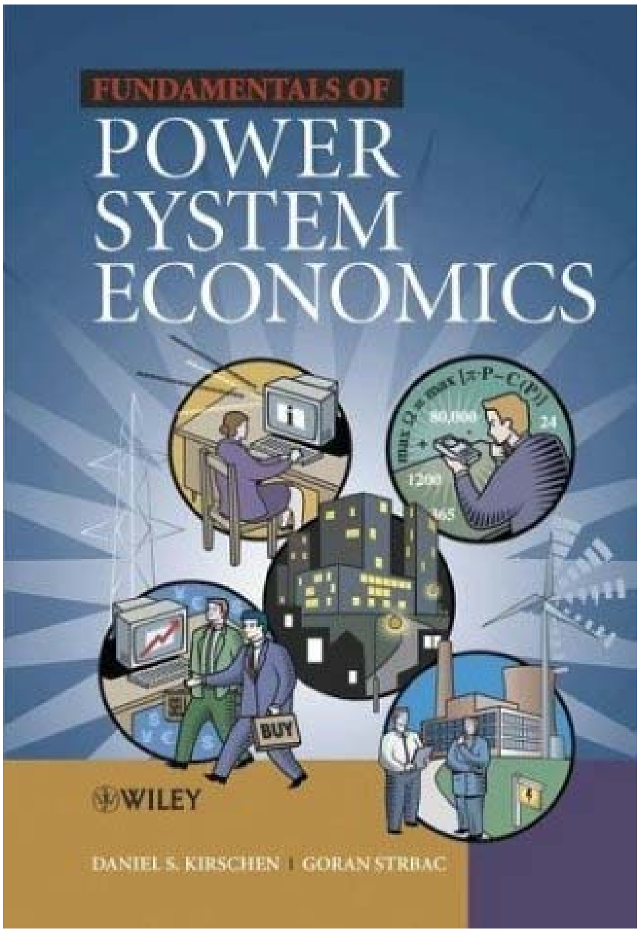
SOLUTIONS MANUAL



CHAPTER 6

TRANSMISSION NETWORKS AND ELECTRICITY MARKETS

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Chapter 6

6.1 Consider the power system shown in Figure P6.1. *Assuming that the only limitations imposed by the network are imposed by the thermal capacity of the transmission lines and that the reactive power flows are negligible, check that the following sets of transactions are simultaneously feasible.*

	Seller	Buyer	Amount
Set 1	B	X	200
	A	Z	400
	C	Y	300
Set 2	B	Z	600
	A	X	300
	A	Y	200
	A	Z	200
Set 3	C	X	1000
	X	Y	400
	B	C	300
	A	C	200
	A	Z	100

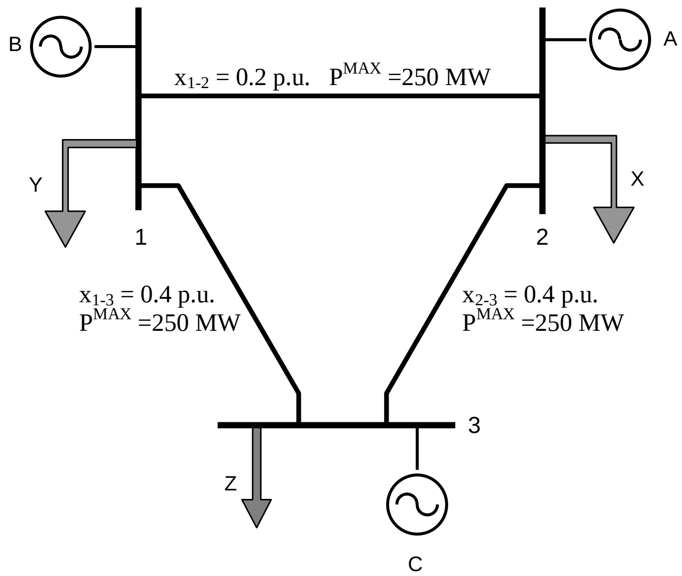

Figure P6.1: Three-bus power system for Problem 6.1

The following power balance equations can be written for this system:

Bus 1: $B - Y = F_{12} + F_{13}$

Bus 2: $A - X = -F_{12} + F_{23}$

Bus 3: $C - Z = -F_{13} - F_{23}$

Using KVL around the loop we can also write the following equation:

$$F_{12}\ x_{12} + F_{23}\ x_{23} - F_{13}\ x_{13} = 0$$

We have three unknowns and four equations. However, as can be seen by adding them all, the three power balance equations are not linearly independent. To get a system of three linearly independent equations that allows us to solve this system, we combine any two power balance equations and the loop equation. For instance, the power balance equations for buses 1 and 2 and the loop equation in matrix form are:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.2 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} B - Y \\ A - X \\ 0 \end{bmatrix}$$

For the first set of transactions, we have:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.2 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} 200 - 300 \\ 400 - 200 \\ 0 \end{bmatrix}$$

Solving this system of equations, we get:

$$F_{12} = -120 \text{ MW} \Rightarrow \left|\mathrm{F}_{12}\right| \leq 250 \text{ MW}$$
$$F_{13} = 20 \text{ MW} \Rightarrow \left|\mathrm{F}_{13}\right| \leq 250 \text{ MW}$$
$$F_{23} = 80 \text{ MW} \Rightarrow \left|\mathrm{F}_{23}\right| \leq 250 \text{ MW}$$

These transactions are thus simultaneously feasible.

For the second set of transactions the linear equations describing the system are:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.2 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} 600 - 200 \\ 700 - 300 \\ 0 \end{bmatrix}$$

Solving this system of equations, we get:

$$F_{12} = 0 \text{ MW} \Rightarrow |F_{12}| \leq 250 \text{ MW}$$
$$F_{13} = 400 \text{ MW} \Rightarrow |F_{13}| \geq 250 \text{ MW}$$
$$F_{23} = 400 \text{ MW} \Rightarrow |F_{23}| \geq 250 \text{ MW}$$

These transactions are thus not simultaneously feasible.

Finally for the third set of transactions, we have:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.2 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} 300-400 \\ 300-600 \\ 0 \end{bmatrix}$$

Solving this system of equations, we get:

$$F_{12} = 80 \text{ MW} \Rightarrow |F_{12}| \leq 250 \text{ MW}$$
$$F_{13} = -180 \text{ MW} \Rightarrow |F_{13}| \leq 250 \text{ MW}$$
$$F_{23} = -220 \text{ MW} \Rightarrow |F_{23}| \leq 250 \text{ MW}$$

These transactions are thus simultaneously feasible.

6.2 *Consider the two-bus power system shown in Figure* P6.2. *The marginal cost of production of the generators connected to buses A and B are given respectively by the following expressions*:

$$MC_A = 20 + 0.03 P_A \text{ [\$/MWh]}$$
$$MC_B = 15 + 0.02 P_B \text{ [\$/MWh]}$$

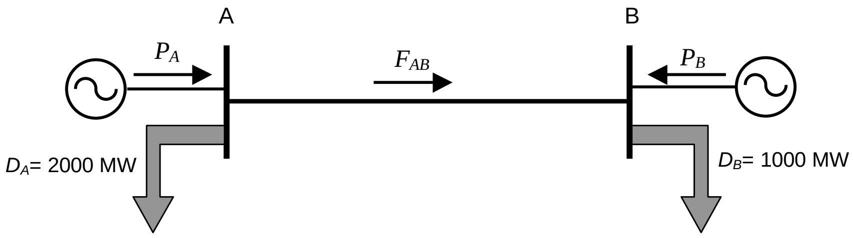

Figure P6.2: Two-bus power system for Problems 6.2, 6.3, 6.4, 6.10 and 6.11

Assume that the demand is constant and insensitive to price, that energy is sold at its marginal cost of production and that there are no limits on the output of the generators. Calculate the price of electricity at each bus, the production of each generator and the flow on the line for the following cases:

a. *The line between buses A and B is disconnected*
b. *The line between buses A and B is in service and has an unlimited capacity*
c. *The line between buses A and B is in service and has an unlimited capacity, but the maximum output of generator B is* 1500 MW
d. *The line between buses A and B is in service and has an unlimited capacity, but the maximum output of generator A is* 900 MW. *The output of generator B is unlimited*.
e. *The line between buses A and B is in service but its capacity is limited to* 600 MW. *The output of the generators is unlimited*.

a. This case can be treated as two independent systems, each with its own load and generation. For system A the amount of power generated is P_A = 2000 MW and the price of electricity is given by: $\pi_A = 20 + 0.03(2000)$ = 80 \$/MWh. For system B the amount of power generated is P_B = 1000 MW and the price is given by: $\pi_B = 15 + 0.02(1000)$ = 35 \$/MWh.

b. In this case the marginal cost of all the generating units is equal to the price of the electricity; and the total production of the generating units is equal to the total load of the system. We can thus write:

$$\pi = 20 + 0.03P_A$$
$$\pi = 15 + 0.02P_B$$
$$P_A + P_B = 3000$$

Writing this set of equations can be written in matrix form:

$$\begin{bmatrix} 0.03 & 0 & -1 \\ 0 & 0.02 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ \pi \end{bmatrix} = \begin{bmatrix} -20 \\ -15 \\ 3000 \end{bmatrix}$$

We get:

P_A = 1100 MW
P_B = 1900 MW
π = 53 \$/MWh

Furthermore, the flow from A to B is: $F_{AB} = P_A - D_A$ = 1100 – 2000 = –900 MW.

c. From the previous case we can see that, if no restriction is imposed on the transfer of power between A and B, the generation in system B is 1900 MW. If this generation is limited to a maximum of 1500 MW, then the generation in A is given by: $P_A = 3000 - P_B$ = 1500 MW. The power flow from A to B is then F_{AB} = 1500 – 2000 = -500 MW. Using the expressions for the marginal production costs, we find that the price at A is π_A = 65

\$/MWh. However, since generator B is producing at its maximum output (1500 MW for this case) it becomes an infra-marginal generator; therefore if the demand at B requires an extra MW it will be provided by A, and therefore the price at B is π_B = 65 \$/MWh as well.

d. The power generated in area B is given by: $P_B = 3000 – P_A = 2100$ MW while the flow from A to B is given by: F_{AB} = 900 – 2000 = –1100 MW. The price at B is $\pi_B = 15 + 0.02(2100) = 57$ \$/MWh. And since the generator at A is producing at its maximum output the next MW supplied at A would have a cost of 57 \$/MWh as well.

e. From case b we know that when no restriction is imposed on any of the components of the system, the flow on the line from A to B is F_{AB} = –900. In this case, it is thus restricted to F_{AB} = –600 MW. Therefore the generation at A is $P_A = F_{AB} + D_A = –600 + 2000 = 1400$ MW. We also have $P_B = 3000 – P_A = 1600$ MW. Using the expressions for the marginal production costs, we get the prices π_A = 62 \$/MWh and π_B = 47 \$/MWh

6.3 *Calculate the generator revenues and the consumer payments for all the cases considered in Problem* 6.2. *Who benefits from the line connecting these two buses*?

	a	b	c	d	e
Payments by consumers at A $E_A = D_A\ \pi_A$ (\$)	160,000	106,000	130,000	114,000	124,000
Payments by consumers at B $E_B = D_B\ \pi_B$ (\$)	35,000	53,000	65,000	57,000	47,000
Revenue of generator A $R_A = P_A\ \pi_A$ (\$)	160,000	58,300	97,500	51,300	86,800
Revenue of generator B $R_B = P_B\ \pi_B$ (\$)	35,000	100,700	97,500	62,700	75,200

The main beneficiaries of the tie line are the generation at B and the load at A because flows on the tie line increase the price at B and lowers it at A.

6.4 *Calculate the congestion surplus for case (e) of Problem 6.2. Check your answer using the results of Problem 6.3. For what values of the flow on the line between buses A and B is the congestion surplus equal to zero*?

The congestion surplus is the difference between the payments and revenues:

$$cs = \left(D_A\ \pi_A + D_B\ \pi_B\right) - \left(P_A\ \pi_A + P_B\ \pi_B\right)$$

Factorizing this expression, we get:

$$cs = \left(D_A - P_A\right)\pi_A + \left(D_B - P_B\right)\pi_B$$

Since $D_A - P_A = -F_{AB}$ and $D_B - P_B = F_{AB}$, we have:

$$cs = F_{AB}\left(\pi_B - \pi_A\right) = -600\left(47 - 62\right) = \$9000$$

From the solution of Problem 6.3, we have:

$$cs = E_A + E_B - R_A - R_B = 124,000 + 47,000 - 86,800 - 75,200 = \$9,000$$

This congestion surplus is equal to zero when the prices at A and B are equal and when the flow from A to B is zero.

6.5 *Consider the three-bus power system shown in Figure* P6.5 *The table below shows the data about the generators connected to this system. Calculate the unconstrained economic dispatch and the nodal prices for the loading conditions shown in Figure* P6.5.

Generator	Capacity [MW]	Marginal Cost [$/MWh]
A	150	12
B	200	15
C	150	10
D	400	8

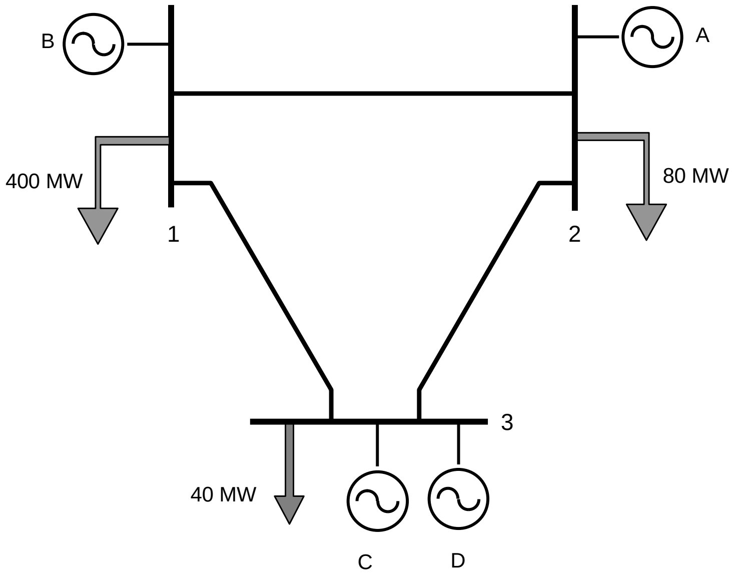

Figure P6.5-a: Three-bus power system for Problems 6.5 to 6.9 and 6.12 to 6.17

Since there are no transmission constraints, the outputs of all the generators can be stacked in order of marginal cost as shown on Figure 6.5:

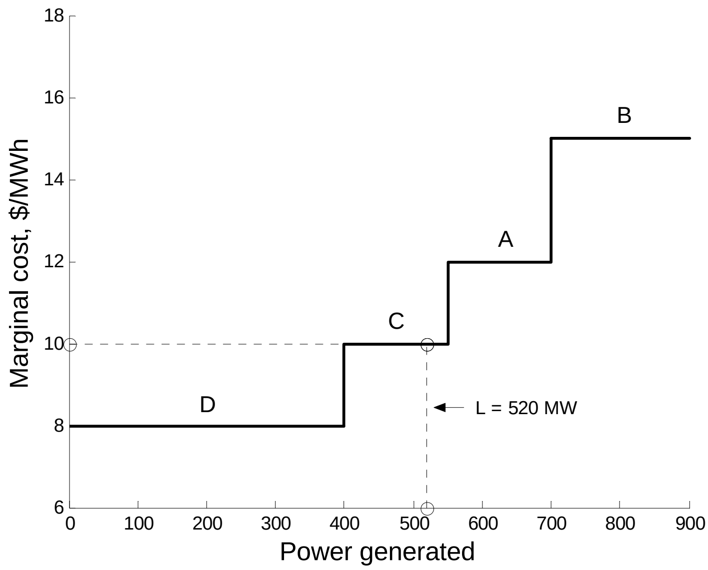

Figure P6.5-b: Stack of generator outputs in order of marginal cost

Using Figure 6.5.b, we see that for a system load of 400 + 40 + 80 = 520 MW the marginal cost (and hence the price) is 10 $/MWh. Furthermore, the units are dispatched as follows: P_D = 400 MW, P_C = 120, and P_A = P_B = 0 MW.

6.6 *The table below gives the branch data for the three-bus power system of Problem 6.5. Using the superposition principle, calculate the flow that would result if the generating units were dispatched as calculated in Problem 6.5. Identify all the violations of security constraints.*

Branch	Reactance [p.u.]	Capacity [MW]
1-2	0.2	250
1-3	0.3	250
2-3	0.3	250

Figure P6.6 shows the injections for the economic dispatch conditions and how this system can be decomposed to make use of the superposition principle to calculate the line flows.

Using Equations. (6.4) and (6.5) and considering the reactances of the various branches, we get:

$$F_1^A = \frac{0.5}{0.8} \times 400 = 250 \text{ MW}$$
$$F_1^B = \frac{0.3}{0.8} \times 400 = 150 \text{ MW}$$
$$F_2^A = \frac{0.5}{0.8} \times 80 = 50 \text{ MW}$$
$$F_2^B = \frac{0.3}{0.8} \times 80 = 30 \text{ MW}$$

Combining these flows as suggested by Figure P6.6, we get:

$$F_{12} = -F_1^B + F_2^B = -120 \text{ MW}$$
$$F_{13} = -F_1^A - F_2^B = -280 \text{ MW}$$
$$F_{23} = -F_1^B - F_2^A = -200 \text{ MW}$$

The flow on line 1-3 thus exceeds its maximum capacity by 30 MW.

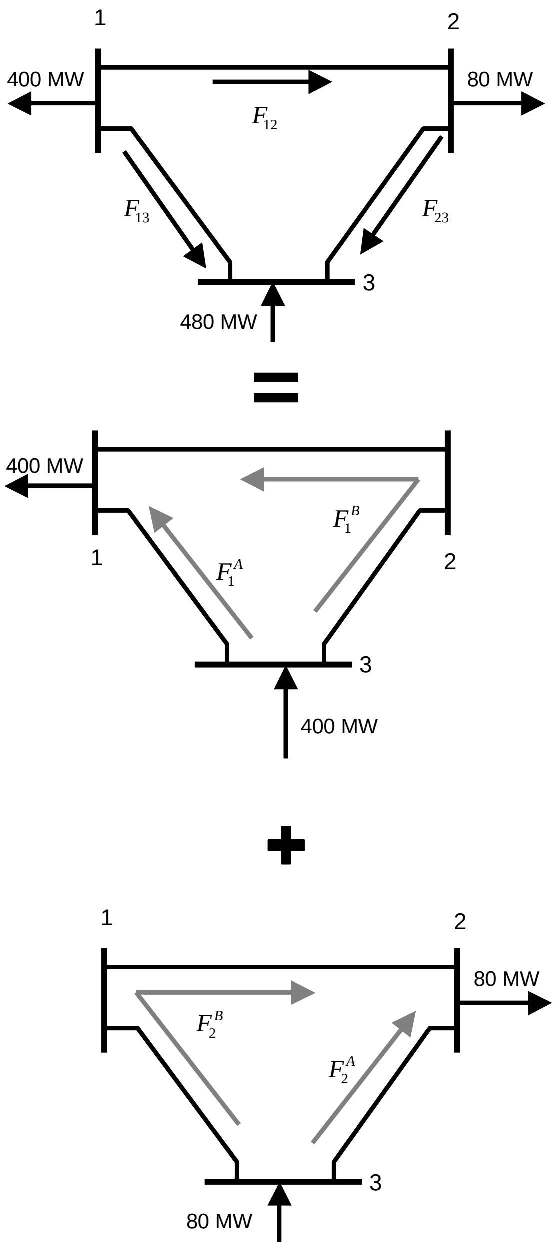

Figure P6.6: Application of the superposition principle to the solution of Problem 6.6

Note that superposition is not the only way to solve this problem. We can also solve it directly. To this effect, we write the power balance equation at two buses and KVL around the loop:

Bus 1: $P_B - 400 = F_{12} + F_{13}$

Bus 2: $P_A - 80 = -F_{12} + F_{23}$

Bus 3: $P_C + P_D - 40 = -F_{13} - F_{23}$

Loop equation: $0.2\,F_{12} + 0.3\,F_{23} - 0.3\,F_{13} = 0$

Putting these equations in matrix form gives:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} P_A - 80 \\ P_C + P_D - 40 \\ 0 \end{bmatrix}$$

Substituting $P_A = 0\,\text{MW};\ P_B = 0\,\text{MW};\ P_C = 120\ \text{MW}; P_D = 400\ \text{MW}$ in these equations, we get:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} -80 \\ 480 \\ 0 \end{bmatrix}$$

Solving these equations, we get $F_{12} = -120\ \text{MW};\quad F_{13} = -280\ \text{MW};\quad F_{23} = -200\ \text{MW}$.

6.7 Determine two ways of removing the constraint violations that you identified in Problem 6.6 by redispatching generating units. Which redispatch is preferable?

The first method consists in increasing the output of generator B and decreasing by the same amount the output of generator C. (Decreasing the output of generator D is not desirable as it is cheaper than generator C). To calculate how big this increase should be to remove the violation of the flow limit on line 3-1, consider an injection of +1 MW at bus 1 and an injection of –1 MW at bus 3. This pair of injection causes a flow in the network that divides itself as follows:

$$\frac{0.3}{(0.2+0.3)+0.3} \times 1 = 0.375\ \text{MW} \ \text{ along the path 1-2-3}$$

$$\frac{(0.2+0.3)}{(0.2+0.3)+0.3} \times 1 = 0.625\ \text{MW} \ \text{ along the path 1-3}$$

Since we use a linear (dc) model, we can say that to remove the 30 MW overload on line 3-1, we therefore need to increase the output of generator B by:

$$\frac{30}{0.625} = 48\ \text{MW}$$

The constrained dispatch is then:

$P_A = 0$
$P_B = 48$ MW
$P_C = 72$ MW
$P_D = 400$ MW

To calculate the flows, we can either use the equations that we developed for Problem 6.6, or compute the changes in flows caused by an injection of +48 MW at bus 1 and an injection of –48 MW at bus 3:

$F_{12} = -120 + 0.375 \times 48 = -102$ MW
$F_{23} = -200 + 0.375 \times 48 = -182$ MW
$F_{13} = -280 + 0.625 \times 48 = -250$ MW

Or solving the linear system generated using the nodal equations and the loop equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} P_B - 400 \\ P_C + P_D - 40 \\ 0 \end{bmatrix}$$

This redispatch does not cause a violation of the line flow constraints on any other line.

The cost of this dispatch is:

$$C_{Total} = 48 \times 15 + 72 \times 10 + 400 \times 8 = \$4,640$$

which represents an increase of \$240 compared to the case where network constraints are not considered.

The other method to remove the constraint violation consists in increasing the output of generator A and decreasing the output of generator C by the same amount. To calculate how big this increase should be to remove the violation of the flow limit on line 3-1, consider an injection of +1 MW at bus 2 and an injection of –1 MW at bus 3. This pair of injection causes a flow in the network that divides itself as follows:

$$\frac{0.3}{(0.2+0.3)+0.3} \times 1 = 0.375 \text{ MW} \text{ along the path 2-1-3}$$

$$\frac{(0.2+0.3)}{(0.2+0.3)+0.3} \times 1 = 0.625 \text{ MW} \text{ along the path 2-3}$$

Since we use a linear (dc) model, we can say that to remove the 30 MW overload on line 3-1, we therefore need to increase the output of generator B by:

$$\frac{30}{0.375} = 80 \text{ MW}$$

The constrained dispatch is then:

$$P_A = 80 \text{ MW}$$
$$P_B = 0 \text{ MW}$$
$$P_C = 40 \text{ MW}$$
$$P_D = 400 \text{ MW}$$

To calculate the flows, we compute the changes caused by an injection of +80 MW at bus 2 and an injection of –80 MW at bus 3:

$$F_{12} = -120 - 0.375 \times 80 = -150 \text{ MW}$$
$$F_{23} = -200 + 0.625 \times 80 = -150 \text{ MW}$$
$$F_{13} = -280 + 0.375 \times 80 = -250 \text{ MW}$$

Or in matrix form:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{13} \\ F_{23} \end{bmatrix} = \begin{bmatrix} P_A - 80 \\ P_C + P_D - 40 \\ 0 \end{bmatrix}$$

Once again, this redispatch does not cause a violation of the line flow constraints on any other line. The cost of this constrained dispatch is:

$$C_{Total} = 80 \times 12 + 40 \times 10 + 400 \times 8 = \$4,560$$

which represents an increase of \$160 compared to the case where network constraints are not considered. Even though it re-dispatches a larger amount of MW, the second constrained is preferable to the first because its cost is smaller. It is thus the optimal constrained dispatch.

6.8 Calculate the nodal prices for the three-bus power system of problems 6.5 and 6.6 when the generating units have been optimally re-dispatched to relieve the constraint violations identified in Problem 6.7. Calculate the merchandising surplus and show that it is equal to the sum of the surpluses of each line.

The nodal price at each bus is given by the cost of one additional MW of load at each node. Therefore, the price at bus 3 is 10 \$/MWh because the next MW of load would be generated locally by generator C because it is the cheapest generator not operating at its upper limit. An additional MW of load at node 2 would have to be produced by generator A. Producing it with generator C would cause a violation of the line flow constraint on line 3-1. Producing it with generator B would be more expensive than with generator A. The price at node 2 is therefore 12 $/MWh. An additional MW of load at bus 1 requires a redispatch of A and C to minimize the cost increase while maintaining the flow on line 3-1 within limits.

Extracting an additional 1 MW at bus 1 and generating it at bus 3 causes the following change in the flow on line 1-3:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Delta F_{31} = 0.625 \text{ MW}$$

Similarly, extracting an additional 1 MW at bus 1 and generating it at bus 2 causes the following change in the flow on line 1-3:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} \Delta F_{12} \\ \Delta F_{13} \\ \Delta F_{23} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Delta F_{31} = 0.250 \text{ MW}$$

Therefore, if we want to the flow on line 3-1 unchanged (because it is already at its limit), we must change the productions by generators C and A in such a way that:

$$\Delta F_{31} = 0 = 0.625 \times \Delta P_C + 0.250 \times \Delta P_A$$

At the same time, since we are increasing the load by 1 MW, we must also have:

$$\Delta P_A + \Delta P_C = 1$$

Solving the system consisting of the previous two equations we get:

$$\Delta P_C = -0.667 \text{ MW}$$

$$\Delta P_A = 1.667 \text{ MW}$$

To supply an additional MW of load at bus 1 without violating network constraints, we must therefore increase the output of generator A and decrease the output of generator C.

The nodal price at bus 1 is thus given by a linear combination of the marginal cost of production of these two generators:

$$\pi_1 = -0.667 \times 10 + 1.667 \times 12 = 13.33 \text{ \$/MWh}$$

The table below summarizes the calculation by bus and by line of the merchandising surplus caused by the congestion in this network.

Surplus by bus				total
Bus	1	2	3	
Production	0	80	440	520
Consumption	400	80	40	520
Price	13.33	12	10	
Consumers' payments	5332	960	400	6692
Producers' revenue	0	960	4400	5360
Congestion surplus				1332

Surplus by line:				
Line	2 to 1	3 to 1	3 to 2	
Flow	150	250	150	
Price at from bus	12	10	10	
Price at to bus	13.33	13.33	12	
Congestion surplus	-199.5	-832.5	-300	1332

6.9 *Consider the three-bus power system described in Problems 6.5 and 6.6. Suppose that the capacity of branch 1-2 is reduced to 140 MW while the capacity of the other lines remains unchanged. Calculate the optimal dispatch and the nodal prices for these conditions.*
[Hint: the optimal solution involves a redispatch of generating units at all three buses]

While we could solve this problem using superposition, this approach is getting a bit difficult for this problem. Instead, let us write the power balance equation for each node:

Bus 1: $P_B - 400 = F_{12} + F_{13}$

Bus 2: $P_A - 80 = -F_{12} + F_{23}$

Bus 3: $P_C + P_D - 40 = -F_{13} - F_{23}$

Again, we must remember that these equations are not linearly independent because of the principle of conservation of energy. We should therefore replace one of these equations by the obtained by applying KVL around the loop:

$$0.2\, F_{12} + 0.3\, F_{23} - 0.3\, F_{13} = 0$$

Before the rating of line 1-2 was changed, the flows were as follows for the optimal constrained dispatch:

$F_{12} = -150$ MW
$F_{13} = -250$ MW
$F_{23} = -150$ MW

If the capacity of line 1-2 is restricted to 140 MW, this constraint is thus likely to be binding as well as the constraint on line 1-3. We will thus have:

$F_{12} = -140$ MW
$F_{13} = -250$ MW

The value of F_{23} can then be calculated using the loop equation:

$$F_{23} = \frac{0.3\,F_{13} - 0.2\,F_{12}}{0.3} = -156.67 \text{ MW}$$

Using the nodal power balance equations, we then get:

$$P_A = -F_{12} + F_{23} + 80 = 63.33 \text{ MW}$$

$$P_C + P_D = -F_{13} - F_{23} + 40 = 446.67 \text{ MW}$$

$$P_B = F_{12} + F_{13} + 400 = 10 \text{ MW}$$

Since generator D has a lower marginal cost than generator C, it should be loaded up to its maximum capacity before loading generator C:

$P_D = 400$ MW
$P_C = 46.67$ MW

(Note that the value given in the appendix of the book for the output of generator C is incorrect.)

Since we have two binding constraints, we have 2 + 1 = 3 partly-loaded generators, i.e. one at each bus of this system. The nodal price at each bus is thus given by the marginal cost of these generators:

$\pi_1 = MC_B = 15$ $/MWh
$\pi_2 = MC_A = 12$ $/MWh
$\pi_3 = MC_C = 10$ $/MWh

6.10 *Consider the two-bus power system of Problem 6.2. Given that* $K = R/V^2 = 0.0001\,\text{MW}^{-1}$ *for the line connecting buses A and B and that there is no limit on the capacity of this line, calculate the value of the flow that minimizes the total variable cost of production. Assuming that a competitive electricity market operates at both buses, calculate the nodal marginal prices and the merchandising surplus.*
[Hint: use a spreadsheet].

The EXCEL® spreadsheet "P6_10.xls" shows how this problem can be solved using a trial and error approach, i.e. calculating the total variable cost of production for various dispatches.

This problem can be solved analytically as follows. The marginal costs of production are:

$$MC_A = 20 + 0.03 P_A \text{ [\$/MWh]}$$
$$MC_B = 15 + 0.02 P_B \text{ [\$/MWh]}$$

The variable costs of production for each system are the integral of the marginal costs:

$$C_X(P_X) = \int_0^{P_X} MC_X(P)\, dP$$

We get:

$$C_A(P_A) = 20 P_A + 0.015 P_A^2$$

$$C_B(P_B) = 15 P_B + 0.01 P_B^2$$

The power balance equation is:

$$P_A + P_B = D_A + D_B + K\, F_{AB}^2$$

Where $F_{AB} = P_A - D_A$. Substituting this expression in the power balance equation, we get:

$$P_A + P_B = D_A + D_B + K\,(P_A - D_A)^2$$

$$P_A + P_B = D_A + D_B + K\,P_A^2 - 2\,K\,P_A\,D_A + K\,D_A^2$$

Since we are trying to minimize the overall production cost, we construct the following Lagrangian function:

$$\ell\left(P_A, P_B, \lambda\right) =$$
$$20P_A + 0.015P_A^2 + 15P_B + 0.01P_B^2 + \lambda\left(D_A + D_B + K\,P_A^2 - 2\,K\,P_A\,D_A + K\,D_A^2 - P_A - P_B\right)$$

The optimality conditions are:

$$f_1 = \frac{\partial \ell}{\partial P_A} = 20 + 0.03P_A + \lambda\left(2\,K\,P_A - 2\,K\,D_A - 1\right) = 0$$
$$f_2 = \frac{\partial \ell}{\partial P_B} = 15 + 0.02P_B - \lambda = 0$$
$$f_3 = \frac{\partial \ell}{\partial \lambda} = D_A + D_B + K\,P_A^2 - 2\,K\,P_A\,D_A + K\,D_A^2 - P_A - P_B = 0$$

These optimality conditions form a non-linear system of equations, and therefore finding a closed-form solution is not straightforward. Instead, we will use an interactive solution based on the Newton algorithm. The Jacobian matrix of this system is:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial P_A} & \frac{\partial f_1}{\partial P_B} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial P_A} & \frac{\partial f_2}{\partial P_B} & \frac{\partial f_2}{\partial \lambda} \\ \frac{\partial f_3}{\partial P_A} & \frac{\partial f_3}{\partial P_B} & \frac{\partial f_3}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0.03 + 2\,\lambda\,K & 0 & 2\,K\,P_A - 2\,K\,D_A - 1 \\ 0 & 0.02 & -1 \\ 2\,K\,P_A - 2\,K\,D_A - 1 & -1 & 0 \end{bmatrix}$$

The vector of mismatches (residuals) is:

$$\mathbf{g} = \begin{bmatrix} 20 + 0.03P_A + \lambda\left(2\,K\,P_A - 2\,K\,D_A - 1\right) \\ 15 + 0.02P_B - \lambda \\ D_A + D_B + K\,P_A^2 - 2\,K\,P_A\,D_A + K\,D_A^2 - P_A - P_B \end{bmatrix}$$

At each iteration, the corrections are given by:

$$\Delta P = -\mathbf{J}^{-1} \times \mathbf{g}$$

If we set the stopping condition of the iterative process to be that the absolute value of the maximum increment should be smaller than 1×10^{-9}, the solution is obtained in 5 iterations and we get: P_A = 1269.4 MW, P_B = 1783.9 MW, λ = 50.679 \$/MWh, F_{AB} = -730.56 MW, π_A = 58.083 \$/MWh, π_B = 50.679, losses = 53.371 MW.

The file P6_10.m contains a MATLAB® implementation of this optimization procedure, for more information, type "`help p6_10`" in the MATLAB® command window within the directory that contains the file P6_10.m.

6.11 *Repeat problem 6.10 for several values of K ranging from 0 to 0.0005. Plot the optimal flow and the losses in the line, as well as the marginal cost of electrical energy at both buses. Discuss your results.*

Using the spreadsheet or the optimization technique described above, problem 6.10 can be repeated for this range of value. File P6_11.m contains a MATLAB® implementation of this repeated optimization. The results obtained are shown graphically below:

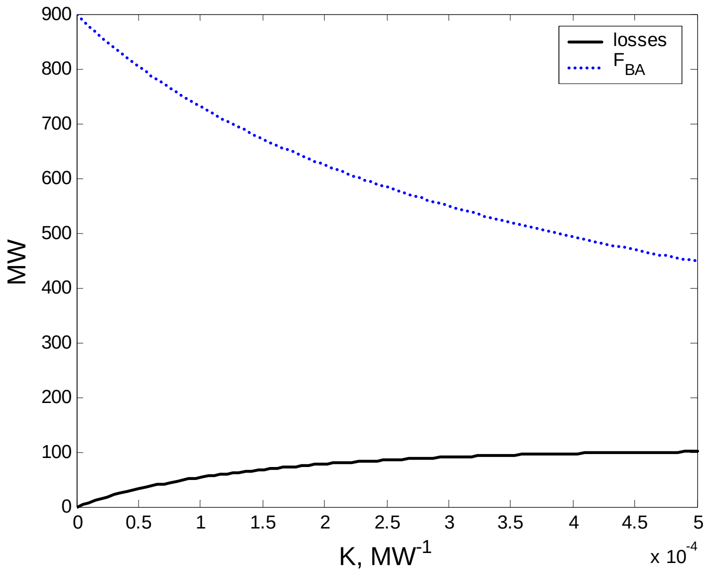

Figure 6.11-a Losses and power flow from B to A as a function of K

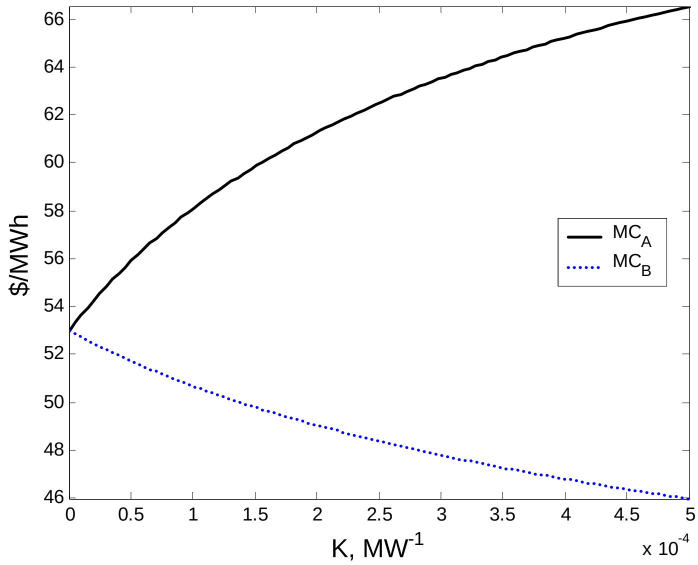

Figure 6.11-b Marginal cost of A and B as a function of *K*

(Note that the label of the secondary vertical axis of Figure P6.11 in the appendix of the book is incorrect. The units of the marginal cost are $/MWh.)

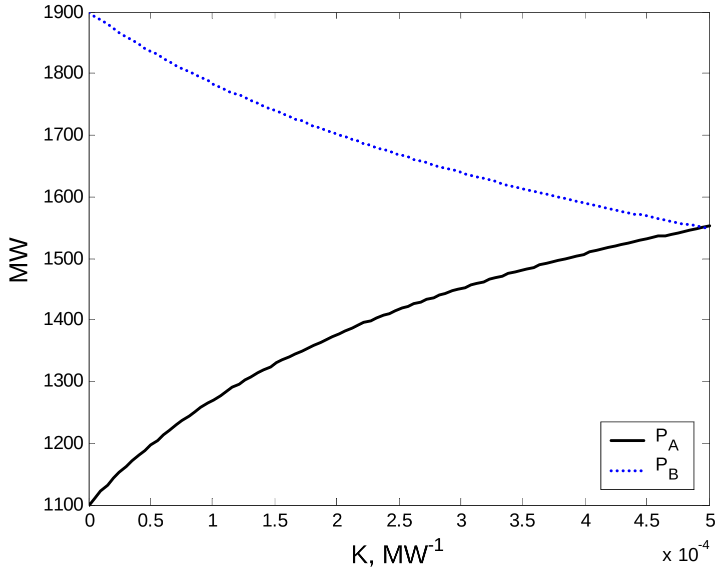

Figure 6.11-c Units generations as a function of *K*

These graphs show that when the resistance of the line is small, generator B should produce more than generator A because its marginal cost of production is smaller. As the resistance increases, so do the losses and the relative advantage of generating unit B decreases.

6.12 Using the linearized mathematical formulation (dc power flow approximation), calculate the nodal prices and the marginal cost of the inequality constraint for the optimal redispatch that you obtained in Problem 6.7. Check that your results are identical to those that you obtained in Problem 6.8. Use bus 3 as the slack bus.

The first step to solve this problem is to build the admittance matrix of the system. Since there is no resistance, the imaginary part of the **Y** matrix is:

$$\mathbf{Y} = \begin{bmatrix} y_{12}+y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12}+y_{23} & -y_{23} \\ -y_{13} & -y_{23} & y_{13}+y_{23} \end{bmatrix}$$

Where the y_{kl} terms are the inverse of the reactance of the branch between k and l. With bus 3 as the slack, instantiating equations 6.137 and 6.138 gives the following equations:

$$\begin{bmatrix} y_{12}+y_{13} & -y_{12} \\ -y_{12} & y_{12}+y_{23} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} y_{13} \\ y_{23} \end{bmatrix} \pi_3 + \begin{bmatrix} y_{13} \\ 0 \end{bmatrix} \mu_{31}$$

There are two unknowns in these equations: π_1 and μ_{13}. Rewriting the equations to put the unknown variables on the left hand side, we get:

$$\left(y_{12}+y_{13}\right)\pi_1 - y_{13}\,\mu_{31} = y_{12}\,\pi_2 + y_{13}\,\pi_3$$
$$-y_{21}\,\pi_1 = -\left(y_{12}+y_{23}\right)\pi_2 + y_{23}\,\pi_3$$

Or in matrix form:

$$\begin{bmatrix} y_{12}+y_{13} & -y_{13} \\ -y_{21} & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \mu_{31} \end{bmatrix} = \begin{bmatrix} y_{12} & y_{13} \\ -y_{12}-y_{23} & y_{23} \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_3 \end{bmatrix}$$

Solving these equations gives: π_1 = 13.33 \$/MWh and μ_{31} = 5.33 \$/MWh

6.13 Show that the choice of slack bus does not influence the nodal prices for the dc power flow approximation by repeating Problem 6.12 using bus 1 and then bus 2 as the slack bus.

Selecting bus 1 as the slack bus, the equations are:

$$-y_{21}\pi_1 = -\left(y_{12}+y_{23}\right)\pi_2 + y_{23}\,\pi_3$$
$$-y_{31}\,\pi_1 + y_{31}\,\mu_{31} = y_{32}\,\pi_2 - \left(y_{13}+y_{23}\right)\pi_3$$

Rewriting in matrix form:

$$\begin{bmatrix} -y_{21} & 0 \\ -y_{31} & y_{31} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \mu_{31} \end{bmatrix} = \begin{bmatrix} -y_{12} - y_{23} & y_{23} \\ y_{32} & -y_{13} - y_{23} \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_3 \end{bmatrix}$$

The results are: π_1 = 13.3333 \$/MWh and μ_{31} = 5.3333 \$/MWh.

Selecting bus 2 as the slack bus the equations are:

$$\begin{aligned} &(y_{12} + y_{13})\pi_1 - y_{13}\mu_{31} = y_{12}\pi_2 + y_{13}\pi_3 \\ &-y_{31}\pi_1 + y_{31}\mu_{31} = y_{32}\pi_2 - (y_{13} + y_{23})\pi_3 \end{aligned}$$

In matrix form:

$$\begin{bmatrix} y_{12} + y_{13} & -y_{13} \\ -y_{31} & y_{31} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \mu_{31} \end{bmatrix} = \begin{bmatrix} y_{12} & y_{13} \\ y_{32} & -y_{13} - y_{23} \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_3 \end{bmatrix}$$

The results are again: π_1 = 13.3333 \$/MWh and μ_{31} = 5.3333 \$/MWh

6.14 Using the linearized mathematical formulation (dc power flow approximation), calculate the marginal costs of the inequality constraints for the conditions of Problem 6.9.

In problem 6.9 the solution consisted of dispatching the units as follows: P_A = 63.33 MW, P_B = 10 MW, P_C = 46.66 MW and P_D = 400 MW. Therefore the prices at each bus are: π_1 = 15 \$/MWh, π_2 = 12 \$/MWh and π_3 = 10 \$/MWh.

Using the dc approximation (Equations 6.137 and 6.138) we can write the following equations:

$$\begin{aligned} y_{21}\mu_{21} &= y_{21}\pi_1 - (y_{12} + y_{23})\pi_2 + y_{23}\pi_3 \\ y_{31}\mu_{31} &= y_{31}\pi_1 + y_{32}\pi_2 - (y_{13} + y_{23})\pi_3 \end{aligned}$$

Since in this case we know the nodal prices, the solution is obtained directly by replacing the nodal prices by the values given above and dividing both sides of the first equation by y_{21} and both sides of the second dividing by y_{31}.

Therefore the marginal costs of the constraints are: μ_{31} = 7.00 \$/MWh and μ_{21} = 1.67 \$/MWh.

6.15 *Consider the three-bus system shown in Figure P6.5. Suppose that generator D and a consumer located at bus 1 have entered into a contract for difference for the delivery of* 100 MW *at a strike price of* 11.00 \$/MWh *with reference to the nodal price at bus 1. Show that purchasing* 100 MW *of point-to-point financial rights between buses* 3 *and* 1 *provides a perfect hedge to generator D for the conditions of Problem 6.8.*

In problem 6.8 the nodal prices were: π_1 = 13.33 \$/MWh, π_2 = 12 \$/MWh and π_3 = 10 \$/MWh. The dispatch corresponding to these prices was as follows: P_A = 80 MW, P_B = 0 MW, P_C = 40 MW and P_D = 400 MW. This dispatch produces the following flows: F_{12} = -150 MW, F_{13} = -250 MW and F_{23} = -150 MW

Therefore the contract would be settled as follows:

The consumer pays 100×13.33 = \$1333.33, for extracting the 100 MWh at bus 1
The generator receives 100×10 = \$1000, for injecting 100 MWh at bus 3
The consumer pays 100×(11-10) = \$100 to the generator to settle the contact for difference
The consumer who owns the point-to-point financial rights of 100 MWh between 3 and 1 collects 100×(13.33-10) = \$ 333.33.

The consumer thus pays a total of 1333.33+100-333.33 = \$1100 for 100 MWh, which is equivalent to a price of 11 \$/MWh.

6.16 *What flowgate rights should generator D purchase to achieve the same perfect hedge as in problem 6.15?*

Because we are using a dc network approximation, the system is linear and transactions can be treated independently. We can therefore analyze the flows that this transaction causes independently of all other transactions. As we have done before, we must use two nodal equations and one loop equation to solve this network. If generator D injects 100 MWh at bus 3 and the consumer extracts 100 MWh at bus 1, we can write the following equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0.2 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} F_{12}^{fr} \\ F_{13}^{fr} \\ F_{23}^{fr} \end{bmatrix} = \begin{bmatrix} -100 \\ 100 \\ 0 \end{bmatrix}$$

Solving these equations, we get: $F_{12}^{fr} = -37.5$ MW, $F_{13}^{fr} = -62.5$ MW and $F_{23}^{fr} = -37.5$ MW. Generator D should therefore purchase 62.5 MW worth of flowgate rights in branch 3-1. Since the flow in line 3-1 is the only binding constraint, the only non-zero Lagrange multiplier is μ_{31} = 5.33 \$/MWh. Generator D will thus collect 62.5 MW × 5.33

\$/MWh = \$ 333.333. This amount is equal to the amount collected in point-to-point financial rights: 100×(13.33-10).

6.17 Repeat problems 6.15 and 6.16 for the conditions of Problem 6.9.

In that case the dispatch is P_A = 63.33 MW, P_B =10 MW, P_D = 400 MW and P_C = 46.67 MW. The resulting flows are: F_{23} = -156.67 MW, F_{12} = -140 MW and F_{13} = -250 MW. The Lagrange multipliers corresponding to binding constraints are μ_{31} = 7 \$/MWh and μ_{21} = 1.67 \$/MWh and the nodal prices are π_1 = 15 \$/MWh, π_2 = 12 \$/MWh and π_3 = 10 \$/MWh.

The consumer pays 100×15 = \$1500, for extracting the 100 MWh at bus 1
The generator receives 100×10 = \$1000, for injecting 100 MWh at bus 3
The consumer pays 100×(11-10) = \$100 to the generator to settle the contact of difference.
If the consumer owns point-to-point financial rights for 100 MWh between 3 and 1, it collects 100×(15-10) = \$ 500.

The consumer thus pays a total of 1500+100-500 = \$1100 for 100 MWh, which is equivalent to a price of 11 \$/MWh.

In this case the flowgate rights that should be obtained are 62.5 MWh on branch 3-1 and 37.5 MWh on branch 2-1. The amount collected from these rights would be 62.5×7 = \$437.5 and 37.5×1.66 = \$62.5, for a total of \$500, which is equal to the amount that would be collected in point-to-point financial rights.