3. Matchings and covers in bipartite graphs

3.1. Matchings, covers, and Gallai's theorem

Let $G = (V, E)$ be a graph. A *stable set* is a subset C of V such that $e \not\subseteq C$ for each edge e of G. A *vertex cover* is a subset W of V such that $e \cap W \neq \emptyset$ for each edge e of G. It is not difficult to show that for each $U \subseteq V$:

(1) $\qquad U$ is a stable set $\iff V \setminus U$ is a vertex cover.

A *matching* is a subset M of E such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$. A matching is called *perfect* if it covers all vertices (that is, has size $\frac{1}{2}|V|$). An *edge cover* is a subset F of E such that for each vertex v there exists $e \in F$ satisfying $v \in e$. Note that an edge cover can exist only if G has no isolated vertices.

Define:

(2)
$$\begin{aligned}
\alpha(G) &:= \max\{|C| \mid C \text{ is a stable set}\},\\
\rho(G) &:= \min\{|F| \mid F \text{ is an edge cover}\},\\
\tau(G) &:= \min\{|W| \mid W \text{ is a vertex cover}\},\\
\nu(G) &:= \max\{|M| \mid M \text{ is a matching}\}.
\end{aligned}$$

These numbers are called the *stable set number*, the *edge cover number*, the *vertex cover number*, and the *matching number* of G, respectively.

It is not difficult to show that:

(3) $\qquad \alpha(G) \leq \rho(G)$ and $\nu(G) \leq \tau(G)$.

The triangle K_3 shows that strict inequalities are possible. In fact, equality in one of the relations (3) implies equality in the other, as Gallai [1958,1959] proved:

Theorem 3.1 (Gallai's theorem)**.** *For any graph $G = (V, E)$ without isolated vertices one has*

(4) $\qquad \alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$.

Proof. The first equality follows directly from (1).

To see the second equality, first let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v missed by M, add to M an edge covering v. We obtain an edge cover of size $|M| + (|V| - 2|M|) = |V| - |M|$. Hence $\rho(G) \leq |V| - \nu(G)$.

Second, let F be an edge cover of size $\rho(G)$. For each $v \in V$ delete from F, $d_F(v)-1$ edges incident with v. We obtain a matching of size at least $|F| - \sum_{v\in V}(d_F(v)-1) = |F| - (2|F| - |V|) = |V| - |F|$. Hence $\nu(G) \geq |V| - \rho(G)$. ∎

This proof also shows that if we have a matching of maximum cardinality in any graph G, then we can derive from it a minimum cardinality edge cover, and conversely.

Exercises

3.1. Let $G = (V, E)$ be a graph without isolated vertices. Define:

(5)

$\alpha_2(G) :=$ the maximum number of vertices such that no edge contains more than two of these vertices;

$\rho_2(G) :=$ the minimum number of edges such that each vertex is contained in at least two of these edges;

$\tau_2(G) :=$ the minimum number of vertices such that each edge contains at least two of these vertices

$\nu_2(G) :=$ the maximum number of edges such that no vertex is contained in more than two of these edges;

possibly taking vertices (edges, respectively) more than once.

(i) Show that $\alpha_2(G) \leq \rho_2(G)$ and that $\nu_2(G) \leq \tau_2(G)$.

(ii) Show that $\alpha_2(G) + \tau_2(G) = 2|V|$.

(iii) Show that $\nu_2(G) + \rho_2(G) = 2|V|$.

3.2. M-augmenting paths

Basic in matching theory are M-augmenting paths, which are defined as follows. Let M be a matching in a graph $G = (V, E)$. A path $P = (v_0, v_1, \ldots, v_t)$ in G is called *M-augmenting* if

(6) (i) t is odd,
(ii) $v_1v_2, v_3v_4, \ldots, v_{t-2}v_{t-1} \in M$,
(iii) $v_0, v_t \notin \bigcup M$.

Note that this implies that $v_0v_1, v_2v_3, \ldots, v_{t-1}v_t$ do not belong to M.

Clearly, if $P = (v_0, v_1, \ldots, v_t)$ is an M-augmenting path, then

(7) $$M' := M \triangle EP$$

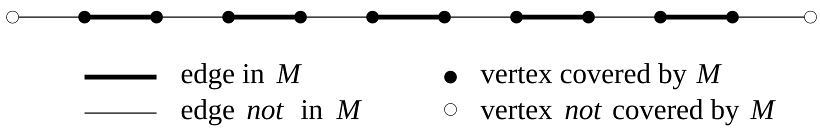

Figure 3.1

is a matching satisfying $|M'| = |M| + 1$.[8]

In fact, it is not difficult to show that:

Theorem 3.2. *Let $G = (V, E)$ be a graph and let M be a matching in G. Then either M is a matching of maximum cardinality, or there exists an M-augmenting path.*

Proof. If M is a maximum-cardinality matching, there cannot exist an M-augmenting path P, since otherwise $M \triangle EP$ would be a larger matching.

If M' is a matching larger than M, consider the components of the graph $G' := (V, M \cup M')$. As G' has maximum valency two, each component of G' is either a path (possibly of length 0) or a circuit. Since $|M'| > |M|$, at least one of these components should contain more edges of M' than of M. Such a component forms an M-augmenting path. ∎

3.3. Kőnig's theorems

A classical min-max relation due to Kőnig [1931] (extending a result of Frobenius [1917]) characterizes the maximum size of a matching in a bipartite graph (we follow de proof of De Caen [1988]):

Theorem 3.3 (Kőnig's matching theorem)**.** *For any bipartite graph $G = (V, E)$ one has*

(8) $$\nu(G) = \tau(G).$$

That is, the maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover.

Proof. By (3) it suffices to show that $\nu(G) \geq \tau(G)$. We may assume that G has at least one edge. Then:

(9) G has a vertex u covered by each maximum-size matching.

[8] EP denotes the set of edges in P. $\triangle$ denotes symmetric difference.

To see this, let $e = uv$ be any edge of G, and suppose that there are maximum-size matchings M and N missing u and v respectively[9]. Let P be the component of $M \cup N$ containing u. So P is a path with end vertex u. Since P is not M-augmenting (as M has maximum size), P has even length, and hence does not traverse v (otherwise, P ends at v, contradicting the bipartiteness of G). So $P \cup e$ would form an N-augmenting path, a contradiction (as N has maximum size). This proves (9).

Now (9) implies that for the graph $G' := G - u$ one has $\nu(G') = \nu(G) - 1$. Moreover, by induction, G' has a vertex cover C of size $\nu(G')$. Then $C \cup \{u\}$ is a vertex cover of G of size $\nu(G') + 1 = \nu(G)$. ∎

Combination of Theorems 3.1 and 3.3 yields the following result of Kőnig [1932].

Corollary 3.3a (Kőnig's edge cover theorem)**.** *For any bipartite graph $G = (V, E)$, without isolated vertices, one has*

$$(10) \qquad \alpha(G) = \rho(G).$$

That is, the maximum cardinality of a stable set in a bipartite graph is equal to the minimum cardinality of an edge cover.

Proof. Directly from Theorems 3.1 and 3.3, as $\alpha(G) = |V| - \tau(G) = |V| - \nu(G) = \rho(G)$. ∎

Exercises

3.2. (i) Prove that a k-regular bipartite graph has a perfect matching (if $k \geq 1$).

(ii) Derive that a k-regular bipartite graph has k disjoint perfect matchings.

(iii) Give for each $k > 1$ an example of a k-regular graph not having a perfect matching.

3.3. Prove that in a matrix, the maximum number of nonzero entries with no two in the same line (=row or column), is equal to the minimum number of lines that include all nonzero entries.

3.4. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of some finite set X. A subset Y of X is called a *transversal* or a *system of distinct representatives* (*SDR*) of $\mathcal{A}$ if there exists a bijection $\pi : \{1, \ldots, n\} \to Y$ such that $\pi(i) \in A_i$ for each $i = 1, \ldots, n$.

Decide if the following collections have an SDR:

(i) $\{3, 4, 5\}, \{2, 5, 6\}, \{1, 2, 5\}, \{1, 2, 3\}, \{1, 3, 6\}$,

(ii) $\{1, 2, 3, 4, 5, 6\}, \{1, 3, 4\}, \{1, 4, 7\}, \{2, 3, 5, 6\}, \{3, 4, 7\}, \{1, 3, 4, 7\}, \{1, 3, 7\}$.

[9] M *misses* a vertex u if $u \notin \bigcup M$. Here $\bigcup M$ denotes the union of the edges in M; that is, the set of vertices covered by the edges in M.

3.5. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of some finite set X. Prove that $\mathcal{A}$ has an SDR if and only if

$$|\bigcup_{i \in I} A_i| \geq |I| \tag{11}$$

for each subset I of $\{1, \ldots, n\}$.

[Hall's 'marriage' theorem (Hall [1935]).]

3.6. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be subsets of the finite set X. A subset Y of X is called a *partial transversal* or a *partial system of distinct representatives* (*partial SDR*) if it is a transversal of some subcollection $(A_{i_1}, \ldots, A_{i_k})$ of $(A_1, \ldots, A_n)$.

Show that the maximum cardinality of a partial SDR of $\mathcal{A}$ is equal to the minimum value of

$$|X \setminus Z| + |\{i \mid A_i \cap Z \neq \emptyset\}|, \tag{12}$$

where Z ranges over all subsets of X.

3.7. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of finite sets and let k be a natural number. Show that $\mathcal{A}$ has k pairwise disjoint SDR's of $\mathcal{A}$ if and only if

$$|\bigcup_{i \in I} A_i| \geq k|I| \tag{13}$$

for each subset I of $\{1, \ldots, n\}$.

3.8. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a finite set X and let k be a natural number. Show that X can be partitioned into k partial SDR's if and only if

$$k \cdot |\{i \mid A_i \cap Y \neq \emptyset\}| \geq |Y| \tag{14}$$

for each subset Y of X.

(*Hint:* Replace each A_i by k copies of A_i and use Exercise 3.6 above.)

3.9. Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be two partitions of the finite set X.

(i) Show that $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ have a *common* SDR if and only if for each subset I of $\{1, \ldots, n\}$, the set $\bigcup_{i \in I} A_i$ intersects at least $|I|$ sets among $B_1, \ldots, B_n$.

(ii) Suppose that $|A_1| = \cdots = |A_n| = |B_1| = \cdots = |B_n|$. Show that the two partitions have a common SDR.

3.10. Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be two partitions of the finite set X. Show that the minimum cardinality of a subset of X intersecting each set among $A_1, \ldots, A_n, B_1, \ldots, B_n$ is equal to the maximum number of pairwise disjoint sets in $A_1, \ldots, A_n, B_1, \ldots, B_n$.

3.11. A matrix is called *doubly stochastic* if it is nonnegative and each row sum and each column sum is equal to 1. A matrix is called a *permutation matrix* if each entry is 0 or 1 and each row and each column contains exactly one 1. Show that each doubly stochastic matrix is a convex linear combination of permutation matrices.

[Birkhoff-von Neumann theorem (Birkhoff [1946], von Neumann [1953]).]

3.12. Let $G = (V, E)$ be a bipartite graph with colour classes U and W. Let $b : V \to \mathbb{Z}_+$ be so that $\sum_{v \in U} b(v) = \sum_{v \in W} b(v) =: t$.

A *b-matching* is a function $c : E \to \mathbb{Z}_+$ so that for each vertex v of G:

$$\sum_{e \in E, v \in e} c(e) = b(v) \tag{15}$$

Show that there exists a b-matching if and only if

$$\sum_{v \in X} b(v) \geq t \tag{16}$$

for each vertex cover X.

3.13. Let $G = (V, E)$ be a bipartite graph with colour classes U and W. Let $b : V \to \mathbb{Z}_+$ be so that $\sum_{v \in U} b(v) = \sum_{v \in W} b(v) = t$.

Show that there exists a subset F of E so that each vertex v of G is incident with exactly $b(v)$ of the edges in F if and only if

$$t + |E(X)| \geq \sum_{v \in X} b(v) \tag{17}$$

for each subset X of V, where $E(X)$ denotes the set of edges contained in X.

3.14. Let $G = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Show that the maximum number of edges in a subset F of E so that each vertex v of G is incident with at most $b(v)$ of the edges in F, is equal to

$$\min_{X \subseteq V} \sum_{v \in X} b(v) + |E(V \setminus X)|. \tag{18}$$

3.15. Let G be a bipartite graph with colour classes U and W satisfying $|U| = |W| = t$. Prove that G has k disjoint perfect matchings if and only if for all $U' \subseteq U$ and $W' \subseteq W$ there are at least $k(|U'| + |W'| - t)$ edges connecting U' and W'.

3.16. Show that each $2k$-regular graph contains a set F of edges so that each vertex is incident with exactly two edges in F.

3.4. Cardinality bipartite matching algorithm

We now focus on the problem of finding a maximum-sized matching in a bipartite graph algorithmically.

In any graph, if we have an algorithm finding an M-augmenting path for any matching M (if it exists), then we can find a maximum cardinality matching: we iteratively find matchings $M_0, M_1, \ldots$, with $|M_i| = i$, until we have a matching M_k such that there does not exist any M_k-augmenting path.

We now describe how to find an M-augmenting path in a bipartite graph.

Matching augmenting algorithm for bipartite graphs

input: a bipartite graph $G = (V, E)$ and a matching M,
output: a matching M' satisfying $|M'| > |M|$ (if there is one).
description of the algorithm: Let G have colour classes U and W. Orient each edge $e = \{u, w\}$ of G (with $u \in U, w \in W$) as follows:

(19) if $e \in M$ then orient e from w to u,
if $e \notin M$ then orient e from u to w.

Let D be the directed graph thus arising. Consider the sets

(20) $U' := U \setminus \bigcup M$ and $W' := W \setminus \bigcup M$.

Now an M-augmenting path (if it exists) can be found by finding a directed path in D from any vertex in U' to any vertex in W'. Hence in this way we can find a matching larger than M. ∎

This implies:

Theorem 3.4. *A maximum-size matching in a bipartite graph can be found in time $O(|V||E|)$.*

Proof. The correctness of the algorithm is immediate. Since a directed path can be found in time $O(|E|)$, we can find an augmenting path in time $O(|E|)$. Hence a maximum cardinality matching in a bipartite graph can be found in time $O(|V||E|)$ (as we do at most $|V|$ iterations). ∎

Hopcroft and Karp [1973] gave an $O(|V|^{1/2}|E|)$ algorithm — see Section 4.2.

Application 3.1: Assignment problem. Suppose we have k machines at our disposal: $m_1, \ldots, m_k$. On a certain day we have to carry out n jobs: $j_1, \ldots, j_n$. Each machines is capable of performing some jobs, but can do only one job a day. E.g., we could have five machines $m_1, \ldots, m_5$ and five jobs $j_1, \ldots, j_5$ and the capabilities of the machines are indicated by crosses in the following table:

	j_1	j_2	j_3	j_4	j_5
m_1	X	X			X
m_2	X	X	X	X	
m_3	X	X			
m_4		X			
m_5		X			

We want to assign the machines to the jobs in such a way that every machine performs at most one job and that a largest number of jobs is carried out.

In order to solve this problem we represent the machines and jobs by vertices $m_1, \ldots, m_k$ and $j_1, \ldots, j_n$ of a bipartite graph $G = (V, E)$, and we make an edge from m_i to j_j if job j can be performed by machine i. Thus the example gives Figure 3.2. Then a maximum-size matching in G corresponds to a maximum assignment of jobs.

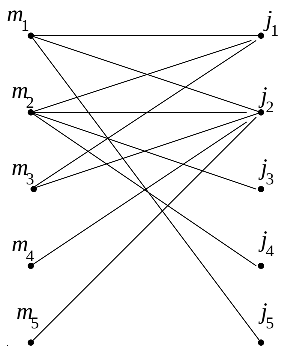

Figure 3.2

Exercises

3.17. Find a maximum-size matching and a minimum vertex cover in the bipartite graph in Figure 3.3.

3.18. Solve the assignment problem given in Application 3.1.

3.19. Derive Kőnig's matching theorem from the cardinality matching algorithm for bipartite graphs.

3.20. Show that a minimum-size vertex cover in a bipartite graph can be found in polynomial time.

3.21. Show that, given a family of sets, a system of distinct representatives can be found in polynomial time (if it exists).

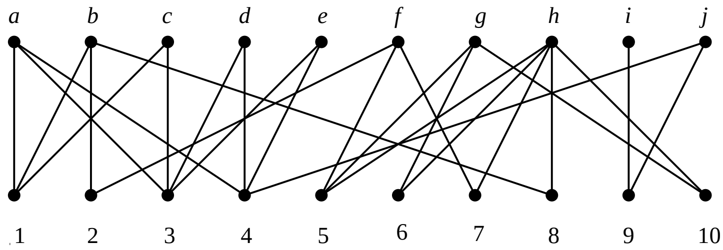

Figure 3.3

3.5. Weighted bipartite matching

We now consider the problem of finding a matching of maximum weight for which we describe the so-called *Hungarian method* developed by Kuhn [1955], using work of Egerváry [1931] (see Corollary 3.7b below).

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{R}$ be a 'weight' function. For any subset M of E define the *weight* $w(M)$ of M by

(21) $$w(M) := \sum_{e \in M} w(e).$$

The maximum-weight matching problem consists of finding a matching of maximum weight.

Again, augmenting paths are of help at this problem. Call a matching M *extreme* if it has maximum weight among all matchings of cardinality $|M|$.

Let M be an extreme matching. Define a 'length' function $l : E \to \mathbb{R}$ as follows:

(22) $$l(e) := \begin{cases} w(e) & \text{if } e \in M, \\ -w(e) & \text{if } e \notin M. \end{cases}$$

Then the following holds:

Proposition 1. *Let P be an M-augmenting path of minimum length. If M is extreme, then $M' := M \triangle EP$ is extreme again.*

Proof. Let N be any extreme matching of size $|M| + 1$. As $|N| > |M|$, $M \cup N$ has a component Q that is an M-augmenting path. As P is a shortest M-augmenting path, we know $l(Q) \geq l(P)$. Moreover, as $N \triangle EQ$ is a matching of size $|M|$, and as M is extreme, we know $w(N \triangle EQ) \leq w(M)$. Hence

(23) $$w(N) = w(N \triangle EQ) - l(Q) \leq w(M) - l(P) = w(M').$$

Hence M' is extreme. ∎

This implies that if we are able to find a minimum-length M-augmenting path in polynomial time, we can find a maximum-weight matching in polynomial time: find iteratively extreme matchings $M_0, M_1, \ldots$ such that $|M_k| = k$ for each k. Then the matching among $M_0, M_1, \ldots$ of maximum weight is a maximum-weight matching.

If G is bipartite, we can find a minimum-length M-augmenting path as follows. Let G have colour classes U and W. Orient the edges of G as in (19), making the directed graph D, and let U' and W' as in (20). Then a minimum-length M-augmenting path can be found by finding a minimum-length path in D from any vertex in U' to any vertex in W'. This can be done in polynomial time, since:

Theorem 3.5. *Let M be an extreme matching. Then D has no directed circuit of negative length.*

Proof. Suppose C is a directed circuit in D with length $l(C) < 0$. We may assume $C = (u_0, w_1, u_1, \ldots, w_t, u_t)$ with $u_0 = u_t$ and $u_1, \ldots, u_t \in U$ and $w_1, \ldots, w_t \in W$. Then the edges $w_1u_1, \ldots, w_tu_t$ belong to M and the edges $u_0w_1, u_1w_2, \ldots, u_{t-1}w_t$ do not belong to M. Then $M'' := M \triangle EC$ is a matching of cardinality k of weight $w(M'') = w(M) - l(C) > w(M)$, contradicting the fact that M is extreme. ∎

This gives a polynomial-time algorithm to find a maximum-weight matching in a bipartite graph. The description above yields:

Theorem 3.6. *A maximum-weight matching in a bipartite graph $G = (V, E)$ can be found in $O(|V|^2|E|)$ time.*

Proof. We do $O(|V|)$ iterations, each consisting of finding a shortest path (in a graph without negative-length directed circuits), which can be done in $O(|V||E|)$ time (with the Bellman-Ford algorithm — see Corollary 1.10a). ∎

In fact, a sharpening of this method (by transmitting a 'potential' $p : V \to \mathbb{Q}$ throughout the matching augmenting iterations, making the length function l nonnegative, so that Dijkstra's method can be used) gives an $O(|V|(|E| + |V| \log |V|))$ algorithm.

Application 3.2: Optimal assignment. Suppose that we have n jobs and m machines and that each job can be done on each machine. Moreover, let a cost function (or cost matrix) $k_{i,j}$ be given, specifying the cost of performing job j by machine i. We want to perform the jobs with a minimum of total costs.

This can be solved with the maximum-weight bipartite matching algorithm. To this end, we make a complete bipartite graph G with colour classes of cardinality m and n. Let K be the maximum of $k_{i,j}$ over all i, j. Define the weight of the edge connecting machine i and job j to be equal to $K - k_{i,j}$. Then a maximum-weight matching in G corresponds to an optimum assignment of machines to jobs.

So the algorithm for solving the assignment problem counters the remarks made by Thorndike [1950] in an Address delivered on September 9, 1949 at a meeting of the American Psychological Association at Denver, Colorado:

> There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

Application 3.3: Transporting earth. Monge [1784] was one of the first to consider the assignment problem, in the role of the problem of transporting earth from one area to another, which he considered as the discontinuous, combinatorial problem of transporting molecules:

> Lorsqu'on doit transporter des terres d'un lieu dans un autre, on a coutime de donner le nom de *Déblai* au volume des terres que l'on doit transporter, & le nom de *Remblai* à l'espace qu'elles doivent occuper après le transport.
>
> Le prix du transport d'une molécule étant, toutes choses d'ailleurs égales, proportionnel à son poids & à l'espace qu'on lui fait parcourir, & par conséquent le prix du transport total devant être proportionnel à la somme des produits des molécules multipliées chacune par l'espace parcouru, il s'ensuit que le déblai & le remblai étant donné de figure & de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, daprès laquelle la somme de ces produits sera la moindre possible, & le prix du transport total sera *minimum.*[10]

Monge describes an interesting geometric method to solve the assignment problem in this case: let l be a line touching the two areas from one side; then transport the earth molecule

[10] When one must transport earth from one place to another, one usually gives the name of *Déblai* to the volume of earth that one must transport, & the name of *Remblai* to the space that they should occupy after the transport.

The price of the transport of one molecule being, if all the rest is equal, proportional to its weight & to the distance that one makes it covering, & hence the price of the total transport having to be proportional to the sum of the products of the molecules each multiplied by the distance covered, it follows that, the déblai & the remblai being given by figure and position, it makes difference if a certain molecule of the déblai is transported to one or to another place of the remblai, but that there is a certain distribution to make of the molcules from the first to the second, after which the sum of these products will be as little as possible, & the price of the total transport will be a *minimum.*

touched in one area to the position touched in the other area. Then repeat, until all molecules are transported.

Exercises

3.22. Five mechanics, stationed in the cities A, B, C, D, E, have to perform jobs in the cities F, G, H, I, J. The jobs must be assigned in such a way to the mechanics that everyone gets one job and that the total distance traveled by them is as small as possible. The distances are given in the tables below. Solve these assignment problems with the weighted matching algorithm.

(i)

	F	G	H	I	J
A	6	17	10	1	3
B	9	23	21	4	5
C	2	8	5	0	1
D	19	31	19	20	9
E	21	25	22	3	9

(ii)

	F	G	H	I	J
A	11	5	21	7	18
B	17	4	20	9	25
C	4	1	3	2	4
D	6	2	19	3	9
E	19	7	23	18	26

3.23. Derive from the weighted matching algorithm for bipartite graphs an algorithm for finding a minimum-weight perfect matching in a bipartite graph $G = (V, E)$. (A matching M is *perfect* if $\bigcup M = V$.)

3.24. Let $A_1, \ldots, A_n$ be subsets of the finite set X and let $w : X \to \mathbb{R}_+$ be a 'weight' function. Derive from the weighted matching algorithm a polynomial-time algorithm to find a minimum-weight SDR.

3.6. The matching polytope

The weighted matching problem is related to the 'matching polytope'. Let $G = (V, E)$ be a graph. For each matching M let the *incidence vector* $\chi^M : E \to \mathbb{R}$ of M be defined by:

(24) $$\begin{aligned} &\chi^M(e) := 1 \text{ if } e \in M, \\ &\chi^M(e) := 0 \text{ if } e \notin M, \end{aligned}$$