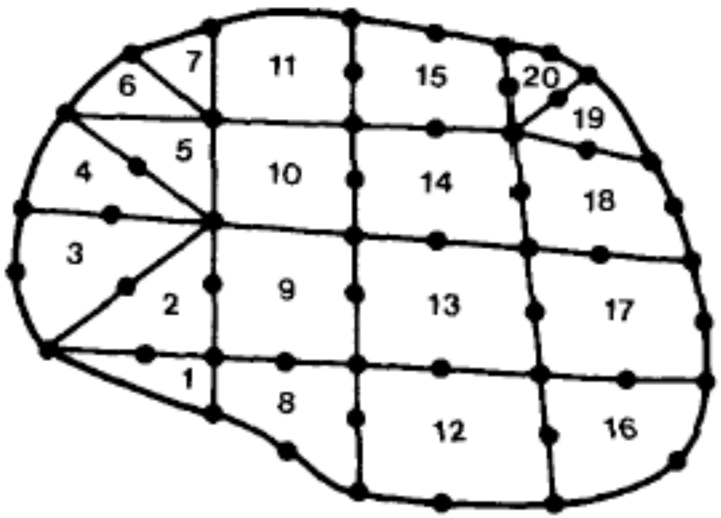
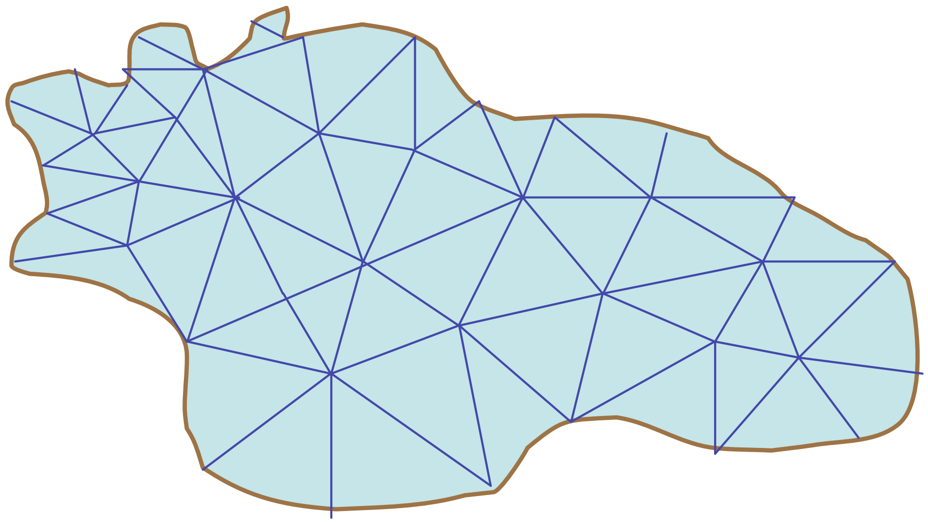
ELEMENTOS FINITOS
2-D

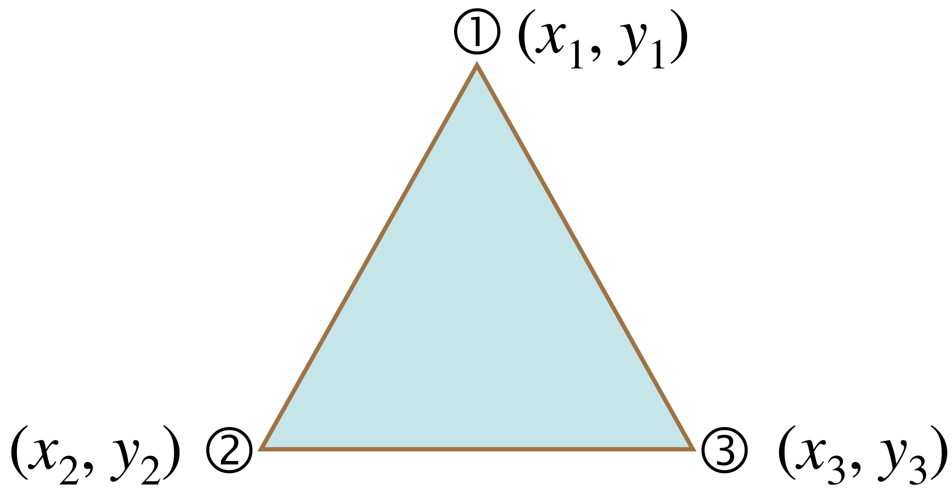
DISCRETIZACION DEL DOMINIO EN ELEMENTOS 2-D



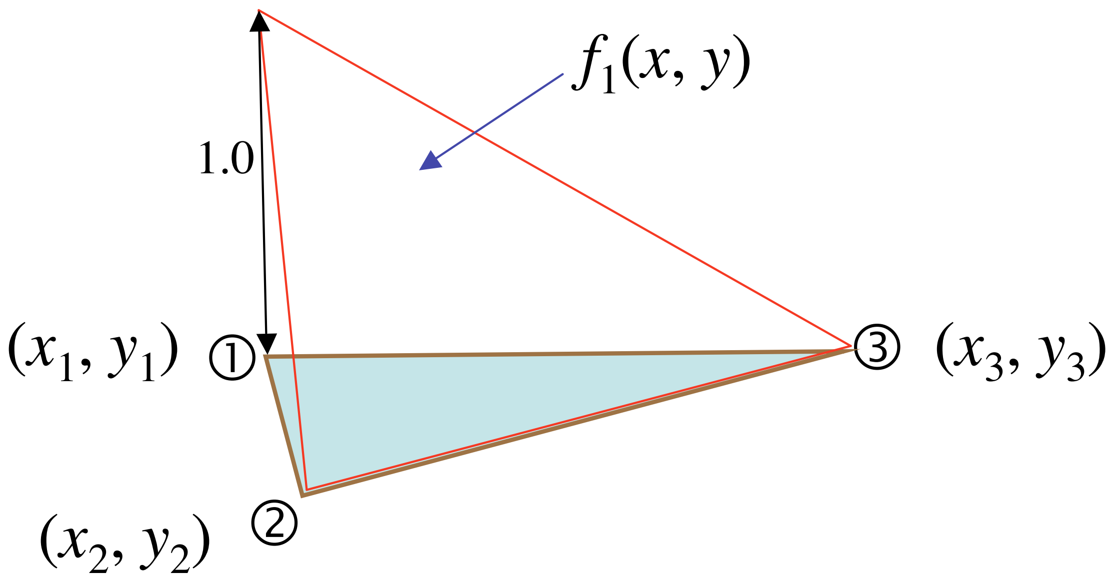
ELEMENTOS TRIANGULARES



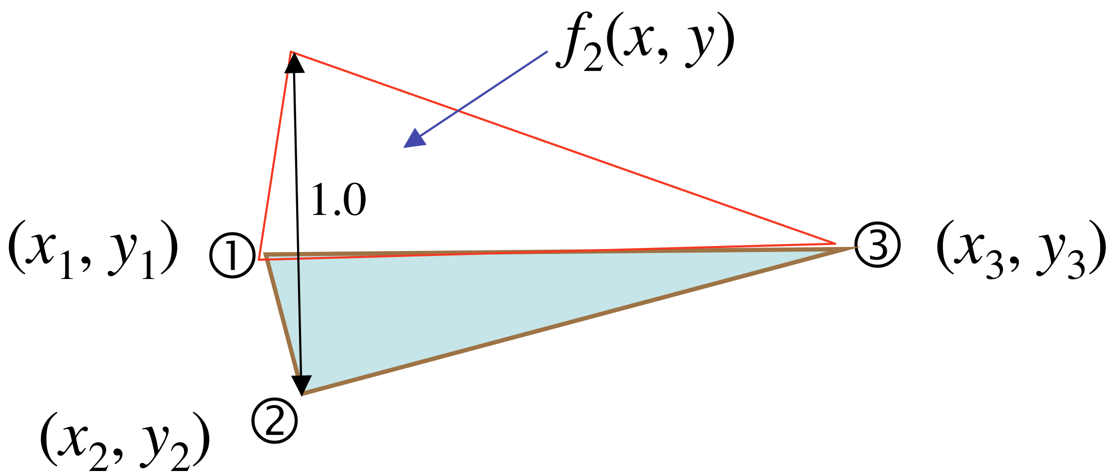
ELEMENTOS TRIANGULARES



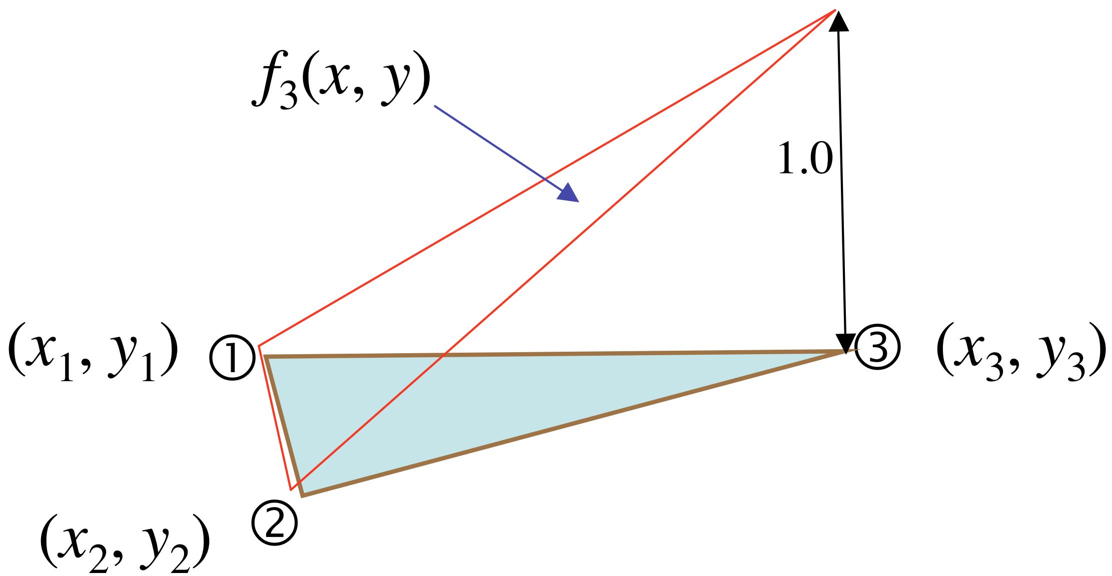
FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES



FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_1(x_1, y_1) = A_1\, x_1 + B_1\, y_1 + C_1 = 1$$

$$f_1(x_2, y_2) = A_1\, x_2 + B_1\, y_2 + C_1 = 0$$

$$f_1(x_3, y_3) = A_1\, x_3 + B_1\, y_3 + C_1 = 0$$

FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_2(x_1, y_1) = A_2\, x_1 + B_2\, y_1 + \mathrm{C}_2 = 0$$

$$f_2(x_2, y_2) = A_2\, x_2 + B_2\, y_2 + \mathrm{C}_2 = 1$$

$$f_2(x_3, y_3) = A_2\, x_3 + B_2\, y_3 + \mathrm{C}_2 = 0$$

FUNCIONES BASE LINEALES

$$f_i(x, y) = A_i\, x + B_i\, y + C_i$$

$$f_3(x_1, y_1) = A_3\, x_1 + B_3\, y_1 + C_3 = 0$$

$$f_3(x_2, y_2) = A_3\, x_2 + B_3\, y_2 + C_3 = 0$$

$$f_3(x_3, y_3) = A_3\, x_3 + B_3\, y_3 + C_3 = 1$$

FUNCIONES BASE LINEALES

$$\tilde{U}(x, y) = f_1(x, y)\, u_1 + f_2(x, y)\, u_2 + f_3(x, y)\, u_3$$

$$\tilde{U}(x, y) = \square_1 + \square_2\, x + \square_3\, y$$

$$u_1 = \square_1 + \square_2\, x_1 + \square_3\, y_1$$

$$u_2 = \square_1 + \square_2\, x_2 + \square_3\, y_2$$

$$u_3 = \square_1 + \square_2\, x_3 + \square_3\, y_3$$

FUNCIONES BASE LINEALES

$$f_1 = \frac{(x_2\, y_3 - x_3\, y_2) + (y_2 - y_3)\, x + (x_3 - x_2)\, y}{x_3\, y_2 - x_2\, y_3 - x_1\, y_2 + x_1\, y_3 + x_2\, y_1 - x_3\, y_1}$$

$$f_2 = \frac{(x_3\, y_1 - x_1\, y_3) + (y_3 - y_1)\, x + (x_1 - x_3)\, y}{x_3\, y_2 - x_2\, y_3 - x_1\, y_2 + x_1\, y_3 + x_2\, y_1 - x_3\, y_1}$$

$$f_3 = \frac{(x_1\, y_2 - x_2\, y_1) + (y_1 - y_2)\, x + (x_2 - x_1)\, y}{x_3\, y_2 - x_2\, y_3 - x_1\, y_2 + x_1\, y_3 + x_2\, y_1 - x_3\, y_1}$$

FUNCIONES BASE LINEALES

$$f_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y)$$

$$a_i = x_j y_m - x_m y_j$$

$$b_i = y_j - y_m$$

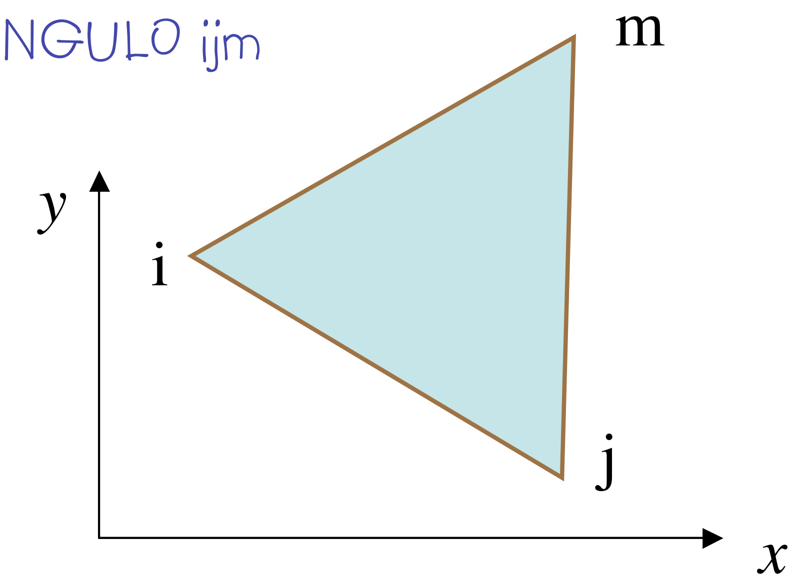
$$c_i = x_m - x_j$$

FUNCIONES BASE LINEALES

$$2\Delta = \det\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

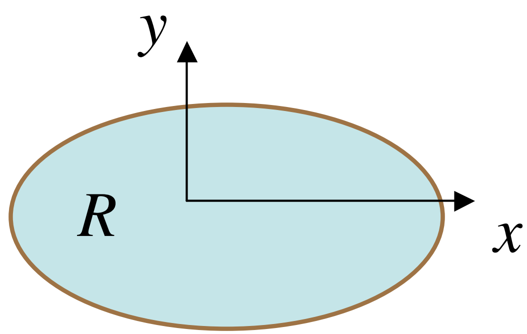
$$2\Delta = x_3\, y_2 - x_2\, y_3 - x_1\, y_2 + x_1\, y_3 + x_2\, y_1 - x_3\, y_1$$

= 2 □ AREA TRIANGULO ijm



EC. DIFUSION 2-D

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k_x \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial U}{\partial y}\right)$$



$$\tilde{U}(x, y; t) = \underset{j}{\Box}\, u_j(t)\, f_j(x, y)$$

$$\frac{\partial \tilde{U}}{\partial t} \Box \frac{\partial}{\partial x}\left(k_x \frac{\partial \tilde{U}}{\partial x}\right) \Box \frac{\partial}{\partial y}\left(k_y \frac{\partial \tilde{U}}{\partial y}\right) = \Box$$

EC. DIFUSION 2-D

$$\int_R W_i \left(\frac{\partial \tilde{U}}{\partial t} - \frac{\partial}{\partial x}\left(\kappa_x \frac{\partial \tilde{U}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\kappa_y \frac{\partial \tilde{U}}{\partial y}\right)\right) dR = 0 \qquad i = 1, \ldots, M$$

$$\tilde{U} = \sum_j u_j \; f_j$$

$$\sum_j \left(\int_R W_i \; f_j \; dR \right) \frac{du_j}{dt} - \sum_j \left[\int_R W_i \; k_{xj} \frac{\partial^2 f_j}{\partial x^2} \, dR + \int_R W_i \; k_{yj} \frac{\partial^2 f_j}{\partial y^2} \, dR \right] u_j = 0$$

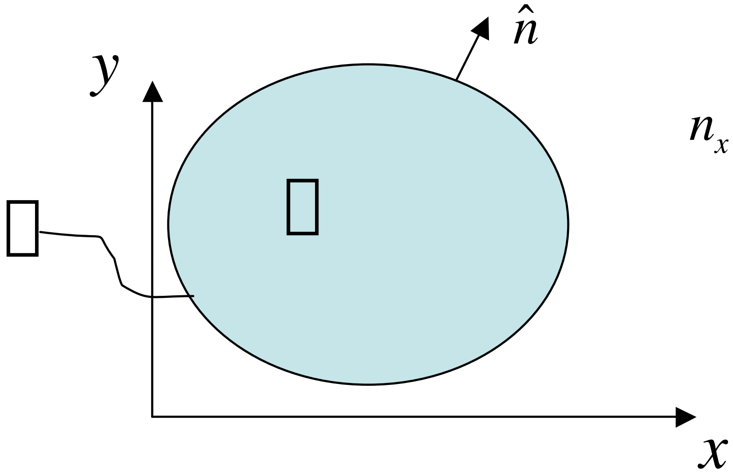
SISTEMA DE EDOS:

$$\sum_{\text{nodos internos}} \left(C_{ij} u_j + M_{ij} \frac{du_j}{dt} \right) = 0$$

TEOREMA DE GREEN

(INTEGRACION x PARTES 2D):

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx \, dy + \oint_{\Gamma} \phi \psi \, n_x \, d\Gamma$$

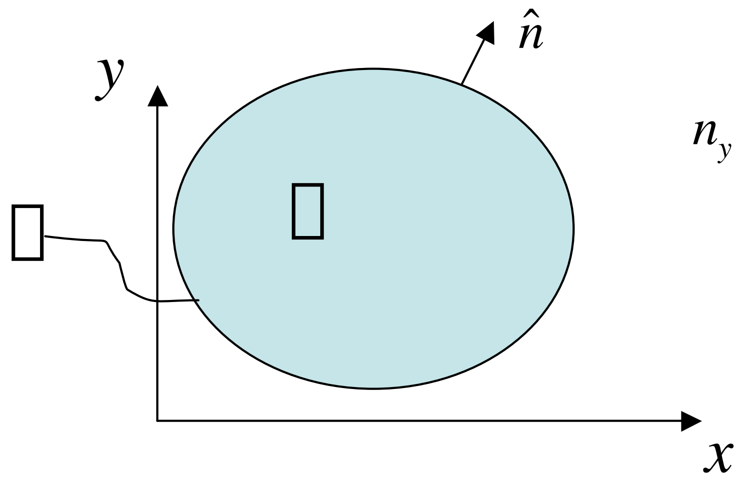


$$n_x = \hat{n} \cdot \hat{i}$$

TEOREMA DE GREEN

(INTEGRACION x PARTES 2D):

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial y} \, dx \, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial y} \psi \, dx \, dy + \oint_{\Gamma} \phi \psi \, n_y \, d\Gamma$$



$$n_y = \hat{n} \cdot \hat{j}$$

PARA LOS NODOS INTERIORES:

$$\sum_j \left[\left(\int_R W_i \, f_j \, dR \right) \frac{du_j}{dt} - \sum_j \left(\int_R W_i \, k_{xj} \frac{\partial^2 f_j}{\partial x^2} \, dR + \int_R W_i \, k_{yj} \frac{\partial^2 f_j}{\partial y^2} \, dR \right) u_j \right] = 0$$

$$\Rightarrow \quad \sum_j \left[\left(\int_R W_i \, f_j \, dx \, dy \right) \frac{du_j}{dt} + \sum_j \left(\int_R k_{xj} \frac{\partial W_i}{\partial x} \frac{\partial f_j}{\partial x} \, dx \, dy + \int_R k_{yj} \frac{\partial W_i}{\partial y} \frac{\partial f_j}{\partial y} \, dx \, dy \right) u_j \right] = 0$$

GALERKIN: $W_{\mathrm{i}} = f_{\mathrm{i}}$

$$\Rightarrow \quad \sum_j \left[\iint_R f_i \, f_j \, dx \, dy \right] \frac{du_j}{dt} + \sum_j \left[\iint_R k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x} \, dx \, dy + \iint_R k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y} \, dx \, dy \right] u_j = 0$$

PERO:

$$f_i = \frac{1}{2\Delta} (a_i + b_i \, x + c_i \, y)$$

$$\Rightarrow \quad \frac{\partial f_i}{\partial x} = \frac{b_i}{2\Delta}$$

$$\Rightarrow \quad \frac{\partial f_i}{\partial y} = \frac{c_i}{2\Delta}$$

$$\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x} = \frac{b_i\, b_j}{(2\,\Box)^2} \qquad \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y} = \frac{c_i\, c_j}{(2\,\Box)^2}$$

$$f_i\, f_j = \frac{1}{(2\,\Box)^2}(a_i + b_i\, x + c_i\, y)(a_j + b_j\, x + c_j\, y)$$

$$f_i\, f_j = \frac{1}{(2\,\Box)^2}(a_i\, a_j + (a_i\, b_j + b_i\, a_j)\, x + (a_i\, c_j + c_i\, a_j)\, y +$$

$$(b_i\, c_j + c_i\, b_j)\, x\, y + b_i\, b_j\, x^2 + c_i\, c_j\, y^2)$$

PARA CADA ELEMENTO:

$$\sum_j \left[\iint_R f_i \, f_j \, dx \, dy \right] \frac{du_j}{dt} + \sum_j \left[\iint_R k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x} \, dx \, dy + \iint_R k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y} \, dx \, dy \right] u_j = 0$$

$$\iint_{\Delta} k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x} \, dx \, dy = k_{xj} \iint_{\Delta} \frac{b_i \, b_j}{(2\Delta)^2} \, dx \, dy = k_{xj} \frac{b_i \, b_j}{(2\Delta)^2} \underbrace{\iint_{\Delta} dx \, dy}_{\Delta}$$

$$\iint_{\Delta} k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x} \, dx \, dy = k_{xj} \frac{b_i \, b_j}{4\Delta}$$

$$\iint_{\Delta} k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y} \, dx \, dy = k_{yj} \frac{c_i \, c_j}{4\Delta}$$

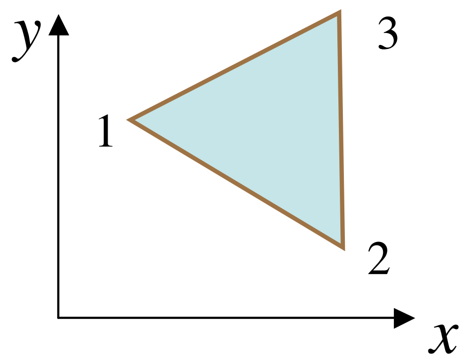
PARA EL CASO PERMANENTE, COEFICIENTES CONSTANTES:

$$\sum_j \left[\underbrace{\int_R k_{xj} \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x}\, dx\, dy}_{k_x \frac{b_i b_j}{4\Delta}} + \underbrace{\int_R k_{yj} \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y}\, dx\, dy}_{k_y \frac{c_i c_j}{4\Delta}} \right] u_j = 0$$

MATRICES ELEMENTALES:

$$\sum_j \left[\int_R k_{xj} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x}\, dx\, dy + \int_R k_{yj} \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y}\, dx\, dy \right] u_j = 0$$

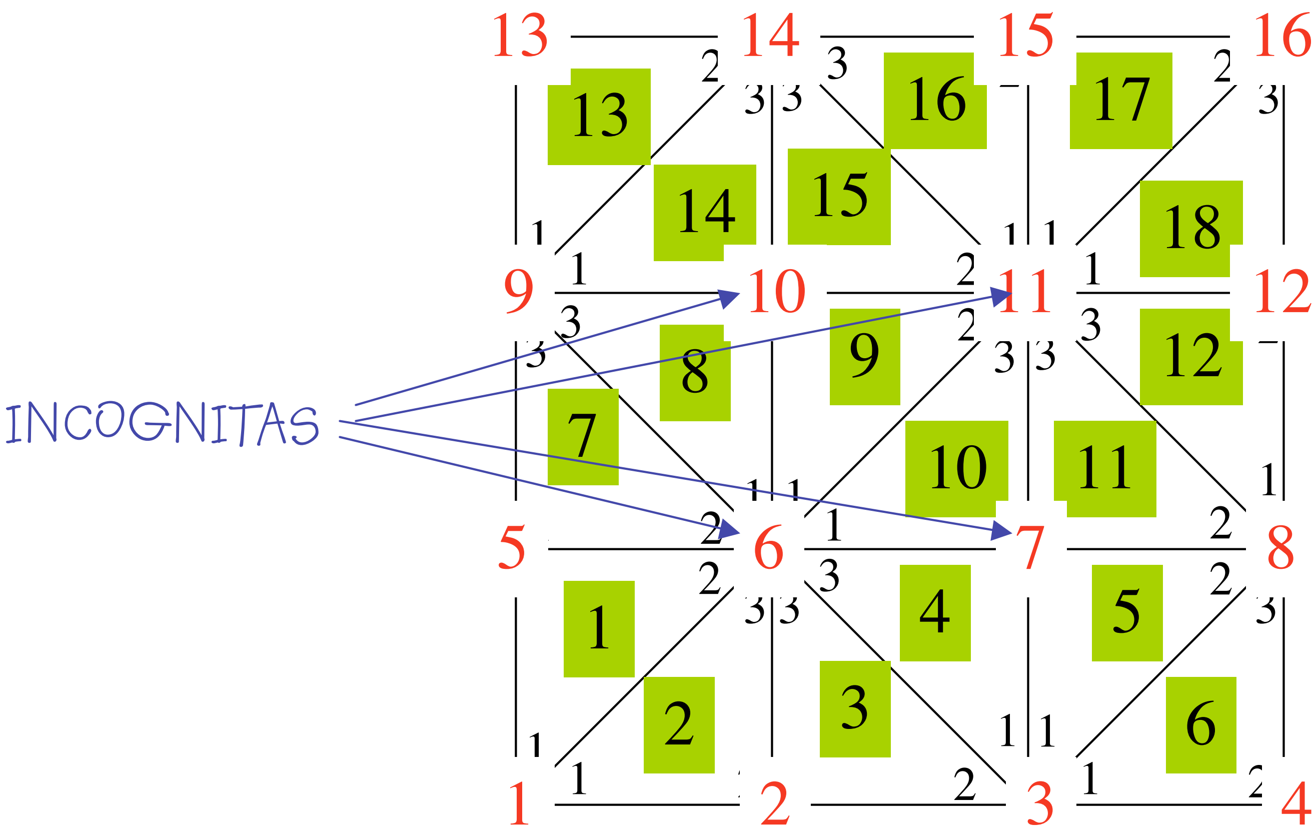
$$k_x \frac{b_i\, b_j}{4\,\Delta} \qquad\qquad k_y \frac{c_i\, c_j}{4\,\Delta}$$

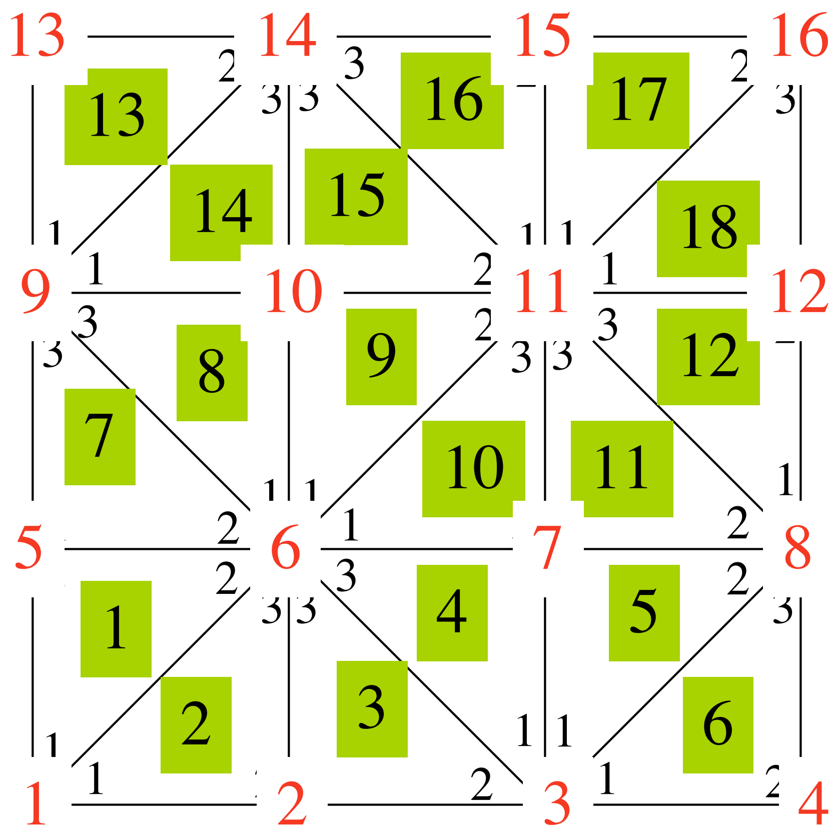


MATRICES ELEMENTALES:

$$\frac{1}{4\,\square}\left\{ k_x \left| \begin{matrix} b_1\,b_1 & b_1\,b_2 & b_1\,b_3 \\ b_2\,b_1 & b_2\,b_2 & b_2\,b_3 \\ b_3\,b_1 & b_3\,b_2 & b_3\,b_3 \end{matrix} \right| + k_y \left| \begin{matrix} c_1\,c_1 & c_1\,c_2 & c_1\,c_3 \\ c_2\,c_1 & c_2\,c_2 & c_2\,c_3 \\ c_3\,c_1 & c_3\,c_2 & c_3\,c_3 \end{matrix} \right| \right\} \left| \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \right| = 0$$

EJEMPLO:





$$f_1\, u_1 + f_2\, u_6 + f_3\, u_5 = \tilde{u}_1$$

$$f_1\, u_1 + f_2\, u_2 + f_3\, u_6 = \tilde{u}_2$$

$$f_1\, u_2 + f_2\, u_3 + f_3\, u_6 = \tilde{u}_3$$

$$f_1\, u_3 + f_2\, u_7 + f_3\, u_6 = \tilde{u}_4$$

$$f_1\, u_3 + f_2\, u_8 + f_3\, u_7 = \tilde{u}_5$$

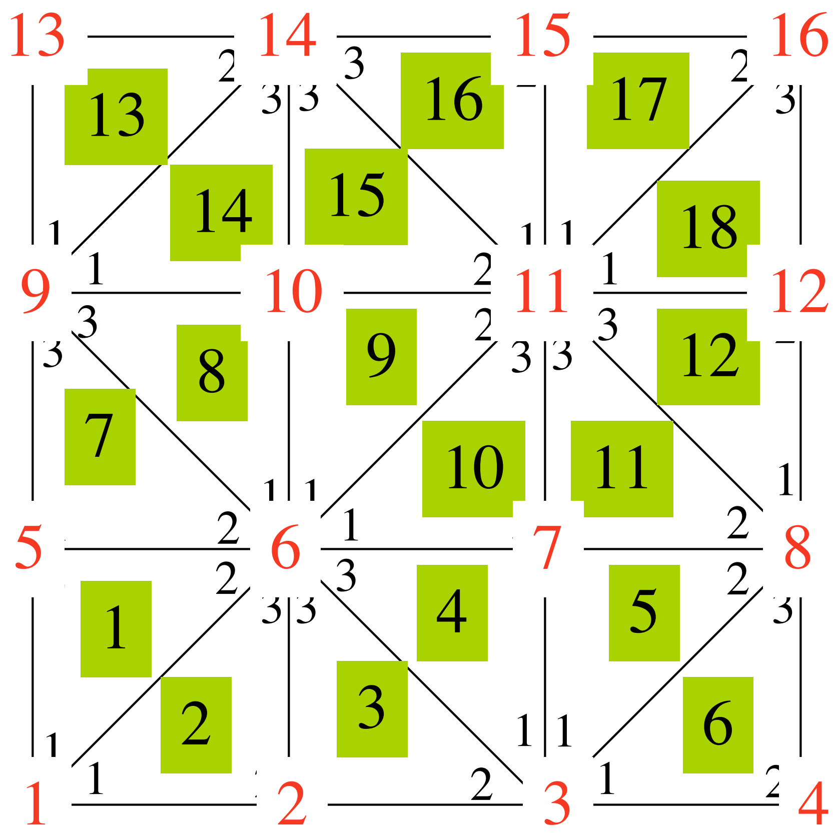
$$f_1\, u_5 + f_2\, u_6 + f_3\, u_9 = \tilde{u}_7$$

$$f_1\, u_6 + f_2\, u_{10} + f_3\, u_9 = \tilde{u}_8$$

$$f_1\, u_6 + f_2\, u_{11} + f_3\, u_{10} = \tilde{u}_9$$

$$f_1\, u_7 + f_2\, u_8 + f_3\, u_{11} = \tilde{u}_{11}$$

$$f_1\, u_8 + f_2\, u_{12} + f_3\, u_{11} = \tilde{u}_{12}$$



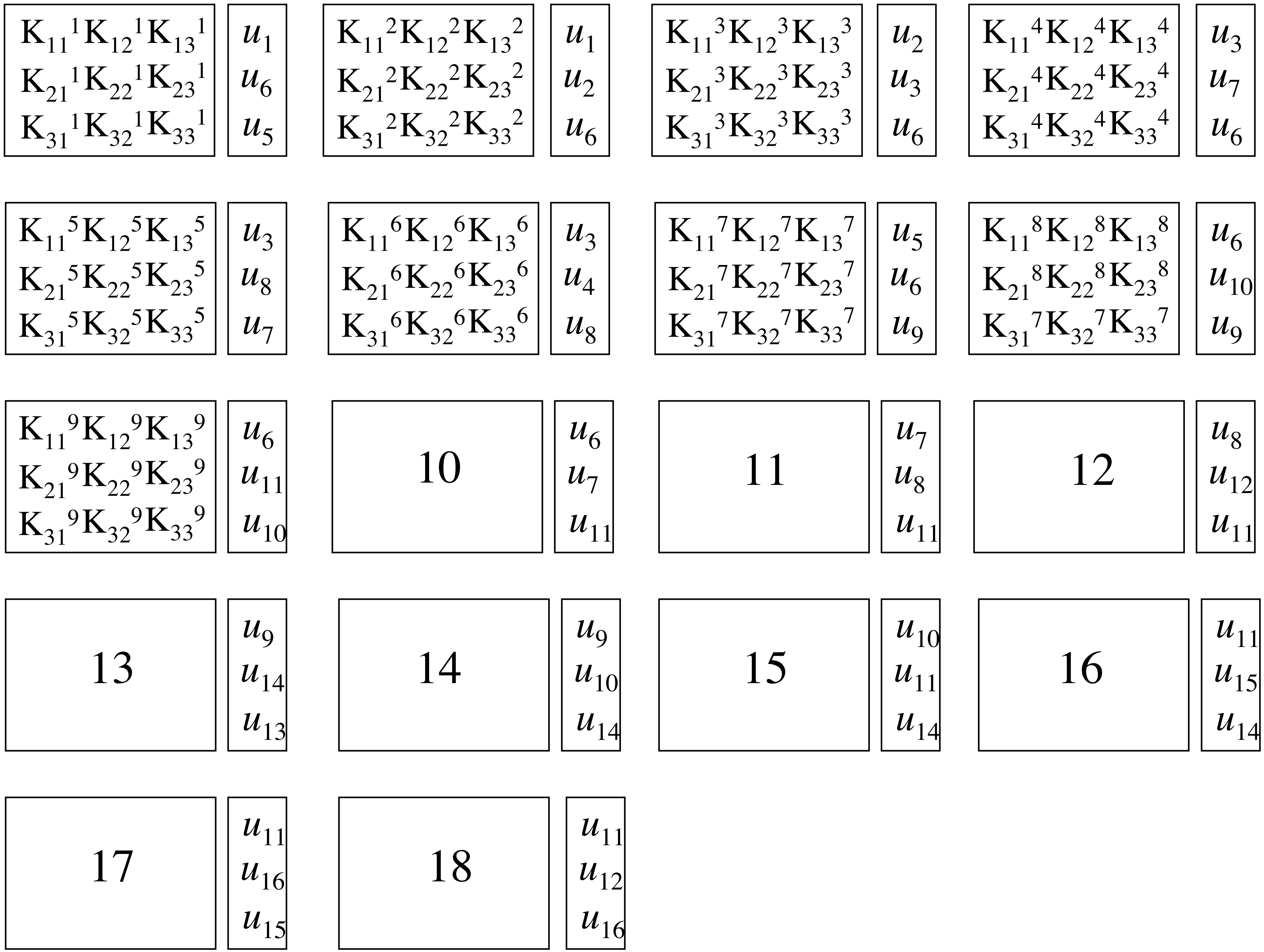
$$f_1\, u_9 + f_2\, u_{10} + f_3\, u_{14} = \tilde{u}_{14}$$

$$f_1\, u_{10} + f_2\, u_{11} + f_3\, u_{14} = \tilde{u}_{15}$$

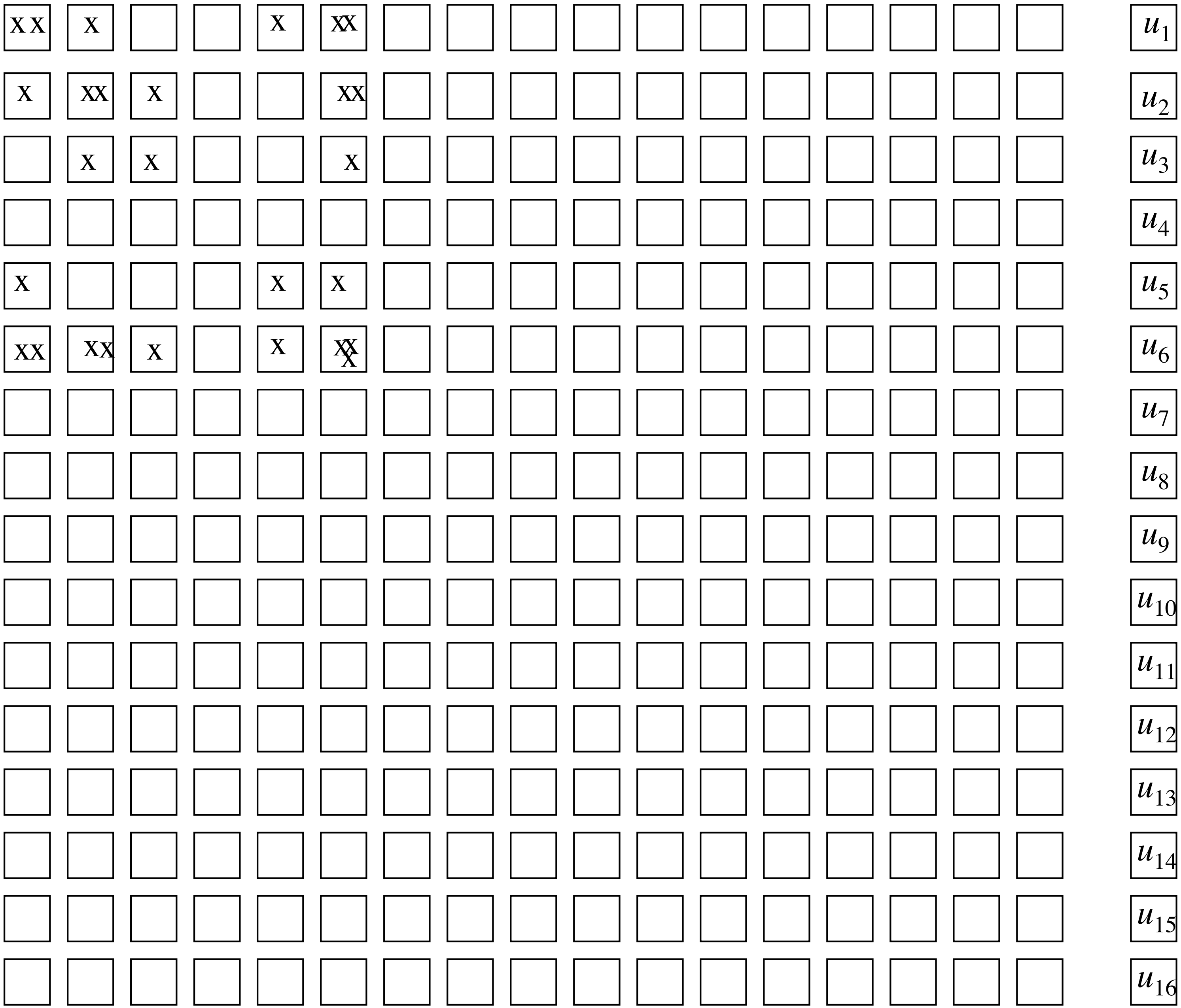
$$f_1\, u_{11} + f_2\, u_{15} + f_3\, u_{14} = \tilde{u}_{16}$$

$$f_1\, u_{11} + f_2\, u_{16} + f_3\, u_{15} = \tilde{u}_{17}$$

$$f_1\, u_{11} + f_2\, u_{12} + f_3\, u_{16} = \tilde{u}_{18}$$



$K_{11}^1 K_{12}^1 K_{13}^1$
$K_{21}^1 K_{22}^1 K_{23}^1$
$K_{31}^1 K_{32}^1 K_{33}^1$
u_1 u_6 u_5
$K_{11}^2 K_{12}^2 K_{13}^2$
$K_{21}^2 K_{22}^2 K_{23}^2$
$K_{31}^2 K_{32}^2 K_{33}^2$
u_1 u_2 u_6
$K_{11}^3 K_{12}^3 K_{13}^3$
$K_{21}^3 K_{22}^3 K_{23}^3$
$K_{31}^3 K_{32}^3 K_{33}^3$
u_2 u_3 u_6
$K_{11}^4 K_{12}^4 K_{13}^4$
$K_{21}^4 K_{22}^4 K_{23}^4$
$K_{31}^4 K_{32}^4 K_{33}^4$
u_3 u_7 u_6
$K_{11}^5 K_{12}^5 K_{13}^5$
$K_{21}^5 K_{22}^5 K_{23}^5$
$K_{31}^5 K_{32}^5 K_{33}^5$
u_3 u_8 u_7
$K_{11}^6 K_{12}^6 K_{13}^6$
$K_{21}^6 K_{22}^6 K_{23}^6$
$K_{31}^6 K_{32}^6 K_{33}^6$
u_3 u_4 u_8
$K_{11}^7 K_{12}^7 K_{13}^7$
$K_{21}^7 K_{22}^7 K_{23}^7$
$K_{31}^7 K_{32}^7 K_{33}^7$
u_5 u_6 u_9
$K_{11}^8 K_{12}^8 K_{13}^8$
$K_{21}^8 K_{22}^8 K_{23}^8$
$K_{31}^7 K_{32}^7 K_{33}^7$
u_6 u_{10} u_9
$K_{11}^9 K_{12}^9 K_{13}^9$
$K_{21}^9 K_{22}^9 K_{23}^9$
$K_{31}^9 K_{32}^9 K_{33}^9$
u_6 u_{11} u_{10}
10
u_6 u_7 u_{11}
11
u_7 u_8 u_{11}
12
u_8 u_{12} u_{11}
13
u_9 u_{14} u_{13}
14
u_9 u_{10} u_{14}
15
u_{10} u_{11} u_{14}
16
u_{11} u_{15} u_{14}
17
u_{11} u_{16} u_{15}
18
u_{11} u_{12} u_{16}



u_1
u_2
u_3
u_4
u_5
u_6
u_7
u_8
u_9
u_{10}
u_{11}
u_{12}
u_{13}
u_{14}
u_{15}
u_{16}

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix} \begin{vmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{vmatrix} = \text{B.C.}$$