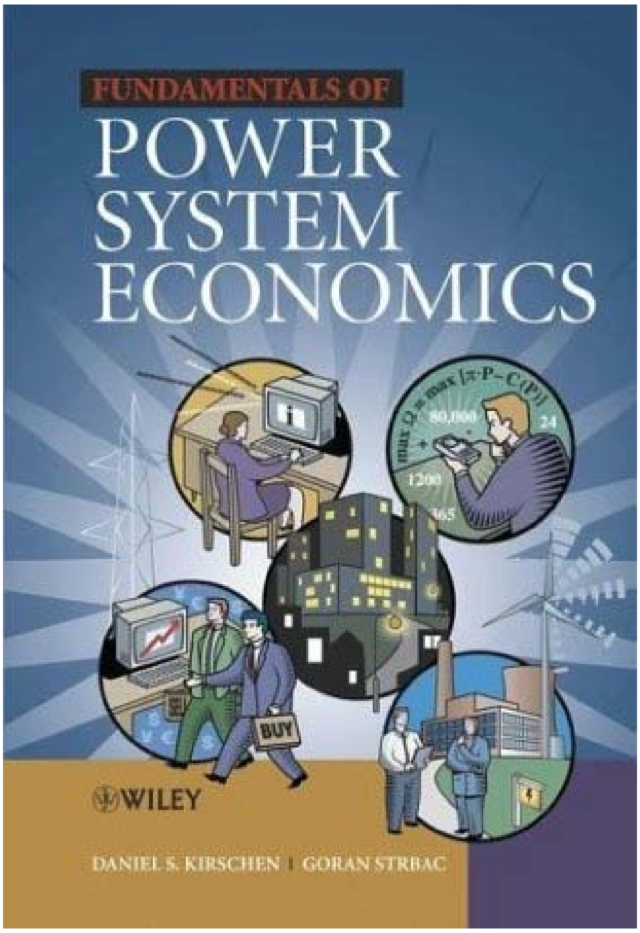
SOLUTIONS MANUAL



CHAPTER 4

PARTICIPATING IN MARKETS FOR ELECTRICAL ENERGY

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Chapter 4

4.1 *Cheapo Electrons is an electricity retailer. The table below shows the load that it forecast its consumers would use over a six-hour period. Cheapo Electrons purchased on the forward market and the power exchange exactly enough energy to cover this forecast. The table shows the average price that it paid for this energy for each hour. As one might expect, the actual consumption of its customers did not exactly match the load forecast and it had to purchase or sell the difference on the spot market at the prices indicated. Assuming that Cheapo Electrons sells energy to its customers at a flat rate of* 24.00$/MWh, *calculate the profit or loss that it made during this six-hour period. What would be the rate that it should have charged its customers to break even*?

Period	1	2	3	4	5	6
Load Forecast [MWh]	120	230	310	240	135	110
Average cost [$/MWh]	22.5	24.5	29.3	25.2	23.1	21.9
Actual load [MWh]	110	225	330	250	125	105
Spot price [$/MWh]	21.6	25.1	32	25.9	22.5	21.5

Table P4.1 shows the payments for energy, the revenue from the sales, the spot balancing transactions and the profit for each of period.

Table P4.1: Summary of the revenues and expenses for Problem 4.1

Period	1	2	3	4	5	6
Payment for energy ($)	-2700	-5635	-9083	-6048	-3118.5	-2409
Revenue from the sales ($)	2640	5400	7920	6000	3000	2520
Spot transactions ($)	216	125.5	-640	-259	225	107.5
Profit ($)	156	-109.5	-1803	-307	106.5	218.5

Summing the profits (or losses) for each of the six periods, we get a loss of $1738.50.

Cheapo Electrons will break even price if the total income it derives from the sale of electricity to consumers is equal to the total payments it makes for the energy it purchases on the forward and spot markets:

Revenue from customer sales = cost forward market purchases + cost of spot market purchases

Using the data from Table P4.1 and summing over all six periods, we get:

$\pi \times 1145 = 28993.50 + 225$

The break-even price is thus $\pi = 25.52$ $/MWh.

4.2 *The input-output curve of a gas-fired generating unit is approximated by the following function:*

$$H(P) = 120 + 9.3P + 0.0025P^2 \text{ [MJ/h]}$$

This unit has a minimum stable generation of 200 MW and a maximum output of 500 MW. The cost of gas is 1.20 \$/MJ. Over a six-hour period, the output of this unit is sold on a market for electrical energy at the prices shown in the table below.

Period	1	2	3	4	5	6
Price [$/MWh]	12.5	10	13	13.5	15	11

Assuming that this unit is optimally dispatched, is initially on-line and cannot be shut down, calculate its operational profit or loss for this period.

The running cost of the unit is:

$$C(P) = F \times H(P) = 144 + 11.16P + 0.003P^2 \text{ [\$/h]}$$

Therefore the marginal cost of production of the unit is:

$$\frac{d}{dP}C(P) = 11.16 + 0.006P$$

Since the unit is optimally dispatched, its operator will adjust its output so that the marginal cost of production of the unit is equal to the market price for each period. Therefore, we have:

$$P = \frac{\pi_{period} - 11.16}{0.006}$$

Using this relation, we can build Table P4.2, which also shows the revenue collected by the unit, the running cost of the unit and the profit for each period.

Table P4.2: Optimal operation of the unit of Problem 4.2

Period		1	2	3	4	5	6
Price, π ($/MWh)		12.5	10	13	13.5	15	11
Production *(P)*	(MW)	233.33	200	306.66	390	500	200
Revenue $(\pi \times P)$	(MW)	2916.66	2000	3986.66	5265	7500	2200
Running cost *C(P)*	($)	2911.29	2496	3848.53	4952.7	6474	2496
Profit	($)	5.376	-496	138.136	312.3	1026	-296

Summing the values in the last row of the table, we get the total operating profit of \$690.07

4.3 *Repeat the calculation of Problem 4.2 assuming that the cost curve is replaced by a three-segment piecewise linear approximation whose values correspond with those given by the quadratic function for* 200 MW, 300 MW, 400 MW *and* 500 MW.

The three-segment piecewise linear approximation of the cost function is shown on Figure P4.3. This function has the following elbow values:

Output [MW]	200	300	400	500
Cost [\$/h]	2496	3762	5088	6474

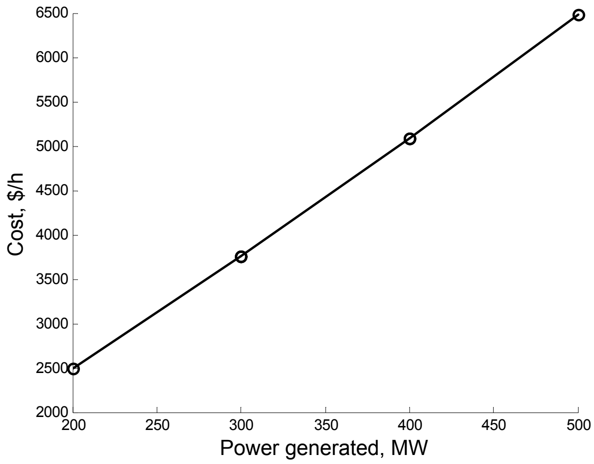

Figure P4.3: Piecewise-linear cost curve for the unit of Problem 4.3

Table P4.3 is calculated in the same fashion as Table P4.2.

Table P4.3: Optimal operation of the unit of Problem 4.3

Period		1	2	3	4	5	6
Price, π (\$/MWh)		12.5	10	13	13.5	15	11
Production *(P)*	(MW)	200	200	300	400	500	200
Revenue ($\pi \times P$)	(MW)	2500	2000	3900	5400	7500	2200
Running cost *C(P)*	(\$)	2496	2496	3762	5088	6474	2496
Profit	(\$)	4	-496	138	312	1026	-296

The total profit for all the periods is \$ 688.00. The error introduced by the linearization of the cost curve is thus very small in terms of the total cost. It is more significant in terms of the dispatch.

4.4 *Assume that the unit of Problem 4.2 has a start-up cost of* \$500 *and that it is initially shutdown. Given the same prices as in problem 4.2, when should this unit be brought on-line and when should it be shutdown to maximize its operational profit? Assume that dynamic constraints do not affect the optimal dispatch of this generating unit.*

From the results obtained in problem 4.2 it can be appreciated that, without considering the start-up cost, the profit for the first period is only around 1% of the start-up cost. In the following period, the unit produces at a loss because of the no-load cost. Therefore, if we consider the start-up cost, the most profitable strategy is to bring the unit on-line for the third period. Since the unit also operates at a loss in period 6, it is desirable to turn the unit off at the end of period 5. Table P4.4 summarizes the operation of the unit under these conditions.

Table P4.4: Optimal operation of the unit taking into account the start-up cost

Period	1	2	3	4	5	6
Price π (\$/MWh)	12.5	10	13	13.5	15	11
Production (*P*) (MW)	-	-	306.66	390	500	-
Revenue ($\pi \times P$) (MW)	-	-	3986.66	5265	7500	-
Running cost *C(P)* (\$)	-	-	3848.53	4952.70	6474	-
Startup cost (\$)	-	-	500	-	-	-
Total cost (\$)	-	-	4348.50	4952.70	6474	-
Profit (\$)	-	-	-361.84	312.30	1026	-

The total profit is thus \$976.43

4.5 *Repeat Problem 4.4 taking into account that the minimum up-time of this unit is four hours.*

If the minimum up time for the unit is 4 hours then it is not possible to shut it down at the end of period 5, therefore the optimal operating procedure is as shown in Table P4.5.

Table P4.5: Optimal operation of the unit taking into account the start-up cost and the minimum up-time

Period	1	2	3	4	5	6
Price, π (\$/MWh)	12.5	10	13	13.5	15	11
Production *(P)* (MW)	-	-	306.66	390	500	200
Revenue *($\pi \times P$)* (MW)	-	-	3986.66	5265	7500	2200
Running cost *C(P)* (\$)	-	-	3848.53	4952.70	6474	
Startup cost (\$)	-	-	500	-	-	-
Total cost (\$)	-	-	4348.50	4952.70	6474	2496
Profit (\$)	-	-	-361.84	312.30	1026	-296

The total profit is \$680.43

4.6 *Borduria Generation owns three generating units that have the following cost functions:*

Unit A: $15 + 1.4\, P_A + 0.04\, P_A^2$ *\$/h*
Unit B: $25 + 1.6\, P_B + 0.05\, P_B^2$ *\$/h*
Unit C: $20 + 1.8\, P_C + 0.02\, P_C^2$ *\$/h*

How should these units be dispatched if Borduria Generation must supply a load of 350 MW *at minimum cost?*

This problem can be formulated as an optimization problem where the objective is to minimize the total operating cost:

$$\min\left\{\sum_{i=1}^{3} C_i\left(P_i\right)\right\}$$

Subject to the load/generation balance constraint:

$$P_A + P_B + P_C = 350 \text{ MW}$$

Therefore we can build the Lagrangian function:

$$\ell\left(P_A, P_B, P_C, \lambda\right) = \sum_{i=1}^{3} C_i\left(P_i\right) + \lambda\left(350 - \sum_{i=1}^{3} P_i\right)$$

and write the optimality conditions:

$$\frac{\partial \ell}{\partial P_A} = 1.4 + 0.08 P_A - \lambda = 0$$
$$\frac{\partial \ell}{\partial P_B} = 1.6 + 0.1 P_B - \lambda = 0$$
$$\frac{\partial \ell}{\partial P_C} = 1.8 + 0.04 P_C - \lambda = 0$$
$$\frac{\partial \ell}{\partial \lambda} = 350 - P_A - P_B - P_C = 0$$

Since these optimality conditions are linear, we can write them in matrix form as follows:

$$\begin{bmatrix} 0.08 & 0 & 0 & -1 \\ 0 & 0.1 & 0 & -1 \\ 0 & 0 & 0.04 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \\ \lambda \end{bmatrix} = \begin{bmatrix} -1.4 \\ -1.6 \\ -1.8 \\ 350 \end{bmatrix}$$

The solution of this system of equations is: P_A = 95.3 MW, P_B = 74.2 MW, P_C = 180.5 MW, λ = 9.02 \$/MWh.

The total production cost is \$1,927.15

4.7 *How would the dispatch of Problem 4.6 change if Borduria Generation had the opportunity to buy some of the energy it must supply on the spot market at a price of* 8.20 \$/MWh?

From the solution of problem 4.6, we see that the marginal cost of production, which is equal to the Lagrange multiplier λ, is equal to 9.02 \$/MWh. Since this is larger than the 8.20 \$/MWh price at which Borduria Generation can purchase energy on the spot market, it would save money by reducing its own production down the point where the marginal cost of production of each generator is equal to the market price. For each unit we get:

$$\frac{\partial C_A}{\partial P_A} = 1.4 + 0.08 P_A = 8.2 \quad \Rightarrow \quad P_A = 85\text{MW}$$
$$\frac{\partial C_B}{\partial P_B} = 1.6 + 0.1 P_B = 8.2 \quad \Rightarrow \quad P_B = 66\text{MW}$$
$$\frac{\partial C_C}{\partial P_C} = 1.8 + 0.04 P_C = 8.2 \quad \Rightarrow \quad P_C = 160\text{MW}$$

Summing these productions, we get:

$$P_A + P_B + P_C = 311 \text{ MW}$$

The remainder of the load (350 – 311= 39 MW) is purchased on the spot market.

This problem can be formulated as a an optimization problem in a different manner if we denote by P_{Spot} the amount of energy purchased on the spot market and by π_{Spot} the price at which this energy is purchased.

The objective function is to minimize the sum of the cost of producing energy with the generating units and of buying energy on the spot market:

$$\min\left\{\sum_{i=1}^{3} C_i\left(P_i\right) + \pi_{Spot} \times P_{Spot}\right\}$$

This minimization is done subject to the following constraint:

$$P_A + P_B + P_C + P_{Spot} = 350 \text{ MW}$$

$$\lambda = 8.20$$

The corresponding Lagrangian function is:

$$\ell\left(P_A, P_B, P_C, P_{Spot}, \lambda\right) = \sum_{i=1}^{3} C_i\left(P_i\right) + \pi_{Spot} \times P_{Spot} + \lambda\left(350 - \sum_{i=1}^{3} P_i - P_{Spot}\right)$$

The optimality conditions are then:

$$\frac{\partial \ell}{\partial P_A} = 1.4 + 0.08 P_A - \lambda = 0$$
$$\frac{\partial \ell}{\partial P_B} = 1.6 + 0.1 P_B - \lambda = 0$$
$$\frac{\partial \ell}{\partial P_C} = 1.8 + 0.04 P_C - \lambda = 0$$
$$\frac{\partial \ell}{\partial P_{Spot}} = \pi_{Spot} - \lambda = 0$$
$$\frac{\partial \ell}{\partial \lambda} = 350 - P_A - P_B - P_C - P_{Spot} = 0$$

The last optimality conditions forces the Lagrange multiplier (and hence the marginal cost of production) to be equal to the spot market price. Rewriting this set of linear equations in matrix form and replacing π_{Spot} by its value, we have:

$$\begin{bmatrix} 0.08 & 0 & 0 & 0 & -1 \\ 0 & 0.1 & 0 & 0 & -1 \\ 0 & 0 & 0.04 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_{Spot} \\ \lambda \end{bmatrix} = \begin{bmatrix} -1.4 \\ -1.6 \\ -1.8 \\ 350 \\ 8.2 \end{bmatrix}$$

Solving this system of equations, we get: P_A = 85 MW, P_B = 66 MW, P_C = 160 MW, P_{Spot} = 39 MW and λ = 8.2 \$/MWh.

4.8 *If, in addition to supplying a* 350 MW *load, Borduria Generation had the opportunity to sell energy on the electricity market at a price of* 10.20 \$/MWh*, what is the optimal amount of power that it should sell? What profit would it derive from this sale?*

This problem is similar to the previous one except that, since the market price is higher than its marginal cost of production, Borduria Generation will want to sell energy. It will increase its production and sell the excess until the marginal cost of production is equal to the market price:

$$MC_A(P_A) = 1.4 + 0.08 P_A = 10.2$$
$$MC_B(P_B) = 1.6 + 0.1 P_B = 10.2$$
$$MC_C(P_C) = 1.8 + 0.04 P_C = 10.2$$

From which we get:

$$P_A = \frac{10.2 - 1.4}{0.08} = 110 \,\text{MW}$$
$$P_B = \frac{10.2 - 1.6}{0.1} = 86 \,\text{MW}$$
$$P_C = \frac{10.2 - 1.8}{0.04} = 210 \,\text{MW}$$

Since the total production is 406 MW and the load is 350 MW, Borduria Generation sells 406 - 350 = 56 MW on the spot market.

As above, we can also write this problem as an optimisation problem:

$$\min \left\{ \sum_{i=1}^{3} C_i\left(P_i\right) - \pi_{Spot} \times P_{Spot} \right\}$$

Subject to:

$$P_A + P_B + P_C - P_{Spot} = 350 \text{ MW}$$

The minus signs in the objective function and the constraint arise because P_{Spot} represents a sale on the spot market instead of a purchase and should therefore appear as a revenue in the objective function and an additional load in the constraint.

The Lagrangian function of this problem is:

$$\ell\left(P_A, P_B, P_C, P_{Spot}, \lambda\right) = \sum_{i=1}^{3} C_i\left(P_i\right) - \pi_{Spot} \times P_{Spot} + \lambda\left(350 - \sum_{i=1}^{3} P_i + P_{Spot}\right)$$

The optimality conditions are as in problem 4.7 except the fourth one, which becomes:

$$\frac{\partial \ell}{\partial \lambda} = 350 - P_A - P_B - P_C + P_{Spot} = 0$$
$$\frac{\partial \ell}{\partial \varphi} = \lambda - 10.2 = 0$$

The linear system of equations that we must solve is then:

$$\begin{bmatrix} 0.08 & 0 & 0 & 0 & -1 \\ 0 & 0.1 & 0 & 0 & -1 \\ 0 & 0 & 0.04 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_{Spot} \\ \lambda \end{bmatrix} = \begin{bmatrix} -1.4 \\ -1.6 \\ -1.8 \\ 350 \\ 10.2 \end{bmatrix}$$

And the solution is: P_A = 110 MW, P_B = 86 MW, P_C = 210 MW, λ = 10.2 \$/MWh, P_{Spot} = 56 MW.

4.9 Repeat Problem 4.8 if the outputs of the generating units are limited as follows:

P_A^{MAX}= 100 MW
P_B^{MAX}= 80 MW
P_C^{MAX}= 250 MW

We will again formulate this problem as an optimization problem. However, while the objective function and the load generation balance constraint are the same, we must consider the inequality constraints on the output of the generators:

$$\min\left\{\sum_{i=1}^{3} C_i\left(P_i\right) - \pi_{Spot} \times P_{Spot}\right\}$$

Subject to:

$$P_A + P_B + P_C - P_{Spot} = 350 \text{ MW}$$

$P_A \leq 100$ MW
$P_B \leq 80$ MW

$P_C \leq 250$ MW

The Lagrangian function is as follows:

$$\begin{aligned}\ell\left(P_A, P_B, P_C, P_{Spot}, \lambda, \mu_1, \mu_2, \mu_3\right) = & \sum_{i=1}^{3} C_i\left(P_i\right) - \pi_{Spot} \times P_{Spot} \\ & + \lambda\left(350 - \sum_{i=1}^{3} P_i + P_{Spot}\right) \\ & + \mu_1(P_A - 100) \\ & + \mu_2(P_B - 80) \\ & + \mu_3(P_C - 250)\end{aligned}$$

The optimality conditions are:

$$\frac{\partial \ell}{\partial P_A} = 1.4 + 0.08 P_A - \lambda + \mu_1 = 0$$
$$\frac{\partial \ell}{\partial P_B} = 1.6 + 0.1 P_B - \lambda + \mu_2 = 0$$
$$\frac{\partial \ell}{\partial P_C} = 1.8 + 0.04 P_C - \lambda + \mu_3 = 0$$
$$\frac{\partial \ell}{\partial P_{Spot}} = \lambda - \pi_{Spot} = 0$$
$$\frac{\partial \ell}{\partial \lambda} = 350 - P_A - P_B - P_C + P_{Spot} = 0$$
$$\frac{\partial \ell}{\partial \mu_1} = P_A - 100 \leq 0$$
$$\frac{\partial \ell}{\partial \mu_2} = P_B - 80 \leq 0$$
$$\frac{\partial \ell}{\partial \mu_3} = P_C - 250 \leq 0$$

Complementary slackness conditions:

$$\mu_1\left(P_A - 100\right) = 0$$
$$\mu_2\left(P_B - 80\right) = 0$$
$$\mu_3\left(P_C - 250\right) = 0$$

As always with optimization problems involving inequality constraints, the difficulty is to determine which of these constraints are actually binding. The only way to determine the set of binding constraints is by trial and error. The process, however, can be made much faster through clever guessing. In this case, for example, we know from Problem 4.8 that the unconstrained solution is:

$P_A = 110$ MW
$P_B = 86$ MW
$P_C = 210$ MW

Since both P_A and P_B are greater than their maximum limit, it is probable that these two constraints will be binding. On the other hand, the constraint on P_C is probably not binding. Based on the complementary slackness conditions, we therefore try a solution with:

$\mu_1 \neq 0$; $\mu_2 \neq 0$; $\mu_3 = 0$

Rewriting the optimality conditions in matrix form, we have:

$$\begin{bmatrix} 0.08 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0.1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0.04 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_{Spot} \\ \lambda \\ \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} -1.4 \\ -1.6 \\ -1.8 \\ 10.2 \\ 350 \\ 100 \\ 80 \end{bmatrix}$$

The solution of this linear system of equations is:

$P_A = 100$ MW
$P_B = 80$ MW
$P_C = 210$ MW
$P_{Spot} = 40$ MW
$\lambda = 10.2$ \$/MWh
$\mu_1 = 0.8$ \$/MWh
$\mu_2 = 0.6$ \$/MWh

This solution is feasible because the Lagrange multipliers μ_1 and μ_2 are positive. This is thus the optimal solution.

Generating unit C operates at a marginal cost equal to the spot market price of 10.2 \$/MWh. The other two generating units operate at a lower marginal cost but at their maximum output. The Lagrange multipliers μ_1 and μ_2 indicate the marginal cost of the constraints on the outputs of generators A and B respectively, i.e. how much money would be saved if these constraints were relaxed by 1 MW.

4.10 *Consider a market for electrical energy that is supplied by two generating companies whose cost functions are:*

$$C_A = 36 \cdot P_A \text{ [\$/h]}$$
$$C_B = 31 \cdot P_B \text{ [\$/h]}$$

The inverse demand curve for this market is estimated to be:

$$\pi = 120 - D \text{ [\$/MWh]}$$

Assuming a Cournot model of competition, use a table similar to the one used in Example 4.8 *to calculate the equilibrium point of this market (price, quantity, production and profit of each firm).*
[Hint: Use a spreadsheet. A resolution of 5 MW *is acceptable]*

In the Cournot model of competition the state of the market is determined by the production decisions made by each firm. We summarize the possible outcomes using a table where all the cells in a column correspond to a given production by generator A while all the cells in a row correspond to a given production by generator B.

Each cell contains four pieces of information arranged in the following format:

D	Ω_A
Ω_B	π

Where:
π price [\$/MWh]
D demand [MWh]
Ω_A profit made by firm A [\$]
Ω_B profit made by firm B [\$]

Given the productions P_A and P_B of the two generators, the other quantities are calculated as follows:

$$D = P_A + P_B$$
$$\pi = 120 - D$$
$$\Omega_A = P_A(\pi - 36)$$
$$\Omega_B = P_B(\pi - 31)$$

File P4-10.xls contains the spreadsheet used to calculate the results shown on Table P4.10. From this table we observe that the cell corresponding to P_A = 25 MW and P_B = 30

MW is an equilibrium point because deviating from those conditions would reduce the profit of the generator that adjusts its output.

Table P4.10: Cournot model of competition for the conditions of Problem P4.10

$P_B \backslash P_A$	5		10		15		20		25		30		35	
5	10	370	15	690	20	960	25	1180	30	1350	35	1470	40	1540
	395	110	370	105	345	100	320	95	295	90	270	85	245	80
10	15	345	20	640	25	885	30	1080	35	1225	40	1320	45	1365
	740	105	690	100	640	95	590	90	540	85	490	80	440	75
15	20	320	25	590	30	810	35	980	40	1100	45	1170	50	1190
	1035	100	960	95	885	90	810	85	735	80	660	75	585	70
20	25	295	30	540	35	735	40	880	45	975	50	1020	55	1015
	1280	95	1180	90	1080	85	980	80	880	75	780	70	680	65
25	30	270	35	490	40	660	45	780	50	850	55	870	60	840
	1475	90	1350	85	1225	80	1100	75	975	70	850	65	725	60
30	35	245	40	440	45	585	50	680	**55**	**725**	60	720	65	665
	1620	85	1470	80	1320	75	1170	70	**1020**	**65**	870	60	720	55
35	40	220	45	390	50	510	55	580	60	600	65	570	70	490
	1715	80	1540	75	1365	70	1190	65	1015	60	840	55	665	50

4.11 *Write and solve the optimality conditions for problem* 4.10.

$$\max\left\{\Omega_A + \Omega_B\right\}$$

Each generating company is trying to maximize its profit, which is equal to the difference between its revenue and its cost:

$$\Omega_A = P_A\,\pi(D) - C_A(P_A)$$
$$\Omega_B = P_B\,\pi(D) - C_B(P_B)$$

However, we cannot treat these two maximizations as separate optimization problems because they are linked through the demand:

$$P_A + P_B = D$$
$$\pi = 120 - D$$

We can get around this difficulty in this case by expressing the price as a function of the production of the two generators:

$$\pi = 120 - P_A - P_B$$

The profits for the two generators can then be expressed as follows:

$$\Omega_A = P_A\left(120 - P_A - P_B\right) - 36P_A = -P_A^2 + \left(84 - P_B\right)P_A$$
$$\Omega_B = P_B\left(120 - P_A - P_B\right) - 31P_B = -P_B^2 + \left(89 - P_A\right)P_B$$

The optimality condition for the maximization problem of generator A is thus:

$$\frac{\partial \Omega_A}{\partial P_A} = -2P_A - P_B + 84 = 0$$

While the optimality condition for the maximization problem of generator B is:

$$\frac{\partial \Omega_B}{\partial P_B} = -P_A - 2P_B + 89 = 0$$

(We only consider the partial derivatives of the profit of a generator with respect to its own output because that is the only variable over which it has control).

Putting these optimality conditions in matrix form, we get:

$$\ell\left(P_A, P_B\right) = -P_A^2 + \left(84 - P_B\right) - P_B^2 + \left(89 - P_A\right)$$

$$\frac{\partial \ell}{\partial P_A} = -2P_A + 84 - P_B$$
$$\frac{\partial \ell}{\partial P_B} = -2P_B + 84 - P_A$$

$$\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} P_A \\ P_B \end{bmatrix} = \begin{bmatrix} -84 \\ -89 \end{bmatrix}$$

Solving this linear system, we get:

P_A = 26.333 MW
P_B = 31.333 MW

From which we can easily find that

π = 62.333 \$/MWh
D = 57.667 MW
Ω_A = \$ 693.44
Ω_B = \$ 981.78

[Note that the answer given in the Appendix of the book is incorrect due to a typographical mistake].

4.12 *Consider the pumped hydro plant of Example 4.10 and the price profile shown in the table below. Assuming that the operator uses the same strategy as in the example (reservoir initially empty, pumping during four hours of lowest prices and turbining during four hours of highest prices), calculate the profit or loss that this plant would make during this cycle of operation. Determine the value of the plant efficiency that would make the profit or loss equal to zero.*

Period	1	2	3	4	5	6
Price [$/MWh]	40.92	39.39	39.18	40.65	45.42	56.34
Period	7	8	9	10	11	12
Price [$/MWh]	58.05	60.15	63.39	59.85	54.54	49.50

The following table can be produced (possibly using the spreadsheet of file P4-12.xls) to summarize the operation of this pumped-hydro station:

Period	Energy prices ($/MWh)	Energy consumed (MWh)	Energy released (MWh)	Cost ($)	Revenue ($)
1	40.92	250	0	10230	0
2	39.39	250	0	9848	0
3	39.18	250	0	9795	0
4	40.65	250	0	10163	0
5	45.42	0	0	0	0
6	56.34	0	0	0	0
7	58.05	0	187.5	0	10885
8	60.15	0	187.5	0	11278
9	63.39	0	187.5	0	11886
10	59.85	0	187.5	0	11222
11	54.54	0	0	0	0
12	49.50	0	0	0	0
Totals		1000	750	40037	45271

This plant thus makes a profit of 45271 – 40037 = $5,235 during this cycle.

The value of the plant efficiency that would reduce the profit to zero is such that the revenue from the sale of the released energy would be equal to the cost of the energy used for storage. Since the released energy is equal to the efficiency times the consumed energy, we have the following expression for the revenue, assuming that energy is released during the four hours with the highest prices:

Revenue = $250\,\eta\,\pi_7 + 250\,\eta\,\pi_8 + 250\,\eta\,\pi_9 + 250\,\eta\,\pi_{10}$

Where π_i represents the energy price at hour *i*.

Since the cost of storing the energy is $40,037, we have the following expression for the breakeven efficiency:

$$\eta = \frac{40037}{250\left(\pi_7 + \pi_8 + \pi_9 + \pi_{10}\right)} = 66.33\%$$

The same result could be obtained by trial and error using the spreadsheet of file P4_12.xls.