being an equation for 3-forms with values in $\mathbb{R}^4$ contains only $4 \times 4 = 16$ linear equations. In two- and three-dimensional space times the counting is different and curvature does not propagate.

**Variation of $\omega$:** We define the spin density

$$\mathcal{S} \in \Omega^3(U, so(1,3)) \tag{6.33}$$

by

$$\mathcal{L}_M[\omega + \chi] - \mathcal{L}_M[\omega] =: -\tfrac{1}{2}\chi^{ab}\mathcal{S}_{ab} + O(\chi^2). \tag{6.34}$$

Of course, the spin density is zero for the Yang-Mills action (6.23). It is non-vanishing, for instance, for the Dirac action describing spin $\frac{1}{2}$ fields, which motivates the name spin density. Varying $\omega$ in the total action (6.26) yields, after an integration by parts, the equation

$$T^c e^d \, \epsilon_{abcd} = -8\pi\, G\, \mathcal{S}_{ab}. \tag{6.35}$$

'Spin is the source of torsion'. If we now count the number of linear equations and unknowns, we find them to match in any dimension. Torsion does not propagate: Vanishing spin density implies vanishing torsion.

## 6.5 A farewell to $\omega$

We now come to the promised elimination of the spin connection as an independent field. There are two possible routes.

**Einstein's point of view:** Einstein puts torsion to zero right from the beginning. By virtue of equation (6.19),

$$0 = T = \mathrm{d}e + \omega e \tag{6.36}$$

is a covariant constraint and therefore it does not spoil the covariance of Einstein's equation. Let us consider this constraint as a system of linear equations with the components of the spin connection $\omega^{ab}{}_{\mu}$ as unknowns. Since $\omega$ is $so$(1,3)-valued, it is antisymmetric in the indices $a$ and $b$ and there are $6 \times 4$ unknowns. On the other hand, (6.36) is an equation for $\mathbb{R}^4$-valued 2-forms and has $4 \times 6$ components $T^a{}_{\mu\nu}$. Consequently, there exists (for any signature and dimension) a unique solution expressing the spin connection as a function of the frame and its first derivatives. This solution is called Riemannian connection. Its explicit form is most conveniently written down expanding $\omega$ with respect to the orthonormal frame $e$:

$$\omega^a{}_b = \omega^a{}_{bc} e^c. \tag{6.37}$$

Then the Riemannian connection is given by

$$\omega^a{}_{bc} = \tfrac{1}{2}(C^a{}_{bc} - C_b{}^a{}_c - C_c{}^a{}_b), \tag{6.38}$$

where the functions $C$ are defined by

$$\mathrm{d}e^a =: \tfrac{1}{2} C^a{}_{bc}\, e^b e^c. \tag{6.39}$$

Substituting the Riemannian connection $\omega(e, \partial e)$ into Einstein's equations they become non-linear second order differential equations for the orthonormal frame. Alternatively they can be obtained by substituting first the Riemannian connection into the Einstein-Hilbert action and then varying with respect to the frame, 'second order formalism'.

Let us make the link between the Riemannian connection with respect to the orthonormal frame $e^a$, the $so(1,3)$ valued 1-form $\omega$ and the *same* Riemannian connection with respect to a holonomic frame $\mathrm{d}x^\mu$, the $gl_4$ valued 1-form $\Gamma$. The link between the two frames is a $GL_4^+$ gauge transformation:

$$e^a = \gamma^a{}_\mu \mathrm{d}x^\mu, \quad \gamma \in^U GL_4^+, \tag{6.40}$$

and consequently the link between the the two expressions of the Riemannian connection with respect to the two frames is:

$$\omega = \gamma \Gamma \gamma^{-1} + \gamma \mathrm{d}\gamma^{-1}. \tag{6.41}$$

In holonomic components this last equation reads:

$$\frac{\partial}{\partial x^\nu} \gamma^a{}_\mu - \gamma^a{}_\alpha \Gamma^\alpha{}_{\mu\nu} + \omega^a{}_\nu \gamma^b{}_\mu = 0. \tag{6.42}$$

The ${}^U GL_4^+$ element $\gamma^a{}_\mu$ is often denoted $e^a{}_\mu$ and called vierbein. (Attention, the lhs of the last equation is often called covariant derivative of the vierbein and the equation is confused with the metricity property of the Riemannian connection by calling the vierbein a square root of the metric, $g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x) e^b{}_\nu(x)$.) The coefficients of the $gl_4$ valued 1-form $\Gamma^\alpha{}_{\mu\nu}\mathrm{d}x^\nu$ of the Riemannian connection with respect to the orthonormal frame are the Christoffel symbols, equation (6.7).

**Cartan's point of view:** Cartan keeps $\omega$ as an independent field which eliminates itself at the end through its own (algebraic) field equation (6.35): $\omega = \omega(e, \partial e, \mathcal{S})$. Therefore in this so-called Einstein-Cartan theory Riemannian geometry is only valid outside matter with spin. Only there it is verified experimentally. Furthermore the observed spin density in the universe is small and torsion couples to it via the universal coupling constant $G$ implying that although different in principle Einstein's and Einstein-Cartan's theories are presently indistinguishable experimentally.

It can be shown [25] that the Einstein-Hilbert action is the unique action that leads to vanishing torsion in the vacuum as field equation, unique of course up to terms containing no spin connection, the cosmological term

$$\tfrac{\Lambda_C}{4!} \int e^a e^b e^c e^d \ \epsilon_{abcd}. \tag{6.43}$$

As promised we now show that a piece of the 2-dimensional unit sphere (chapter 2) cannot have a holonomic and orthonormal frame.

**Theorem:** An open subset $U$ of $\mathbb{R}^n$ with a metric $g$ admits a holonomic and orthonormal frame if and only if its Riemannian connection has everywhere vanishing curvature.

We use equation (6.38) to calculate the Riemannian connection from

$$e^1 = \sin\theta \ \mathrm{d}\varphi \ , \quad e^2 = d\theta \tag{6.44}$$

and

$$\mathrm{d}e^1 = -\frac{\cos\theta}{\sin\theta}\, e^1 e^2, \quad \mathrm{d}e^2 = 0. \tag{6.45}$$

Therefore

$$C^1{}_{12} = -C^1{}_{21} = -\frac{\cos\theta}{\sin\theta}, \tag{6.46}$$

and all other $C'$s vanish. Consequently, the Riemannian connection is

$$\omega^1{}_2 = \frac{\cos\theta}{\sin\theta}\, e^1 = \cos\theta\, \mathrm{d}\varphi \tag{6.47}$$

and its curvature

$$R^1{}_2 = e^1 e^2 \tag{6.48}$$

is different from zero.

To conclude, following Cartan we have presented general relativity using orthonormal frames. This may be somewhat unfamiliar because Einstein formulated his theory with the help of holonomic frames. Of course, both approaches have advantages and inconveniences. Two major shortcomings of holonomic frames are: Their invariance group is $GL_4$ which does not admit spinor representations [26] therefore excluding fields with half integer spin. Holonomic frames break the gauge invariance of general relativity, ignoring today's belief that all fundamental interactions are described by gauge theories.

## 6.6 The Dirac operator

Quantum mechanical experiments with neutrons teach us that interference patterns repeat themselves only after a rotation through $720^o$ of one of the two neutrons [27]. Mathematically this means that the relevant group for spin $\frac{1}{2}$ is not the rotation group $SO(3)$ put its universal cover $SU(2)$. In relativistic theories the rotation group is embedded in the Lorentz group $SO(1,3)$ and we need its universal cover, the Clifford group $Spin(1,3)$. The Dirac spinor is a vector in the fundamental representation of the Clifford group. In curved spacetime the Lorentz group is gauged and so we must gauge the Clifford group in order to define the Dirac operator there. You will not be surprised that in the gauged case, we need a covariant derivative. The connection takes values in the Lie algebra of the group, here the Clifford group. By definition the Lie algebra of a Lie group is the same as the Lie algebra of its universal cover. This is the short cut that we use to avoid developing the theory of Clifford algebras and groups. All we need is the representation of an infinitesimal Lorentz transformation $X^a{}_b \in so(1,3)$ on a spinor $\psi$:

$$\tilde{\rho}(X)\,\psi = \tfrac{1}{4} X_{ab}\gamma^a\gamma^b\psi = \tfrac{1}{8} X_{ab}[\gamma^a,\gamma^b]\psi. \tag{6.49}$$

This transformation law tells us that the Dirac spinor has spin $\frac{1}{2}$ and this is the transformation law that we should have given already in section 3.4 to prove the Lorentz invariance of the Dirac equation. We recall that we use the flat metric $\eta_{aa'}$ to lower latin indices and that $X_{ab} = X^{a'}{}_b\eta_{aa'}$ is antisymmetric. The $\gamma$ matrices with latin indices are the $x$-independent Dirac matrices introduced in section 3.4. To write down the Dirac operator we need partial derivatives. They are calculated in a holonomic frame. On the other hand we need an orthonormal frame to represent Lorentz transformations. The link between the two frames is a $GL_4^+$ gauge transformation, the (inverse) vierbein $e^c{}_\mu(x)$:

$$\mathrm{d}x^\mu = e^\mu{}_c e^c. \tag{6.50}$$

We use it to define $x$-dependent $\gamma$ matrices,

$$\gamma^\mu(x) := e^\mu{}_c(x)\,\gamma^c. \tag{6.51}$$

We are ready to define the Dirac operator,

$$\partial\!\!\!/ \psi \;:=\; i\gamma^\mu(x)\left(\frac{\partial}{\partial x^\mu}+\tilde\rho(\omega_\mu)\right)\psi \;=\; \gamma^\mu(x)\left(\frac{\partial}{\partial x^\mu}+\tfrac{1}{4}\omega_{ab\mu}\gamma^a\gamma^b\right)\psi. \tag{6.52}$$

In flat Minkowski space with inertial coordinates $x^\mu$, the holonomic frame is orthonormal, $e^\mu{}_c=\delta^\mu{}_c$, the spin connection $\omega$ vanishes and we retrieve the flat Dirac operator.

## 6.7 The Dirac action

To derive the Dirac equation from an action principle we need a pseudo scalar product on the space of spinors, invariant under the Clifford group. At this point the signature of spacetime matters. With Minkowskian signature and unitary Dirac matrices, this product is,

$$(\psi,\chi)=\bar\psi\chi=\psi^*\gamma^0\chi, \tag{6.53}$$

where here the star $\cdot^*$ denotes the transposed, complex conjugate. With Euclidean signature, we have a genuine scalar product,

$$(\psi,\chi)=\psi^*\chi. \tag{6.54}$$

In both signatures, the Dirac action reads:

$$S_{\mathrm{Dirac}}[e,\omega,\psi]=\int *(\psi,\,\partial\!\!\!/\psi)=\tfrac{1}{3!}\int(\psi,\gamma^a\mathrm{D}\psi)\,e^be^ce^d\epsilon_{abcd}, \tag{6.55}$$

with the exterior covariant derivative,

$$\mathrm{D}\psi=\mathrm{d}\psi+\tilde\rho(\omega)\psi=\mathrm{d}\psi+\tfrac{1}{4}\omega_{ab}\gamma^a\gamma^b. \tag{6.56}$$

Two remarks are in order. If the torsion vanishes the Dirac action is real, the Dirac operator is selfadjoint in Euclidean signature. Second remark, in the Euclidean, due to the missing $\gamma^0$ in the scalar product, the Dirac action for a chiral, say left-handed, fermion vanishes. We shall have to pay due attention to this last point during the 'Wick rotation'.

## 6.8 The Lichnérowicz formula

Dirac's first motivation for his operator was a square root of the wave operator. Indeed, in flat Minkowski space we have $\partial\!\!\!/^2=-\Box 1_4$. Let us generalize this formula to curved space. We suppose vanishing torsion but allow the spinor to couple minimally also to a Yang-Mills potential $A$ and to a Higgs scalar $\Phi\in\mathcal{H}_L^*\otimes\mathcal{H}_R$,

$$\mathcal{D}_{t,\mathrm{cov}}=\begin{pmatrix}[\partial\!\!\!/\otimes 1_L+ie^\mu{}_j\gamma^j\otimes\rho_L(A_\mu)] & \gamma_5\otimes\Phi\\ \gamma_5\otimes\Phi^* & [\partial\!\!\!/\otimes 1_R+ie^\mu{}_j\gamma^j\otimes\rho_R(A_\mu)]\end{pmatrix}. \tag{6.57}$$

To keep notations simple we have left out the antiparticle part. The square of this total covariant Dirac operator is

$$\mathcal{D}_{t,\mathrm{cov}}^2 = -\,\Box + E, \tag{6.58}$$

$\Box$ is the covariant wave operator

$$\begin{aligned}\Box \;=\; g^{\mu\tilde\nu}&\left[\left(\frac{\partial}{\partial x^\mu} 1_4 \otimes 1_{\mathcal{H}} + \tfrac{1}{4}\omega_{ab\mu}\gamma^a\gamma^b \otimes 1_{\mathcal{H}} + 1_4 \otimes \rho(A_\mu)\right)\delta^\nu{}_{\tilde\nu} - \Gamma^\nu{}_{\tilde\nu\mu} 1_4 \otimes 1_{\mathcal{H}}\right] \\ &\times\left[\frac{\partial}{\partial x^\nu} 1_4 \otimes 1_{\mathcal{H}} + \tfrac{1}{4}\omega_{ab\nu}\gamma^a\gamma^b \otimes 1_{\mathcal{H}} + 1_4 \otimes \rho(A_\nu)\right]\end{aligned} \tag{6.59}$$

with the internal representation $\rho := \rho_L \oplus \rho_R$ on $\mathcal{H} := \mathcal{H}_L \oplus \mathcal{H}_R$. $E$, for endomorphism, is a zero order operator, that is a matrix of size $4 \dim \mathcal{H}$ whose entries are functions constructed from the bosonic fields and their first and second derivatives,

$$E = \tfrac{1}{2}\left[\gamma^\mu\gamma^\nu \otimes 1_{\mathcal{H}}\right] \mathbb{R}_{\mu\nu} + \begin{pmatrix} 1_4 \otimes \Phi\Phi^* & -i\gamma_5\gamma^\mu \otimes \mathrm{D}_\mu\Phi \\ -i\gamma_5\gamma^\mu \otimes (\mathrm{D}_\mu\Phi)^* & 1_4 \otimes \Phi^*\Phi \end{pmatrix}. \tag{6.60}$$

$\mathbb{R}$ is the total curvature, a 2-form with values in the (Lorentz $\oplus$ internal) Lie algebra represented on (spinors $\otimes$ $\mathcal{H}$). It contains the curvature 2-form $R = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ and the field strength 2-form $F = \mathrm{d}A + \frac{1}{2}[A, A]$, in components

$$\mathbb{R}_{\mu\nu} = \tfrac{1}{4} R_{ab\mu\nu}\gamma^a\gamma^b \otimes 1_{\mathcal{H}} + 1_4 \otimes \rho(F_{\mu\nu}). \tag{6.61}$$

An easy calculation shows that the first term in equation (6.60) produces the curvature scalar that we also (!) denote by $R$,

$$\tfrac{1}{2}\left[e^\mu_c e^\nu_d \gamma^c\gamma^d\right] \tfrac{1}{4} R_{ab\mu\nu}\gamma^{ab} = \tfrac{1}{4} R 1_4. \tag{6.62}$$

In our conventions, the curvature scalar is positive on spheres (with signature ++). Finally D is the covariant derivative appropriate to the representation of the scalars.

The Lichnérowicz formula with arbitrary torsion can be found in [28].

## 6.9 Wick rotation

In this section we put together the action of gravity and of the standard model with emphasis on the relative signs. We also indicate the changes when passing from Minkowskian to Euclidean signature.

In 1983 the meter disappeared as fundamental unit of science and technology. The conceptual revolution of general relativity, the abandon of length in favour of time, had made its way up to the domain of technology. Said differently, general relativity is not really geo-metry, but chrono-metry. Hence our natural choice of Minkowskian signature is $+---$.

With this choice and the conventions,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{6.63}$$

$$\Gamma^\lambda{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\kappa} \left[\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right], \tag{6.64}$$

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\nu \Gamma^\lambda{}_{\mu\kappa} - \partial_\kappa \Gamma^\lambda{}_{\mu\nu} + \Gamma^\eta{}_{\mu\kappa}\Gamma^\lambda{}_{\nu\eta} - \Gamma^\eta{}_{\mu\nu}\Gamma^\lambda{}_{\kappa\eta}, \tag{6.65}$$

$$R_{\mu\kappa} = R^\lambda{}_{\mu\lambda\kappa}, \tag{6.66}$$

$$R = R_{\mu\nu} g^{\mu\nu}, \tag{6.67}$$

$$\gamma^{a=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^{a=1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{6.68}$$

$$\gamma^{a=2} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^{a=3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{6.69}$$

$$\gamma^\mu(x) = e^\mu{}_a(x)\gamma^a, \qquad \gamma_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{6.70}$$

the combined Einstein-Hilbert Maxwell Higgs Dirac Lagrangian reads,

$$\begin{aligned} &\{-\tfrac{1}{16\pi} m_P^2\, R \,-\, \tfrac{1}{4g^2} \mathrm{tr}(F^*_{\mu\nu}F^{\mu\nu}) \,+\, \tfrac{1}{2g^2} m_A^2 \mathrm{tr}(A^*_\mu A^\mu) \\ &+ \tfrac{1}{2}\,(\mathrm{D}_\mu\varphi)^*\mathrm{D}^\mu\varphi \,-\, \tfrac{1}{2}\, m_\varphi^2 |\varphi|^2 \,+\, \tfrac{1}{2}\,\mu^2|\varphi|^2 \,-\, \lambda|\varphi|^4 \\ &+ \psi^*\gamma^{a=0}\left[i\gamma^\mu \mathrm{D}_\mu \,-\, m_\psi 1_4\right]\psi\}\ |\det g_{\cdot\cdot}|^{1/2}. \end{aligned} \tag{6.71}$$

This Lagrangian is real if we suppose that all fields vanish at infinity. The relative coeffients between kinetic terms and mass terms are chosen as to reproduce the correct energy momentum relations from the free field equations using Fourier transform and the de Broglie relations as explained after equation (3.29). With the chiral decomposition

$$\psi_L = \tfrac{1-\gamma_5}{2}\,\psi, \quad \psi_R = \tfrac{1+\gamma_5}{2}\,\psi, \tag{6.72}$$

the Dirac Lagrangian reads

$$\begin{aligned} &\psi^*\gamma^0\left[i\gamma^\mu \mathrm{D}_\mu \,-\, m_\psi 1_4\right]\psi \\ &\quad = \psi_L^*\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\,\psi_L \,+\, \psi_R^*\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\,\psi_R \,-\, m_\psi \psi_L^*\gamma^0\psi_R \,-\, m_\psi\psi_R^*\gamma^0\psi_L. \end{aligned} \tag{6.73}$$

The relativistic energy momentum relations are quadratic in the masses. Therefore the sign of the fermion mass $m_\psi$ is conventional and merely reflects the choice: who is particle and who is antiparticle. We can even adopt one choice for the left-handed fermions and the opposite choice for the right-handed fermions. Formally this can be seen by the change of field variable (chiral transformation):

$$\psi := \exp(i\alpha\gamma_5)\,\psi'. \tag{6.74}$$

It leaves invariant the kinetic term and the mass term transforms as,

$$-\,m_\psi \psi'^*\gamma^0[\cos(2\alpha)\,1_4 + i\sin(2\alpha)\,\gamma_5]\psi'. \tag{6.75}$$

With $\alpha = -\pi/4$ the Dirac Lagrangian becomes:

$$\begin{aligned}
&\psi'^*\gamma^0[\,i\gamma^\mu \mathrm{D}_\mu\, + im_\psi\gamma_5]\psi' \\
&\quad = \psi'^*_L\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\, \psi'_L\, +\, \psi'^*_R\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\, \psi'_R\, +\, m_\psi\psi'^*_L\gamma^0 i\gamma_5\psi'_R\, +\, m_\psi\psi'^*_R\gamma^0 i\gamma_5\psi'_L \\
&\quad = \psi'^*_L\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\, \psi'_L\, +\, \psi'^*_R\gamma^0\, i\gamma^\mu \mathrm{D}_\mu\, \psi'_R\, +\, im_\psi\psi'^*_L\gamma^0\psi'_R\, -\, im_\psi\psi'^*_R\gamma^0\psi'_L.
\end{aligned} \tag{6.76}$$

We have seen that gauge invariance forbids massive gauge bosons, $m_A = 0$, and that parity violation forbids massive fermions, $m_\psi = 0$. This is fixed by spontaneous symmetry breaking, where we take the scalar mass term with wrong sign, $m_\varphi = 0,\ \mu > 0$. The shift of the scalar then induces masses for the gauge bosons, the fermions and the physical scalars. These masses are calculable in terms of the gauge, Yukawa and Higgs couplings.

The other relative signs in the combined Lagrangian are fixed by the requirement that the energy density of the non-gravitational part $\tau_{00}$ be positive (up to a cosmological constant) and that gravity in the Newtonian limit be attractive. In particular this implies that the Higgs potential must be bounded from below, $\lambda > 0$. The sign of the Einstein-Hilbert action may also be obtained from an asymptotically flat space of weak curvature, where we can define gravitational energy density. Then the requirement is that the kinetic terms of all physical bosons, spin 0, 1 and 2, be of the same sign. Take the metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{6.77}$$

$h_{\mu\nu}$ small. Then the Einstein-Hilbert Lagrangian becomes [29],

$$\begin{aligned}
-\,\tfrac{1}{16\pi G}\,R\,|\det g_{\cdot\cdot}|^{1/2} \;=\;& \tfrac{1}{16\pi G}\{\tfrac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \tfrac{1}{8}\partial_\mu h_\alpha{}^\alpha\partial^\mu h_\beta{}^\beta \\
& -[\partial_\nu h_\mu{}^\nu - \tfrac{1}{2}\partial_\mu h_\nu{}^\nu][\partial_{\nu'} h^{\mu\nu'} - \tfrac{1}{2}\partial^\mu h_{\nu'}{}^{\nu'}] + O(h^3)\}.
\end{aligned} \tag{6.78}$$

Here indices are raised with $\eta^{\cdot\cdot}$. After an appropriate choice of coordinates, 'harmonic coordinates', the bracket $\left[\partial_\nu h_\mu{}^\nu - \tfrac{1}{2}\partial_\mu h_\nu{}^\nu\right]$ vanishes and only two independent components of $h_{\mu\nu}$ remain, $h_{11} = -h_{22}$ and $h_{12}$. They represent the two physical states of the graviton, helicity $\pm 2$. Their kinetic terms are both positive, e.g.:

$$+\,\tfrac{1}{16\pi G}\tfrac{1}{4}\partial_\mu h_{12}\partial^\mu h_{12}. \tag{6.79}$$

Likewise, by an appropriate gauge transformation, we can achieve $\partial_\mu A^\mu = 0$, 'Lorentz gauge', and remain with only two, 'transverse' components $A_1,\ A_2$ of helicity $\pm 1$. They have positive kinetic terms, e.g.:

$$+\ \tfrac{1}{2g^2}\mathrm{tr}(\partial_\mu A_1^* \partial^\mu A_1). \tag{6.80}$$

Finally the kinetic term of the scalar is positive:

$$+\ \tfrac{1}{2}\partial_\mu \varphi^* \partial^\mu \varphi. \tag{6.81}$$

An old recipe from quantum field theory, 'Wick rotation', amounts to replace spacetime by a compact Riemannian manifold with Euclidean signature. Then certain calculations become feasible or easier. One of the reasons for this is that Euclidean quantum field theory resembles statistical mechanics, the imaginary time playing formally the role of the inverse temperature. Only at the end of the calculation the result is 'rotated back' to real time. In some cases, this recipe can be justified rigorously. The precise formulation of the recipe is that the $n$-point functions computed from the Euclidean Lagrangian be the analytic continuations in the complex time plane of the Minkowskian $n$-point functions. We shall indicate a hand waving formulation of the recipe that for our purpose is sufficient: In a first stroke we pass to the signature $-+++$. In the second stroke we replace $t$ by $it$ and replace all Minkowskian scalar products by the corresponding Euclidean ones.

The first stroke amounts simply to replacing the metric by its negative. This leaves invariant the Christoffel symbols, the Riemann and Ricci tensors, but reverses the sign of the curvature scalar. Likewise, in the other terms of the Lagrangian we get a minus sign for every contraction of indices, e.g.: $\partial_\mu\varphi^*\partial^\mu\varphi = \partial_\mu\varphi^*\partial_{\mu'}\varphi g^{\mu\mu'}$ becomes $\partial_\mu\varphi^*\partial_{\mu'}\varphi(-g^{\mu\mu'}) = -\partial_\mu\varphi^*\partial^\mu\varphi$. After multiplication by a conventional overall minus sign the combined Lagrangian reads now,

$$\begin{aligned}\{-\tfrac{1}{16\pi}m_P^2\,R\ +\ \tfrac{1}{4g^2}\mathrm{tr}(F^*_{\mu\nu}F^{\mu\nu})\ +\ \tfrac12\,(\mathrm{D}_\mu\varphi)^*\mathrm{D}^\mu\varphi\ -\ \tfrac12\,\mu^2|\varphi|^2\ +\ \lambda|\varphi|^4 \\ +\,\psi^*\gamma^0[\,i\gamma^\mu\mathrm{D}_\mu\ +\ m_\psi 1_4\,]\psi\,\}\ |\det g_{\cdot\cdot}|^{1/2}.\end{aligned} \tag{6.82}$$

To pass to the Euclidean signature, we multiply time, energy and mass by $i$. This amounts to $\eta^{\mu\nu} = \delta^{\mu\nu}$ in the scalar product. In order to have the Euclidean anticommutation relations,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}1, \tag{6.83}$$

we change the Dirac matrices to the Euclidean ones, (4.4), (4.5), (4.8), that are all self adjoint. The Minkowskian scalar product for spinors has a $\gamma^0$. This $\gamma^0$ is needed for the correct physical interpretation of the energy of antiparticles and for Lorentz invariance, $Spin(1,3)$. In the Euclidean, there is no physical interpretation and we can only retain the requirement of a

$Spin(4)$ invariant scalar product. This scalar product has no $\gamma^0$. But then we have a problem if we want to write the Dirac Lagrangian in terms of chiral spinors as above. For instance, $\psi_L^* \, i\gamma^\mu \mathrm{D}_\mu \, \psi_L$ vanishes identically because $\gamma_5$ anticommutes with the four $\gamma^\mu$. The standard trick of Euclidean field theoreticians is fermion doubling, $\psi_L$ and $\psi_R$ are treated as two *independent*, four component spinors. They are not chiral projections of one four component spinor as in the Minkowskian, equation (6.72). The spurious degrees of freedom in the Euclidean are kept all the way through the calculation. They are projected out only after the Wick rotation back to Minkowskian, by imposing $\gamma_5\psi_L = -\psi_L, \gamma_5\psi_R = \psi_R$.

In noncommutative geometry the Dirac operator must be self adjoint, which is not the case of the Euclidean Dirac operator $i\gamma^\mu \mathrm{D}_\mu + im_\psi 1_4$ we get from the Lagrangian (6.82) after multiplication of the mass by $i$. We therefore prefer the primed spinor variables $\psi'$ producing the self adjoint Euclidean Dirac operator $i\gamma^\mu \mathrm{D}_\mu + m_\psi\gamma_5$. Dropping the prime, the combined Lagrangian in the Euclidean then reads:

$$\begin{aligned}&\{-\tfrac{1}{16\pi} m_P^2 \, R + \tfrac{1}{4g^2} \mathrm{tr}(F_{\mu\nu}^* F^{\mu\nu}) + \tfrac{1}{2} (\mathrm{D}_\mu\varphi)^* \mathrm{D}^\mu\varphi - \tfrac{1}{2}\mu^2|\varphi|^2 + \lambda|\varphi|^4 \\ &+ \psi_L^* \, i\gamma^\mu \mathrm{D}_\mu \, \psi_L + \psi_R^* \, i\gamma^\mu \mathrm{D}_\mu \, \psi_R + m_\psi \psi_L^* \gamma_5 \psi_R + m_\psi \psi_R^* \gamma_5 \psi_L\} \, (\det g_{\cdot\cdot})^{1/2}. \qquad (6.84)\end{aligned}$$

In flat space, this is precisely the Yang-Mills-Higgs Lagrangian (4.71) and the Dirac Lagrangian (4.75) in the form obtained from Connes' first dreisatz.

# Chapter 7

# Connes' second dreisatz

Again our starting point is the one–to–one correspondence between *commutative* spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and compact Riemannian manifolds $(M, g)$ with spin structure. Noncommutative or *fuzzy* spaces are defined by relaxing the condition of commutativity. In these spaces the Dirac operator $\mathcal{D}$ plays several important roles:

- It defines the differential structure in terms of the exterior derivative $\mathrm{d} = [\mathcal{D}, \cdot]$.
- The dimension of the space can be read in the spectrum of $\mathcal{D}$, the eigenvalues $\lambda_n$ grow like $n^{1/\dim}$.
- The Dirac operator allows to define integration by regularizing the scalar product of differential forms $\kappa$, $\varphi$:

$$(\kappa, \varphi) := \tfrac{1}{2}\mathrm{Re}\ \mathrm{tr}_\omega\left([\kappa + J\kappa J^{-1}]^*[\varphi + J\varphi J^{-1}]\,|\mathcal{D}|^{-\dim}\right). \tag{7.1}$$

- The Dirac operator generalizes the metric. Indeed on commutative spaces $M$, the metric $g$ can be retrieved from the Dirac operator via the geodesic distance between two points $x_1, x_2 \in M$,

$$d(x_1, x_2) = \mathrm{Sup}\{|f(x_1) - f(x_2)|;\ f \in \mathcal{A},\ ||[\mathcal{D}, \rho(f)]|| \leq 1\}, \tag{7.2}$$

  with $\mathcal{A} = \mathcal{C}^\infty(M)$, $(\rho(f)\psi)(x) = f(x)\psi(x)$ and $\mathcal{D} = \partial\!\!\!/$.

For gravity the last role is vital because the metric is the dynamical variable on spacetime $M$.

## 7.1 The spectral principle

Einstein used the matrix $g_{\mu\nu}(x)$ of the metric $g$ with respect to a holonomic frame $\partial/\partial x^\mu$ to parameterize the set of all metrics on a fixed spacetime $M$. The coordinate system $x^\mu$ being unphysical, Einstein required his field equations for the metric to be covariant under coordinate transformations, the principle of general relativity. Following physicists' habits we will confuse coordinate transformations and diffeomorphisms. Elie Cartan used orthonormal frames, *repères mobiles*, to parameterize the set of all metrics. This parameterization allowed to generalize the Dirac operator $\mathcal{D}$ to curved space-times and also reformulated general relativity as a gauge theory under the Lorentz group. Connes [7] goes one step further by relating the set of all metrics to the set of all Dirac operators. The Einstein-Hilbert action, from this point of view, is the Wodzicki residue of the second inverse power of the Dirac operator [30] and is computed most conveniently from the second coefficient of the heat kernel expansion of the Dirac operator squared.

The natural question now is: what becomes the principle of general relativity in Connes' point of view? Connes' answer is as natural: Invariance under the group of automorphisms of the algebra $\mathcal{A}$. Indeed in the commutative case, $\mathcal{A} = \mathcal{C}^\infty(M)$, this group is the group of diffeomorphisms Diff($M$). And what is an intrinsic property of the Dirac operator, a property invariant under algebra automorphisms? It is the spectrum of $\mathcal{D}$ and Connes proposes to generalize the principle of general relativity in terms of *the spectral principle:*

- Physics is coded in the spectrum of the Dirac operator.

If instead of the Dirac operator we take its square, the Laplace operator, on a flat two dimensional space, then the spectral principle asks an old question:

- Can you hear the shape of a drum?

Let us apply the spectral principle to almost commutative geometries, $\mathcal{A}_t = \mathcal{C}^\infty(M) \otimes \mathcal{A}_f$. Its group of automorphisms is the semidirect product of the group of diffeomorphisms with a gauge group,

$$\mathrm{Diff}(M) \circledS {}^M G, \tag{7.3}$$

where $G$ is the automorphism group of $\mathcal{A}_f$. Up to discrete symmetries, all automorphisms of the inner space $\mathcal{A}_f$ are inner automorphisms,

$$\varphi_u(a) = uau^{-1}, \quad \text{for all } a \in \mathcal{A}_f, \tag{7.4}$$

for a unitary element $u \in U(\mathcal{A}_f)$. Consequently (up to discrete symmetries) the automorphism group of $\mathcal{A}_f$ is a subgroup of its group of unitaries, $G \subset U(\mathcal{A}_f)$. For instance, $\mathcal{A}_f = \mathbb{H}$, $G = U(\mathcal{A}_f) = SU(2)$, and $\mathcal{A}_f = M_3(\mathbb{C})$, $G = SU(3)$, $U(\mathcal{A}_f) = U(3)$. Therefore the spectral principle explains the invariance group of the combined actions of gravity with certain non-Abelian

Yang-Mills theories, the above semidirect product, in terms of almost commutative geometries. It was precisely these geometries, that explained the Higgs and spontaneous symmetry breaking in Connes' first dreisatz. In other words, as quantum mechanics is behind the (Abelian) $U(1)$ in the gauge dreisatz, almost commutative geometries are behind certain non-Abelian Lie groups in the same dreisatz.

## 7.2 First stroke

Let us now follow the Riemannian dreisatz in two strokes to derive the field variables [7] and their dynamics [31] from the spectral principle and almost commutative geometry.

Of course the matter equation we use in the first stroke is the Dirac equation for a free, massive fermion $\psi$ in inertial coordinates (coordinates whose holonomic frame is orthonormal) rather than Newton's equation for a free point mass in inertial coordinates. We have to ask how the Dirac equation changes under an automorphism. In almost commutative geometry an automorphism has two parts. An outer part which is a spacetime diffeomorphism – $\mathcal{C}^\infty(M)$ being commutative has no inner automorphism – and an inner part which is a gauge transformation. We already know how the naked Dirac operator $\partial\!\!\!/$ changes under a diffeomorphism, it becomes covariant with respect to the flat spin connection $\omega(e)$ induced by the diffeomorphism. This is the gravitational coupling that the principle of general relativity orders. The inner Dirac operator $\mathcal{D}$ or fermionic mass matrix is invariant. Let us now see how the inner automorphism $\varphi_u$, $u \in U(\mathcal{A}_t)$ being a gauged unitary, modifies the naked, total Dirac operator $\mathcal{D}_t = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D}_f$ . Since the spinor transforms under this unitary as, cf. section 4.5,

$$\rho_{\text{spinor}}(u)\,\psi \;=\; \rho_f(u)\,J_f\rho_f(u)J_f^{-1}\;\psi, \quad u \in U(\mathcal{A}_t) = {}^M U(\mathcal{A}_f), \tag{7.5}$$

the naked, Dirac $\mathcal{D}_t$ becomes:

$$\begin{aligned}
&\left(\rho_t(u)\,J_t\rho_t(u)J_t^{-1}\right)\,\mathcal{D}_t\,\left(\rho_t(u)\,J_t\rho_t(u)J_t^{-1}\right)^{-1}\\
&= \rho_t(u)\,J_t\rho_t(u)J_t^{-1}\mathcal{D}_t\,\rho_t(u^{-1})\,J_t\rho_t(u^{-1})J_t^{-1}\\
&= \rho_t(u)J_t\rho_t(u)J_t^{-1}(\rho_t(u^{-1})\mathcal{D}_t + [\mathcal{D}_t, \rho_t(u^{-1})])J_t\rho_t(u^{-1})J_t^{-1}\\
&= J_t\rho_t(u)J_t^{-1}\mathcal{D}_tJ_t\rho_t(u^{-1})J_t^{-1} + \rho_t(u)[\mathcal{D}_t, \rho_t(u^{-1})] = J_t\rho_t(u)\mathcal{D}_t\rho_t(u^{-1})J_t^{-1} + \rho_t(u)[\mathcal{D}_t, \rho_t(u^{-1})]\\
&= J_t(\rho_t(u)[\mathcal{D}_t, \rho_t(u^{-1})] + \mathcal{D}_t)J_t^{-1} + \rho_t(u)[\mathcal{D}_t, \rho_t(u^{-1})]\\
&= \mathcal{D}_t \;-\; \pi_t(A_t) - J_t\pi_t(A_t)J_t^{-1},
\end{aligned} \tag{7.6}$$

with the flat connection:

$$A_t = u\delta_t u^* = u\mathrm{d}u^* + u\delta_f u^* = A + H \;\in \Omega^1(M, u(\mathcal{A}_f)) \;\oplus\; \mathcal{C}^\infty(M) \otimes \Omega^1_{\mathcal{D}_f}\mathcal{A}_f. \tag{7.7}$$

In the chain (7.6) we have used successively the following three axioms of spectral triples, $[\rho(u_1), J\rho(u_2)J^{-1}] = 0$, the first order condition $[[\mathcal{D}, \rho(u_1)], J\rho(u_2)J^{-1}] = 0$ and $[\mathcal{D}, J] = 0$. The result means that the naked Dirac operator becomes covariant with respect to the Yang-Mills potential $A$ and with respect to the Higgs scalar $H$. The spectral principle implies that in almost commutative geometry, the gravitational field coded in the metric or equivalently in the Dirac operator is necessarily accompanied by the spin 1 field $A$ and the spin 0 field $H$.

So far the three connections $\omega(e)$, $A$, $H$ have no curvature. We now promote them to general fields. Then we have the total, covariant Dirac operator,

$$\mathcal{D}_{t,\mathrm{cov}} = \mathcal{D}_t \,-\, \pi_t(A_t) - J_t\pi_t(A_t)J_t^{-1}, \tag{7.8}$$

which is precisely the one of Connes' first dreisatz, section 4.5.

## 7.3 Second stroke

So far the gravitational, Yang-Mills and Higgs fields are adynamical, only the fermion $\psi$ propagates in the fixed background $((e, \omega(e)), A, H)$. In the second stroke, Chamseddine & Connes [31] develop the full power of the spectral principle to derive the dynamics of the spin 2, 1 and 0 fields from the total, covariant Dirac operator $\mathcal{D}_{t,\mathrm{cov}}$.

In even dimensions, the spectrum of the Dirac operator is even and it is sufficient to consider the positive part of the spectrum which in the Euclidean is conveniently characterized by a distribution function

$$S = \mathrm{tr} f(\mathcal{D}_{t,\mathrm{cov}}^2/\Lambda^2), \tag{7.9}$$

where $\Lambda$ is an energy cutoff and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive, smooth function with finite 'momenta',

$$f_0 \;=\; \int_0^\infty u f(u)\mathrm{d}u, \tag{7.10}$$

$$f_2 \;=\; \int_0^\infty f(u)\mathrm{d}u, \tag{7.11}$$

$$f_4 \;=\; f(0), \tag{7.12}$$

$$f_6 \;=\; -f'(0), \tag{7.13}$$

$$f_8 \;=\; f''(0), ... \tag{7.14}$$

Asymptotically, for large $\Lambda$, the distribution function of the spectrum is given in terms of the heat kernel expansion [32]:

$$S = \mathrm{tr} f(\mathcal{D}_{t,\mathrm{cov}}^2/\Lambda^2) = \frac{1}{16\pi^2}\int_M [\Lambda^4 f_0 a_0 + \Lambda^2 f_2 a_2 + f_4 a_4 + \Lambda^{-2} f_6 a_6 + ...]\sqrt{\det g}\,\mathrm{d}^4x, \tag{7.15}$$

where the $a_j$ are the coefficients of the heat kernel expansion of the Dirac operator squared [33],

$$\begin{aligned} a_0 &= \mathrm{tr}(1_4 \otimes 1_{\mathcal{H}}), && (7.16)\\ a_2 &= \tfrac{1}{6} R\,\mathrm{tr}(1_4 \otimes 1_{\mathcal{H}}) - \mathrm{tr} E, && (7.17)\\ a_4 &= \tfrac{1}{72} R^2 \mathrm{tr}(1_4 \otimes 1_{\mathcal{H}}) - \tfrac{1}{180} R_{\mu\nu} R^{\mu\nu} \mathrm{tr}(1_4 \otimes 1_{\mathcal{H}}) + \tfrac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \mathrm{tr}(1_4 \otimes 1_{\mathcal{H}}) \\ &\quad + \tfrac{1}{12} \mathrm{tr}(\mathbb{R}_{\mu\nu} \mathbb{R}^{\mu\nu}) - \tfrac{1}{6} R\,\mathrm{tr} E + \tfrac{1}{2} \mathrm{tr} E^2 + \text{surface terms}. && (7.18) \end{aligned}$$

We have used the Lichnérowicz formula for the square of the Dirac operator, $\mathcal{D}_{t,\mathrm{cov}}^2 = -\Delta + E$. Note that for large $\Lambda$ the positive function $f$ is universal in the sense that only the three first momenta, $f_0,\ f_2$ and $f_4$ matter.

Let us first check the normalization $16\pi^2$ of equation (7.15). Again we take $M$ to be the flat 4-torus with unit radii, $\mathcal{H}_L = \mathbb{C}$, $\mathcal{H}_R = 0$ and $A = \varphi = 0$. Remember from section 4.3 that for large $\Lambda$ there are $4 B_4 \Lambda^4$ eigenvalues (counted with their multiplicity) whose absolute values are smaller than $\Lambda$. $B_4 = \pi^2/2$ denotes the volume of the unit ball in $\mathbb{R}^4$. On the other hand if we take for $f$ a smooth approximation of the characteristic function of the unit interval, then $f_0 = \frac{1}{2}$ and $S$ simply counts the eigenvalues of the square of the Dirac operator less than $\Lambda^2$:

$$S = 4 \tfrac{1}{2} \pi^2 \Lambda^4 = \frac{1}{16\pi^2}\, \Lambda^4 \tfrac{1}{2} 4 (2\pi)^4. \qquad (7.19)$$

The computation of the Chamseddine-Connes action $S$ for the Dirac operator of the standard model is straightforward. We give a few intermediate steps, a full account can be found in [34].

$$\begin{aligned} a_0 &= 4 \dim \mathcal{H}, && (7.20)\\ \mathrm{tr} E &= \dim \mathcal{H}\, R + 8 \mathrm{tr} \Phi^* \Phi = \dim \mathcal{H}\, R + 8 L |\varphi/v|^2, && (7.21)\\ L &:= 3 \mathrm{tr}(M_u^* M_u) + 3 \mathrm{tr}(M_d^* M_d) + \mathrm{tr}(M_e^* M_e) \\ &= 3(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + m_\tau^2 + m_\mu^2 + m_e^2, && (7.22)\\ a_2 &= \tfrac{4}{6} \dim \mathcal{H}\, R - \dim \mathcal{H}\, R - 8 L |\varphi/v|^2 \\ &= -\tfrac{1}{3} \dim \mathcal{H}\, R - 8 L |\varphi/v|^2, && (7.23)\\ \mathrm{tr}\left(\tfrac{1}{2}[\gamma^a, \gamma^b] \tfrac{1}{2}[\gamma^c, \gamma^d]\right) &= 4 \left[\eta^{ad} \eta^{bc} - \eta^{ac} \eta^{bd}\right], && (7.24)\\ \mathrm{tr} \mathbb{R}_{\mu\nu} \mathbb{R}^{\mu\nu} &= -\tfrac{1}{2} \dim \mathcal{H}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 \mathrm{tr} \rho(F_{\mu\nu})^* \rho(F^{\mu\nu}), && (7.25)\\ \mathrm{tr} E^2 &= \tfrac{1}{4} \dim \mathcal{H}\, R^2 + 2 \mathrm{tr} \rho(F_{\mu\nu})^* \rho(F^{\mu\nu}) \\ &\quad + 8 Q |\varphi/v|^4 + 8 L (\mathrm{D}_\mu \varphi/v)^* (\mathrm{D}^\mu \varphi/v) + 4 L |\varphi/v|^2 R, && (7.26)\\ Q &:= 3 \mathrm{tr} \left[M_u^* M_u\right]^2 + 3 \mathrm{tr} \left[M_d^* M_d\right]^2 + \mathrm{tr} \left[M_e^* M_e\right]^2 \\ &= 3(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + m_\tau^4 + m_\mu^4 + m_e^4. && (7.27) \end{aligned}$$

Using the Weyl tensor,

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \tfrac{1}{2}(g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\rho} R_{\mu\sigma}) + \tfrac{1}{6}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R, \qquad (7.28)$$

we can assemble all higher derivative gravity terms in $a_4$ to form the square of the Weyl tensor

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 = 2R_{\mu\nu}R^{\mu\nu} - \tfrac{2}{3}R^2 + \text{surface terms}, \tag{7.29}$$

because $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is proportional to the Euler characteristic of $M$. Then, up to this surface term, we have

$$-\tfrac{1}{360}\dim\mathcal{H}\left[7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2\right] = -\tfrac{1}{20}\dim\mathcal{H}\; C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{7.30}$$

Finally we have up to surface terms,

$$\begin{aligned} a_4 \;=\;& -\tfrac{1}{20}\dim\mathcal{H}\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tfrac{2}{3}\mathrm{tr}\rho(F_{\mu\nu})^*\rho(F^{\mu\nu}) \\ & +4Q|\varphi/v|^4 + 4L(\mathrm{D}_\mu\varphi/v)^*(\mathrm{D}^\mu\varphi/v) + \tfrac{2}{3}L|\varphi/v|^2R. \end{aligned} \tag{7.31}$$

We have used a trick to compute the second and forth power of the homogeneous scalar variable $\Phi$, a trick proper to the noncommutative formulation of the standard model. Remember from section 4.7 the embedding of the scalar doublet $\varphi = {}^t(\varphi_1,\varphi_2)$ in $\mathcal{H}_L^*\otimes\mathcal{H}_R \oplus \mathcal{H}_L\otimes\mathcal{H}_R^*$:

$$\Phi = \frac{1}{v}\begin{pmatrix} \begin{pmatrix} \varphi_1 M_u & -\bar\varphi_2 M_d \\ \varphi_2 M_u & \bar\varphi_1 M_d \end{pmatrix}\otimes 1_3 & 0 \\ 0 & \begin{pmatrix} -\bar\varphi_2 M_e \\ \bar\varphi_1 M_e \end{pmatrix} \end{pmatrix}, \tag{7.32}$$

with $v$ denoting the vacuum expectation value. This embedding, which is nothing but the Yukawa couplings, takes the form of a matrix product,

$$\Phi = \rho_{Lw}(\phi)\mathcal{M}/v, \quad \phi = \begin{pmatrix} \varphi_1 & -\bar\varphi_2 \\ \varphi_2 & \bar\varphi_1 \end{pmatrix} \in \mathbb{H}, \tag{7.33}$$

and the powers of $\Phi$ follow easily from the identity

$$\phi^*\phi = \phi\phi^* = (|\varphi_1|^2 + |\varphi_2|^2)1_2 = |\varphi|^2 1_2. \tag{7.34}$$

## 7.4 The unified action

Chamseddine & Connes' distribution function $S$ or *spectral action* unifies the Einstein-Hilbert action, the Yang-Mills action, the Klein-Gordon action and the Higgs potential.

- relativity + noncommutative geometry = Einstein-Hilbert-Yang-Mills-Higgs.

We still have to properly normalize the kinetic terms of the gravitational, Yang-Mills and Higgs fields to deduce their couplings, Newton's constant $G$, the gauge couplings $g_i$ and the

Higgs couplings $\lambda$ and $\mu$. We also have a cosmological constant $\Lambda_C$, the conformal scalar gravity coupling and a higher derivative gravity term with coefficient $a$ in the spectral action,

$$\begin{aligned}\mathrm{tr} f(\mathcal{D}^2_{t,\mathrm{cov}}/\Lambda^2) \;=\; \int_M [ & -\tfrac{1}{16\pi} m_P(\Lambda)^2\, R \,-\, \Lambda_C(\Lambda) \\ & + \tfrac{1}{2} g_3(\Lambda)^{-2}\, \mathrm{tr} F^{(3)*}_{\mu\nu} F^{(3)\mu\nu} + \tfrac{1}{2} g_2(\Lambda)^{-2}\, \mathrm{tr} F^{(2)*}_{\mu\nu} F^{(2)\mu\nu} + \tfrac{1}{4} g_1(\Lambda)^{-2}\, F^{(1)*}_{\mu\nu} F^{(1)\mu\nu} \\ & + \tfrac{1}{2}\, (\mathrm{D}_\mu\varphi)^* \mathrm{D}^\mu \varphi \,+\, \lambda(\Lambda)|\varphi|^4 \,-\, \tfrac{1}{2}\, \mu(\Lambda)^2 |\varphi|^2 \\ & - a(\Lambda) C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \,+\, \tfrac{1}{12}\, |\varphi|^2 R \quad ]\, (\det g_{\cdot\cdot})^{1/2}\, \mathrm{d}^4 x \,+\, O(\Lambda^{-2}). \end{aligned} \tag{7.35}$$

Before identifying Newton's constant $G = \hbar c\, m_P^{-2}$ and the cosmological constant $\Lambda_C$, we have to shift the Higgs field by its vacuum expectation value, $|\varphi| = v(\Lambda) = \mu(\Lambda)/(2\sqrt{\lambda}(\Lambda))$. With $N$ generations of quarks and leptons, $N = 3$, we have:

$$m_P(\Lambda)^2 \;=\; \tfrac{1}{\pi} f_2 \left[5N - \frac{2}{3}\frac{L^2}{Q}\right] \Lambda^2 \;\approx\; \tfrac{1}{\pi} f_2 [5N - 2] \Lambda^2, \tag{7.36}$$

$$\Lambda_C(\Lambda) \;=\; \tfrac{1}{4\pi^2} \left[\frac{f_2^2}{f_4}\frac{L^2}{Q} \,-\, 15N f_0\right] \Lambda^4 \;\approx\; \tfrac{3}{4\pi^2} \left[\frac{f_2^2}{f_4} \,-\, 5N f_0\right] \Lambda^4, \tag{7.37}$$

$$g_3(\Lambda)^{-2} \;=\; \tfrac{N}{3\pi^2} f_4, \tag{7.38}$$

$$g_2(\Lambda)^{-2} \;=\; \tfrac{N}{3\pi^2} f_4, \tag{7.39}$$

$$g_1(\Lambda)^{-2} \;=\; \tfrac{5}{3}\tfrac{N}{3\pi^2} f_4, \tag{7.40}$$

$$\lambda(\Lambda)^{-1} \;=\; \tfrac{1}{\pi^2}\, f_4\, \frac{L^2}{Q} \;\approx\; \tfrac{3}{\pi^2}\, f_4, \tag{7.41}$$

$$\mu(\Lambda)^2 \;=\; 2\, \frac{f_2}{f_4}\, \Lambda^2, \tag{7.42}$$

$$a(\Lambda) \;=\; \tfrac{3N}{64\pi^2}\; f_4. \tag{7.43}$$

The indicated approximation concerns the dominating top mass. Comparing with the combined Euclidean action (6.84), we see that each relevant term comes with its correct sign!

Identifying $f_4 = \frac{3}{2} z$ the constraints on the three gauge couplings from noncommutative Yang-Mills coincide with the constraints from noncommutative relativity. This is not an accident. In noncommutative Yang-Mills, we have chosen the scalar product symmetrized with respect to charge conjugation,

$$<\kappa, \varphi> \;:=\; \tfrac{z}{2} \mathrm{Re}\ \mathrm{tr}_\omega \left([\kappa + J\kappa J^{-1}]^* [\varphi + J\varphi J^{-1}]\, |\mathcal{D}|^{-\dim}\right), \quad \kappa, \varphi \in \pi(\Omega^p \mathcal{A}). \tag{7.44}$$

In the spectral action this scalar product is induced from the symmetrized covariant Dirac operator $\mathcal{D} - \pi(A) - J\pi(A)J^{-1}$. The non-symmetrized covariant Dirac operator $\mathcal{D} - \pi(A)$ would induce the non-symmetrized scalar product,

$$<\kappa, \varphi> \;:=\; z \mathrm{Re}\ \mathrm{tr}_\omega (\kappa^* \varphi |\mathcal{D}|^{-\dim}), \tag{7.45}$$

in the spectral action. Physics requires the use of the symmetrized Dirac operator in the fermionic action, $\psi^*(\mathcal{D} - \pi(A) - J\pi(A)J^{-1})\psi$. In the noncommutative Yang-Mills setting we were still free to use either Dirac operator – symmetrized or not – in the bosonic action. This is no longer true in noncommutative relativity where the spectral principle requires one and the same Dirac operator in both actions, the fermionic and the bosonic. This is why we committed to the symmetrized scalar product already in noncommutative Yang-Mills. Here there is no choice and we are forced to swallow the big desert and to extrapolate running couplings to energies $\Lambda = 10^{13} - 10^{17}$ GeV where $f_4 = \frac{3}{2}z = (0.80 - 0.94)4\pi^2$. This of course means that we have to return humblely to flat space because, despite the higher derivative term $a$, gravity remains unrenormalizable. Fortunately, thanks to $f_0$ and $f_2$, the Planck mass and the cosmological constant decouple from the gauge couplings. Since the evolution of $\mu$ strongly depends on the regularization scheme there is only one more unambiguous constraint from noncommutative relativity,

$$\lambda(\Lambda) = \tfrac{N}{9}g_2(\Lambda)^2. \quad \text{nc relat.} \tag{7.46}$$

Remember the corresponding constraint from noncommutative Yang-Mills,

$$\lambda(\Lambda) = \tfrac{3N-2}{24}g_2(\Lambda)^2. \quad \text{nc YM} \tag{7.47}$$

They would coincide for $N = 6$ generations. For $N = 3$, their mismatch is still acceptable, in terms of the resulting Higgs mass, we have,

$$m_H = 182 \pm 10 \pm 7 \text{ GeV}. \quad \text{nc relat.} \tag{7.48}$$

The first error is from the uncertainty in $\Lambda = 10^{13} - 10^{17}$ GeV. The second is from the present experimental uncertainty in the top mass, $m_t = 175 \pm 6$ GeV. Indeed we must admit that noncommutative relativity does not constrain the Yukawa coupling or equivalently the top mass as was the case in noncommutative Yang-Mills where we had

$$\begin{aligned} m_H &= 197 \pm 9 \pm 0 \text{ GeV}, \\ m_t &= 187 \pm 14 \pm 0 \text{ GeV}. \quad \text{nc YM} \end{aligned} \tag{7.49}$$

The mismatch between the two Higgs couplings or masses from noncommutative Yang-Mills and from noncommutative relativity is of the same order of magnitude as the mismatch between the experimental and theoretical values of the three gauge couplings. We blame this mismatch on the enormous extrapolation through the big desert. We take the mismatch as indication that at energies $\Lambda = 10^{13} - 10^{17}$ GeV almost commutative geometry will merge into a truely noncommutative geometry and that gravitational quantum effects will no longer be small. In any case we find it encouraging that noncommutative Yang-Mills and noncommutative relativity produce comparable results for the standard model. This is another miracle of the standard model. Indeed applied to the commutative example of section 4.4, the two dreisätze produce quite different outputs, the first has a photon the second does not. Similarly the minimax model, 4.6, with one generation of leptons, has no spontaneous symmetry break down in noncommutative Yang-Mills, but does enjoy spontaneous break down in noncommutative relativity because there junk does not happen.

In the standard model with $N = 3$ generations, the two Higgs mass predictions have a non-empty intersection . This intersection is $m_H = 188 - 199$ GeV, an energy range experimentally accessible to the Large Hadron Collider LHC in Geneva within ten years.

## 7.5 Outlook

Connes' noncommutative geometry has impressive unification power. Almost commutative geometry unifies the non-Abelian gauge dreisatz with the Riemannian dreisatz. At the same time it indicates a sequence of dreisätze, the Minkowskian, Riemannian and Connes' second dreisatz indexed by the nested invariance groups, the Lorentz, diffeomorphism and $\mathcal{A}_t-$automorphism groups. It seems natural to pursue this sequence to truely noncommutative geometries. Indeed $\mathcal{A}_t = \mathcal{C}^\infty(M) \otimes (\mathbb{H} \oplus \mathbb{C} \otimes M_3(\mathbb{C}))$ is almost as ugly as $\mathrm{Diff}(M) \circledS {}^M(SU(2) \times U(1) \times SU(3))$. Noncommutative geometry grew out of quantum mechanics. Almost commutative geometry unifies gravity with the subnuclear forces. We expect noncommutative geometry to reconcile gravity with quantum field theory.

The basic variable of noncommutative geometry is the Dirac operator acting on fermions. The fermions must define a representation of an associative algebra and are constrained by the axioms of noncommutative geometry, i.e. of spectral triples. These axioms still leave many choices, one of which the quarks and leptons of the standard model with their mass matrix taken from experiment. Of course, we want an explanation for this choice. To define the Dirac operator in Riemannian geometry, the spin group is essential. There is no generalization of the spin group to noncommutative geometry yet. According to Connes [8], this generalization should be a quantum group and it should help us to get a handle on the arbitrariness of the fermion representation.

Minkowskian geometry explains the magnetic field, Riemannian geometry explains gravity. Both geometries have operated revolutions on spacetime that today are well established experimentally: the loss of absolute time and the loss of universal time. Can we observe the noncommutative nature of time, its uncertainty or 'fuzziness', despite its ridiculously small scale $\hbar/\Lambda = 10^{-40}$ s?

So far noncommutative geometry is developed in Euclidean, compact spacetimes, so 'Wick rotation' and 3+1 split remain to be understood [35]. After this, we expect noncommutative geometry to change our picture of black holes in a similar fashion that Heisenberg's uncertainty relation has cured the Coulomb singularity of the hydrogen atom. Also our picture of the big bang, cosmology and the origin of time is expected to be revised [36].

Planatary motion has degraded circles to epicycles and dismissed them all together in favour of ellipses. Particle physics is about to dismiss Riemannian geometry in favour of noncommutative geometry and the question is, what dynamics is behind these new ellipses?

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