CÁLCULO EN VARIAS VARIABLES - MA22A

P.1.

(i) Lo que nos conviene hacer es derivar la función $\overline{V}(r,\theta) = V(T(r,\theta)) = V(r\cos(\theta), r\sin(\theta))$
Calculemos:

$$\frac{\partial \overline{V}}{\partial r} = \frac{\partial V}{\partial r}(r\cos(\theta), r\sin(\theta)) = \frac{\partial V}{\partial x}(r\cos(\theta), r\sin(\theta))\cos(\theta) + \frac{\partial V}{\partial y}(r\cos(\theta), r\sin(\theta))\sin(\theta)$$

Donde x, denota la primera variable, e y denota la segunda. Desde ahora omitiremos que V va siempre evaluado en: $(r\cos(\theta), r\sin(\theta))$, para simplificar notación.

$$\frac{\partial^2 \overline{V}}{\partial r^2} = \frac{\partial^2 V}{\partial r^2}(r\cos(\theta), r\sin(\theta)) = (\frac{\partial^2 V}{\partial x^2}\cos(\theta) + \frac{\partial^2 V}{\partial x \partial y}\sin(\theta))\cos(\theta) + (\frac{\partial^2 V}{\partial y \partial x}\cos(\theta) + \frac{\partial^2 V}{\partial y^2}\sin(\theta))\sin(\theta)$$

$$\frac{\partial^2 \overline{V}}{\partial r^2} = \frac{\partial^2 V}{\partial x^2}\cos^2(\theta) + 2\frac{\partial^2 V}{\partial x \partial y}\sin(\theta)\cos(\theta) + \frac{\partial^2 V}{\partial y^2}\sin^2(\theta)$$

$$\frac{\partial \overline{V}}{\partial \theta} = \frac{\partial V}{\partial \theta}(r\cos(\theta), r\sin(\theta)) = \frac{\partial V}{\partial x}(-r\sin(\theta)) + \frac{\partial V}{\partial y}(r\cos(\theta))$$

$$\frac{\partial^2 \overline{V}}{\partial \theta^2} = \frac{\partial^2 V}{\partial \theta^2}(r\cos(\theta), r\sin(\theta)) = \left\{\frac{\partial^2 V}{\partial x^2}(-r\sin(\theta)) + \frac{\partial^2 V}{\partial y \partial x}(r\cos(\theta))\right\}(-r\sin(\theta)) + \frac{\partial V}{\partial x}(-r\cos(\theta))$$
$$+\left\{\frac{\partial^2 V}{\partial y \partial x}(-r\sin(\theta)) + \frac{\partial^2 V}{\partial y^2}(r\cos(\theta))\right\}(r\cos(\theta)) + \frac{\partial V}{\partial y}(-r\sin(\theta))$$

$$\frac{\partial^2 \overline{V}}{\partial \theta^2} = \frac{\partial^2 V}{\partial x^2}(r^2\sin^2(\theta)) - 2\frac{\partial^2 V}{\partial y \partial x}r^2\cos(\theta)\sin(\theta) + \frac{\partial^2 V}{\partial y^2}(r^2\cos^2(\theta)) - \frac{\partial V}{\partial x}(r\cos(\theta)) - \frac{\partial V}{\partial y}(r\sin(\theta))$$

Ahora debemos multiplicar como dice el enunciado y sumar para obtener la igualdad pedida.

(ii) Usando la ecuación obtenida anteriormente:

$$\Delta V(r,\theta) = \frac{\partial^2 \overline{V}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \overline{V}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \overline{V}}{\partial r} = 0$$

Como suponemos que el potencial no depende del angulo, entonces:

$$V(r,\theta) = V(r) \Rightarrow \frac{\partial \overline{V}}{\partial \theta} = 0 \Rightarrow \frac{\partial^2 \overline{V}}{\partial \theta^2} = 0$$

Con esto la ecuación de Poisson queda resumida en una E.D.O. en la variable r.

$$\Delta V(r) = \frac{\partial^2 \overline{V}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{V}}{\partial r} = 0$$

Despejando para integrar, se obtiene:

$$r\frac{d[V'(r)]'}{dr} + [V'(r)]' = 0 \Rightarrow \frac{d[V'(r)]'}{V'(r)} = -\frac{dr}{r}$$

Integrando, se obtiene que:

$$Ln(V'(r)) = -Ln(r) + C \Rightarrow V'(r) = C_1\frac{1}{r} \Rightarrow V(r) = C_1 Ln(r) + C_2 = \alpha Ln(\frac{r}{\beta})$$