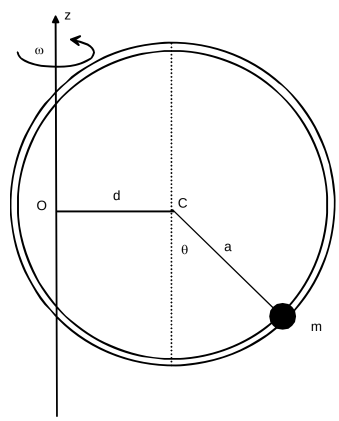
Problema 1



Colocando sistema fijo al aro con origen en C que rota con $\vec{\omega} = \omega\hat{k}$ constante

$$\vec{a}_C + 2\omega\hat{k} \times \vec{v} + \omega\hat{k} \times (\omega\hat{k} \times \vec{r}) + \vec{a} = \frac{\vec{F}}{m} = \frac{\vec{N}}{m}$$

aquí

$$\begin{aligned}
\vec{r} &= a\hat{r} \\
\vec{v} &= a\dot{\theta}\hat{\theta} \\
\vec{a} &= a\ddot{\theta}\hat{\theta} - a\dot{\theta}^2\hat{r}
\end{aligned}$$

los productos cruz

$$\begin{aligned}
\hat{k} \times \hat{\theta} &= \cos\theta\hat{\phi} \\
\hat{k} \times \hat{r} &= \sin\theta\hat{\phi}
\end{aligned}$$

$$d\omega^2\hat{k} \times \hat{\phi} + 2\omega a\dot{\theta}\cos\theta\hat{\phi} + \omega^2 a\sin\theta\hat{k} \times \hat{\phi} + a\ddot{\theta}\hat{\theta} - a\dot{\theta}^2\hat{r} = \frac{\vec{N}}{m}$$

proyectando en las direcciones $\hat{r}, \hat{\theta}, \hat{\phi}$ resultan

$$\begin{aligned}
d\omega^2\hat{k} \times \hat{\phi} \cdot \hat{r} + \omega^2 a\sin\theta\hat{k} \times \hat{\phi} \cdot \hat{r} - a\dot{\theta}^2 &= \frac{\vec{N} \cdot \hat{r}}{m} \\
d\omega^2\hat{k} \times \hat{\phi} \cdot \hat{\theta} + \omega^2 a\sin\theta\hat{k} \times \hat{\phi} \cdot \hat{\theta} + a\ddot{\theta} &= 0 \\
2\omega a\dot{\theta}\cos\theta &= \frac{\vec{N} \cdot \hat{\phi}}{m}
\end{aligned}$$

pero

$$\begin{aligned}
\hat{k} \times \hat{\phi} \cdot \hat{r} &= -\sin\theta \\
\hat{k} \times \hat{\phi} \cdot \hat{\theta} &= -\cos\theta
\end{aligned}$$

luego

$$\begin{aligned}
-\sin\theta d\omega^2 - \omega^2 a \sin^2\theta - a\dot{\theta}^2 &= \frac{\vec{N}\cdot\hat{r}}{m} \\
-d\omega^2\cos\theta - \omega^2 a \sin\theta\cos\theta + a\ddot{\theta} &= 0 \\
2\omega a\dot{\theta}\cos\theta &= \frac{\vec{N}\cdot\hat{\phi}}{m}
\end{aligned}$$

la segunda es

$$a\ddot{\theta} = d\omega^2\cos\theta + \omega^2 a\sin\theta\cos\theta = F(\theta)$$

ángulo de equilibrio θ_0

$$d\omega^2\cos\theta_0 + \omega^2 a\sin\theta_0\cos\theta_0 = 0$$

sea

$$\theta = \theta_0 + \xi$$

luego

$$\ddot{\xi} = F(\theta_0 + \xi) = F(\theta_0) + \xi F'(\theta_0)$$

o sea

$$\ddot{\xi} = \xi F(\theta_0) = \xi(-d\omega^2\sin\theta_0 - \omega^2 a\sin^2\theta_0 + \omega^2 a\cos^2\theta_0)$$

ángulo de equilibrio θ_0

$$\begin{aligned}
d\omega^2\cos\theta_0 + \omega^2 a\sin\theta_0\cos\theta_0 &= 0 \\
\theta_0 &= \pm\frac{\pi}{2} \\
\sin\theta_0 &= -\frac{d}{a}
\end{aligned}$$

sea

$$\theta = \theta_0 + \xi$$

luego

$$\ddot{\xi} = F(\theta_0 + \xi) = F(\theta_0) + \xi F'(\theta_0)$$

o sea

$$\begin{aligned}
\ddot{\xi} &= \xi F(\theta_0) = \xi(-d\omega^2\sin\theta_0 - \omega^2 a\sin^2\theta_0 + \omega^2 a\cos^2\theta_0) \\
+\pi/2 \quad \ddot{\xi} &= -\xi(d+a)\omega^2 \text{ estable} \\
+\pi/2 \quad \ddot{\xi} &= -\xi(a-d)\omega^2 \text{ estable} \\
\sin\theta_0 &= -\frac{d}{a}, \quad \ddot{\xi} = \xi F(\theta_0) = \xi\frac{\omega^2}{a}\left(a^2 - d^2\right) \text{ inestable}
\end{aligned}$$

Potencial efectivo (por si alguien lo hace así...)

$$\begin{aligned}
a\ddot{\theta} &= d\omega^2\cos\theta + \omega^2 a\sin\theta\cos\theta = F(\theta) \\
a\ddot{\theta} &= -\frac{d}{d\theta}V \\
V &= -d\omega^2\sin\theta + \frac{1}{2}\omega^2 a\cos^2\theta
\end{aligned}$$

Problema 2

Podemos usar directamente

$$\vec{r}(t) = \vec{r}(0) + \vec{v}(0)t - \frac{1}{2}gt^2\hat{k} - t^2\vec{\omega} \times \vec{v}(0) + \frac{1}{3}gt^3\vec{\omega} \times \hat{k}$$

obtenirndo

$$\vec{r}(t) = v_0\hat{k}t - \frac{1}{2}gt^2\hat{k} - t^2\vec{\omega} \times v_0\hat{k} + \frac{1}{3}gt^3\vec{\omega} \times \hat{k}$$

pero

$$\vec{\omega} = -\omega\cos\lambda\hat{\imath} + \omega\sin\lambda\hat{k}$$

entonces

$$\vec{\omega} \times \hat{k} = \omega\cos\lambda\hat{\jmath}$$

de manera que

$$\begin{aligned} z &= v_0 t - \frac{1}{2}gt^2 \\ y &= -v_0 t^2\omega\cos\lambda + \frac{1}{3}gt^3\omega\cos\lambda \end{aligned}$$

: $-\frac{1}{3}t^2\omega(\cos\lambda)(-gt + 3v_0)$ el punto de caida será

$$\begin{aligned} z &= 0, \quad t = \frac{2v_0}{g} \\ y &= -\frac{1}{3}(\frac{2v_0}{g})^2\omega(\cos\lambda)v_0 \end{aligned}$$

la máxima desviación hacia el Oeste será el máximo de

$$v_0 t^2\omega\cos\lambda - \frac{1}{3}gt^3\omega\cos\lambda$$

derivamos respecto a t

$$\begin{aligned} 2v_0\omega\cos\lambda - gt\omega\cos\lambda &= 0 \\ t &= 2\frac{v_0}{g} \end{aligned}$$

luego el máximo es

$$\frac{1}{3}(\frac{2v_0}{g})^2\omega(\cos\lambda)v_0$$

Problema 3

Llamemos θ, ϕ los ángulos pequeños que forman las cuerdas con la vertical. Para oscilaciones pequeñas las tensiones son prácticamente las de equilibrio $T_1 = 2mg$, $T_2 = mg$ y las aceleraciones son practicamente horizontales (física) de manera que resultan

$$\begin{aligned} ma\ddot{\theta} &= -2mg\theta + mg\phi \\ ma(\ddot{\theta} + \ddot{\phi}) &= -mg\phi \end{aligned}$$

o bien

$$\begin{aligned}\ddot{\theta}+2\frac{g}{a}\theta-\frac{g}{a}\phi &= 0\\ \ddot{\theta}+\ddot{\phi}+\frac{g}{a}\phi &= 0\end{aligned}$$

al sustituir

$$\begin{aligned}\theta &= Ae^{-i\omega t}\\ \phi &= Be^{-i\omega t}\end{aligned}$$

$$\begin{aligned}(-\omega^2+2\frac{g}{a})A-\frac{g}{a}B &= 0\\ -\omega^2 A-\omega^2 B+\frac{g}{a}B &= 0\end{aligned}$$

$$\omega^4 a^2-4g\omega^2 a+2g^2=0$$

$$\begin{aligned}\omega_1^2 &= \left(2+\sqrt{2}\right)\frac{g}{a}\\ \omega_2^2 &= \left(2-\sqrt{2}\right)\frac{g}{a}\end{aligned}$$

$$\frac{A}{B}=\frac{\frac{g}{a}}{(-\omega^2+2\frac{g}{a})}$$

$$\begin{aligned}\frac{A_1}{B_1} &= -\frac{1}{2}\sqrt{2}\\ \frac{A_2}{B_2} &= \frac{1}{2}\sqrt{2}\end{aligned}$$

$$\begin{aligned}\theta &= A_1 e^{-i\omega_1 t}+A_2 e^{-i\omega_2 t}\\ \theta &= -\frac{1}{2}\sqrt{2}B_1 e^{-i\omega_1 t}+\frac{1}{2}\sqrt{2}B_2 e^{-i\omega_2 t}\\ \phi &= B_1 e^{-i\omega_1 t}+B_2 e^{-i\omega_2 t}\end{aligned}$$

de donde las coordenadas normales son

$$\begin{aligned}\xi_1 &= \frac{1}{2}\phi+\frac{1}{2}\sqrt{2}\theta\\ \xi_2 &= \frac{1}{2}\phi-\frac{1}{2}\sqrt{2}\theta\end{aligned}$$

Problema 4

las ecuaciones de movimiento son

$$\begin{aligned}M\ddot{x}_1 &= -kx_1+2k(x_2-x_1)=-3kx_1+2kx_2\\ M\ddot{x}_2 &= -kx_2-2k(x_2-x_1)=-3kx_2+2kx_1\end{aligned}$$

sumándolas y restándolas

$$\begin{aligned} M(\ddot{x}_1 + \ddot{x}_2) &= -k\,(x_1 + x_2) \\ M(\ddot{x}_1 - \ddot{x}_2) &= -5k\,(x_1 - x_2) \end{aligned}$$

de donde las coordenadas normales y frecuencias propias son

$$\begin{aligned} \xi_1 &= x_1 + x_2 \\ \omega_1 &= \sqrt{\frac{k}{M}} \end{aligned}$$

$$\begin{aligned} \xi_2 &= x_1 - x_2 \\ \omega_1 &= \sqrt{\frac{5k}{M}} \end{aligned}$$