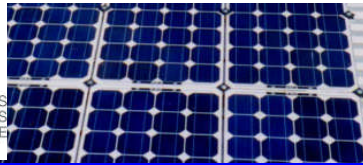




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FI33A ELECTROMAGNETISMO

Clase 8

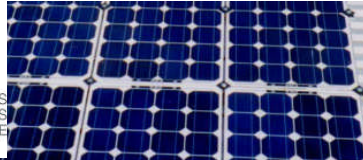
Medios Materiales III

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Area de Energía
Departamento de Ingeniería Eléctrica
Universidad de Chile



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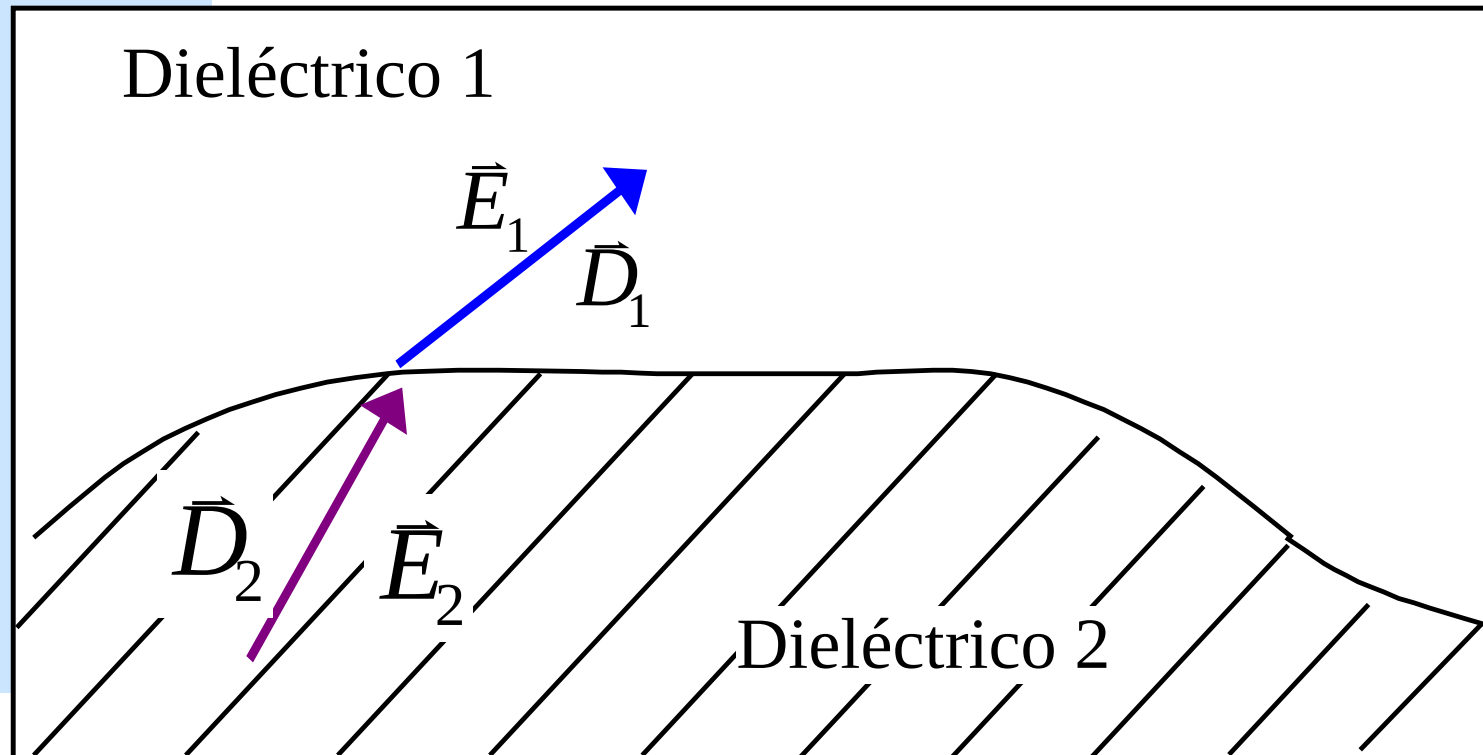


INDICE

- Condiciones de borde para el campo eléctrico
- Ejemplo
- Refracción del campo eléctrico
- Consideraciones sobre Simetría



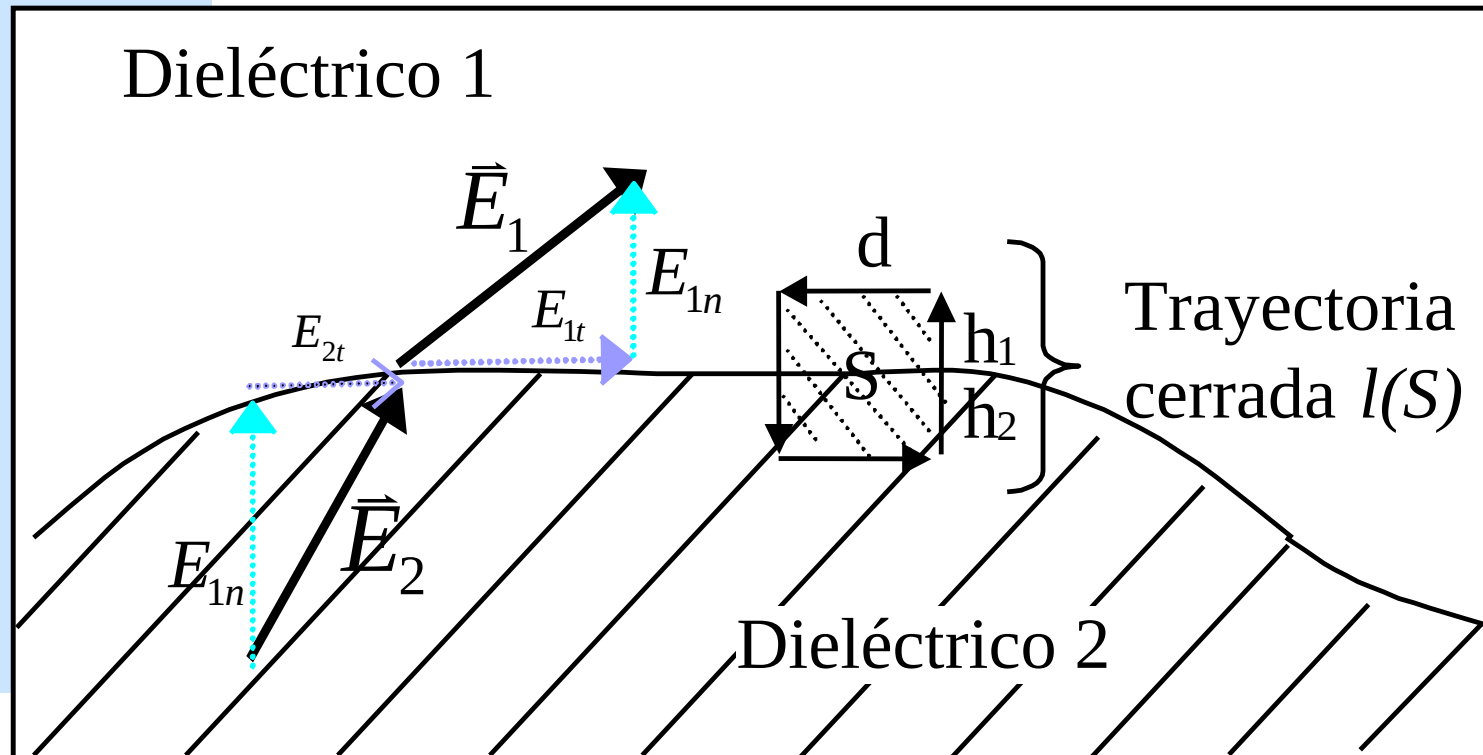
Condiciones de borde



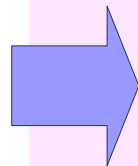
Usaremos dos ecuaciones $\nabla \times \vec{E} = 0$ y $\nabla \cdot \vec{D} = \rho$



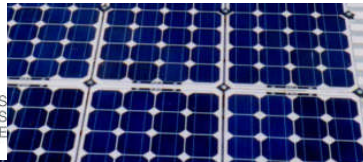
Condiciones de borde para el campo eléctrico



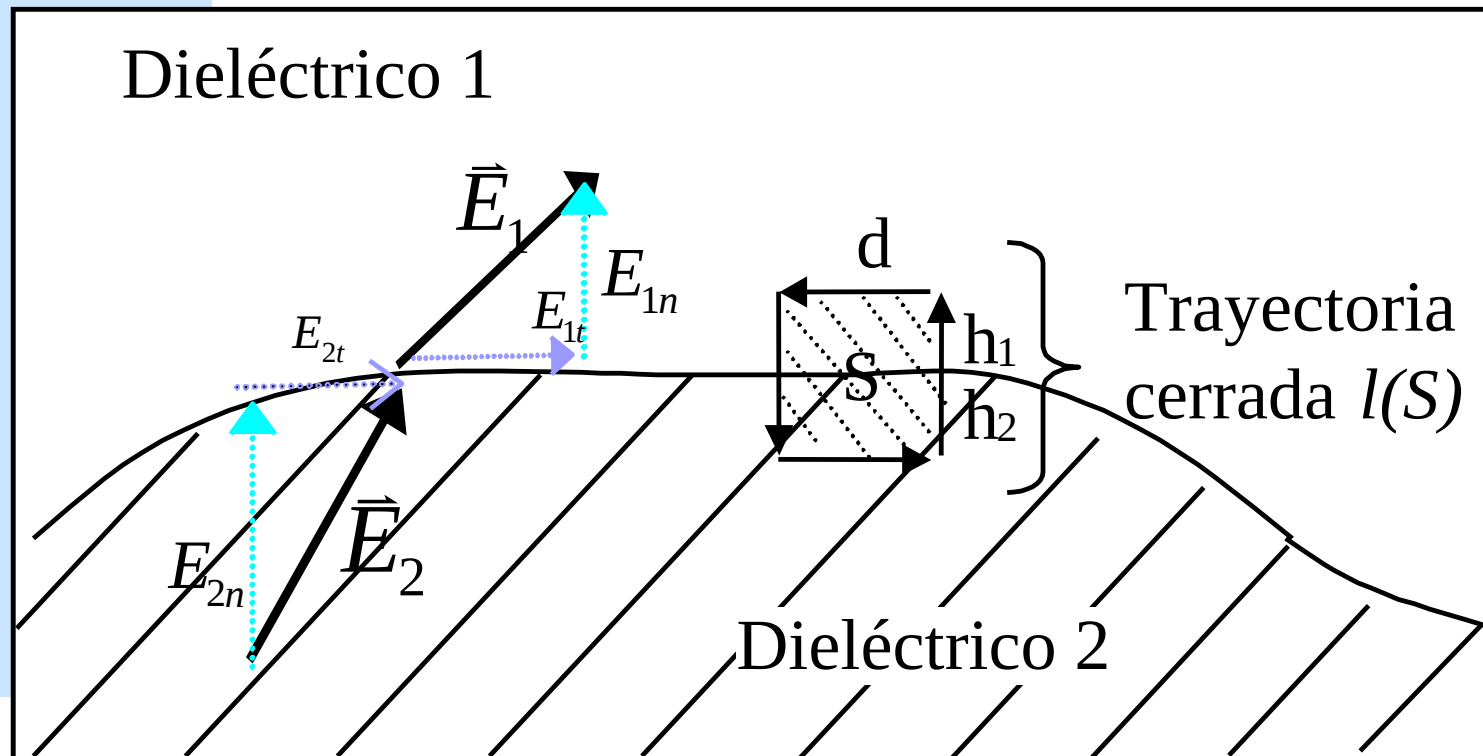
$$\nabla \times \vec{E} = 0$$



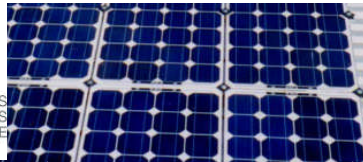
$$\iint_S \nabla \times \vec{E} \cdot d\vec{s} = 0 \Rightarrow \oint_{l(S)} \vec{E} \cdot d\vec{l} = 0$$



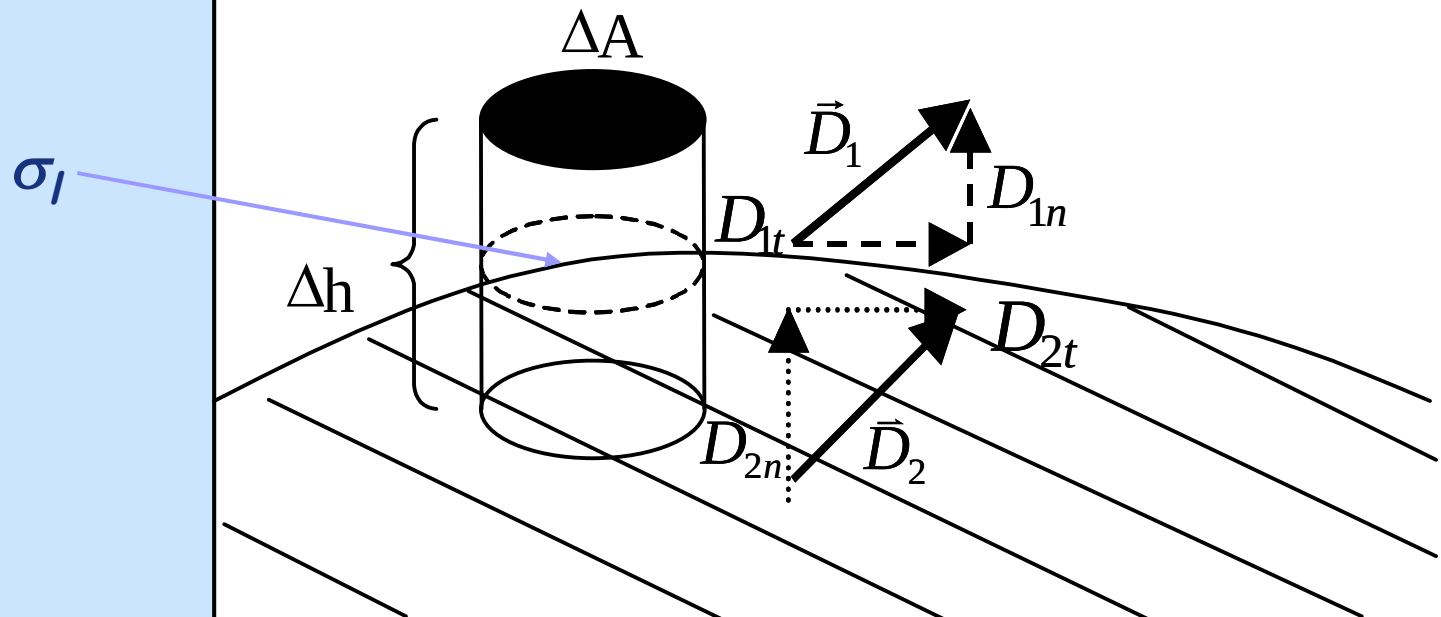
Condiciones de borde para el campo eléctrico



$$\oint_{l(S)} \vec{E} \cdot d\vec{l} = 0 \quad \Rightarrow \quad -E_{1t}d - E_{1n}h_1 - E_{2n}h_2 + E_{2t}d + E_{2n}h_2 + E_{1n}h_1 = 0$$
$$l(S) \quad h_1 \rightarrow 0, \quad h_2 \rightarrow 0 \Rightarrow -E_{1t}d + E_{2t}d = 0 \quad \therefore E_{1t} = E_{2t} \quad \therefore \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$



Condiciones de borde para el campo eléctrico



$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \quad \text{y} \quad Q_{\text{libre}} = \sigma_l \Delta A \quad \Rightarrow \quad D_{1n} \Delta A - D_{2n} \Delta A + \iint_{\text{manto}} \vec{D} \cdot d\vec{S} = \sigma_l \Delta A$$

$$\Delta h \rightarrow 0 \Rightarrow \iint_{\text{manto}} \vec{D} \cdot d\vec{S} = 0 \Rightarrow D_{1n} - D_{2n} = \sigma_l$$

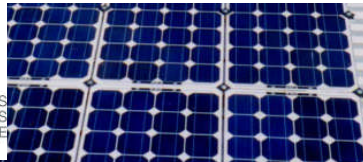
$$\text{si } \sigma_l = 0 \quad \left\{ \begin{array}{l} D_{1n} = D_{2n} \\ \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \end{array} \right.$$

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \sigma_l$$



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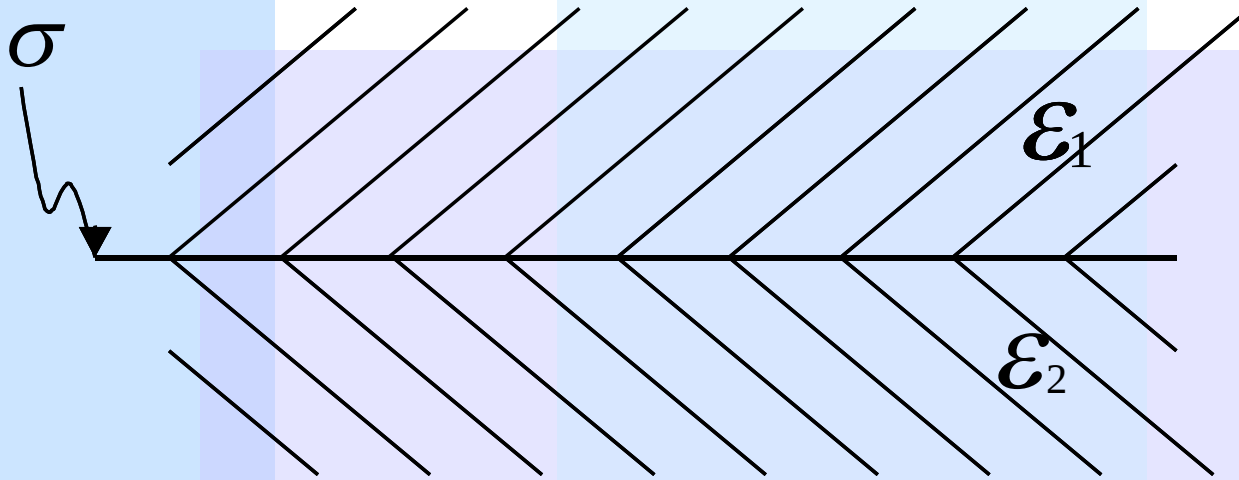
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Ejemplo

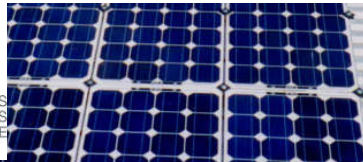
Ejemplo

σ

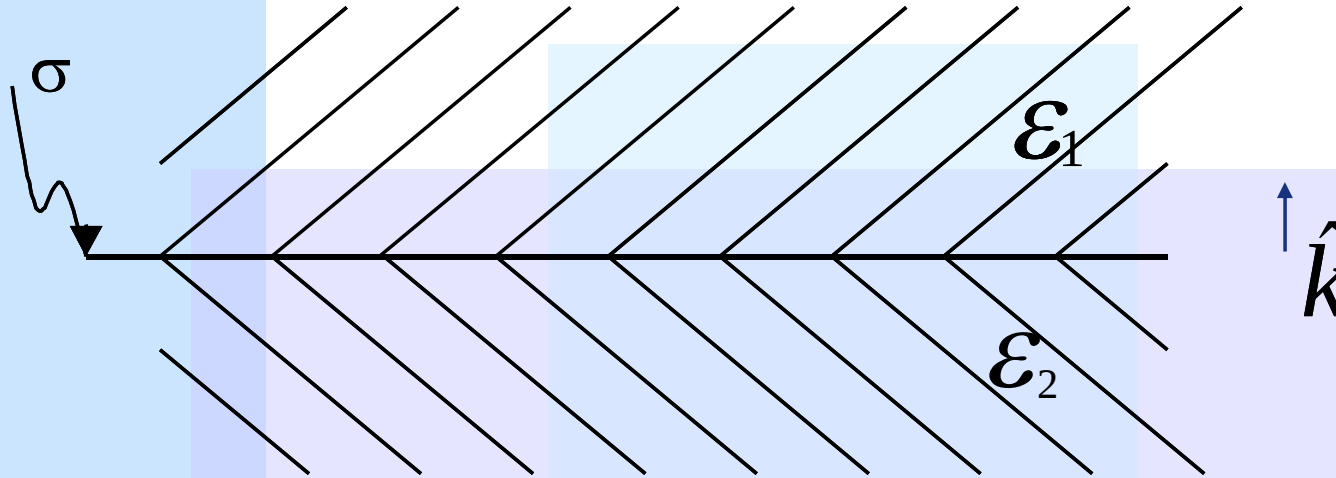


ϵ_1, ϵ_2 constantes

$\vec{E}, \vec{D}$ en todo el espacio?



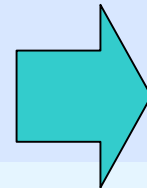
Ejemplo



$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

y

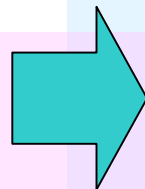
$$\vec{D} = \epsilon \vec{E}$$



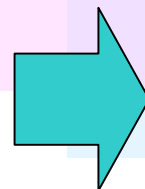
$$\vec{P} = \vec{D} - \epsilon_0 \frac{\vec{D}}{\epsilon} = \left(\frac{\epsilon - \epsilon_0}{\epsilon} \right) \vec{D}$$

además

$$\rho_L = \nabla \cdot \vec{D} = 0$$



Fuentes de $\vec{D}$ son sólo cargas libres σ , luego $\vec{D} = D_i \hat{k}$ con D_i constante



$$\vec{D} = D \hat{k}, \quad \vec{E} = E \hat{k} \quad \text{y} \quad \vec{P} = P \hat{k}$$

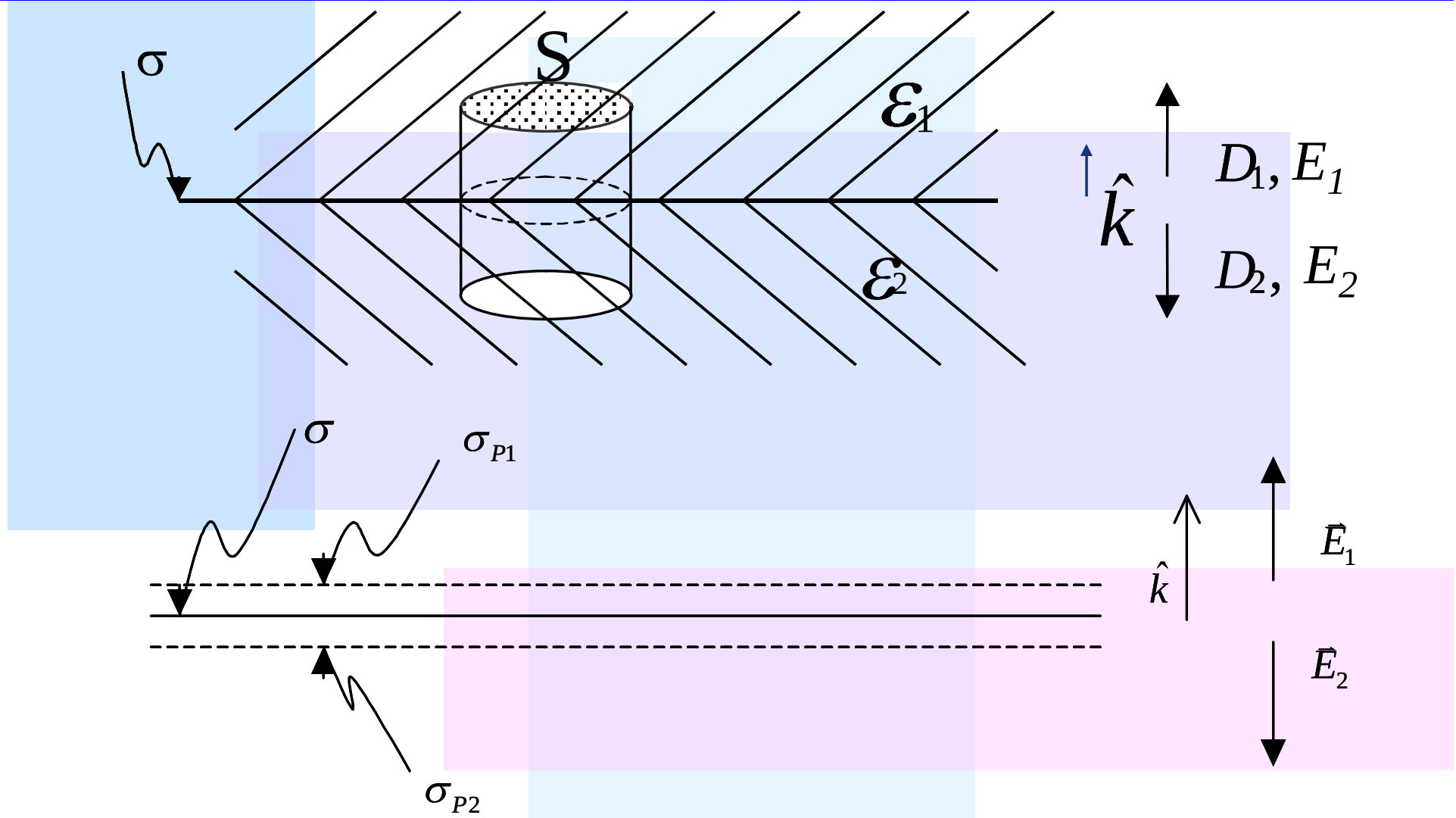


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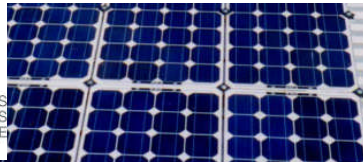
Ejemplo





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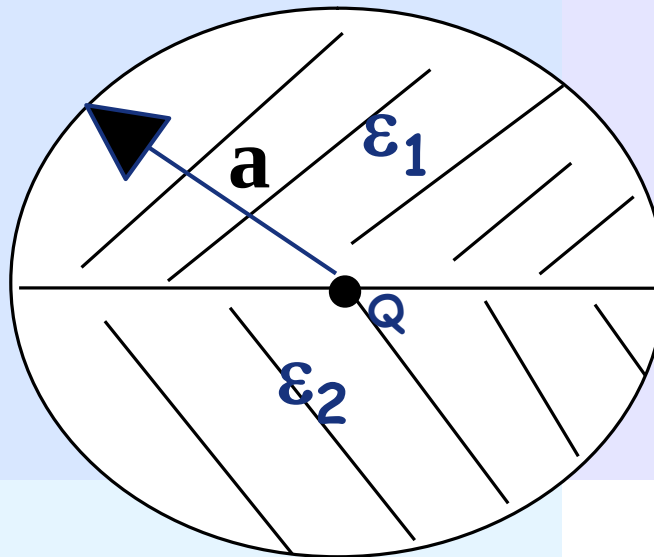
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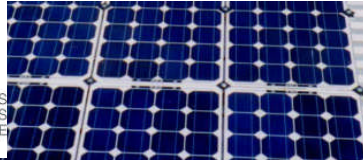


Consideraciones sobre Simetría

I. Caso dos medios con carga puntual Q en el centro

Calcular E y D
dentro de la
esfera de radio a





Consideraciones sobre Simetría

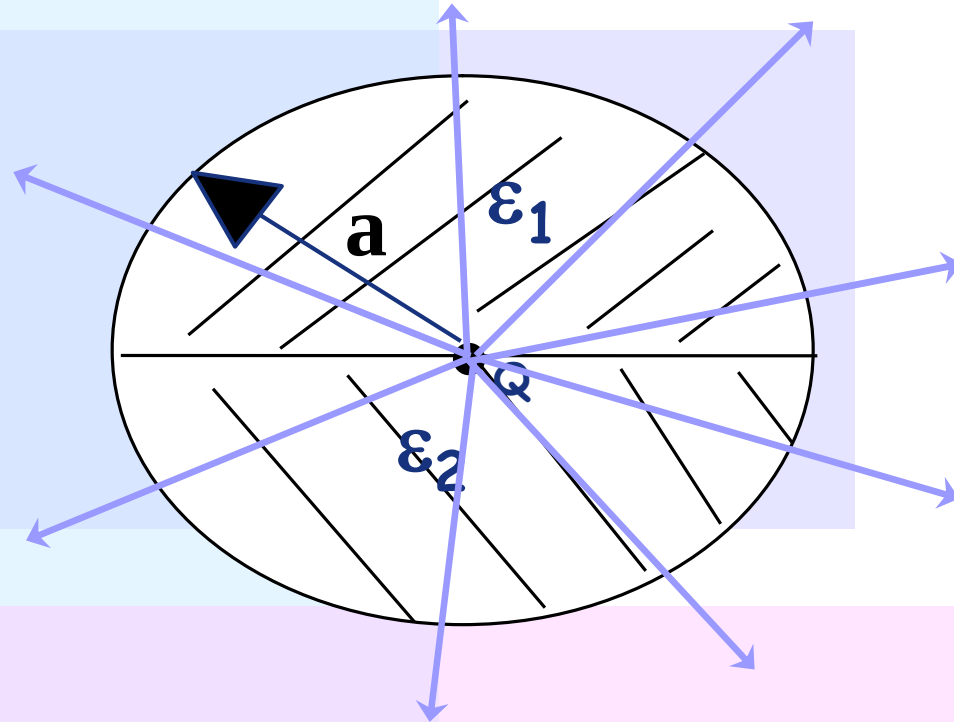
I. Caso dos medios con carga puntual Q en el centro

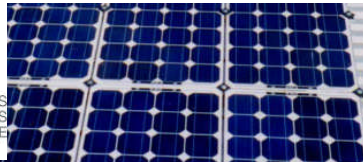
Supuesto:

Campos son radiales

$$\vec{D}_1 = D_1(r)\hat{r}, \quad \vec{D}_2 = D_2(r)\hat{r},$$

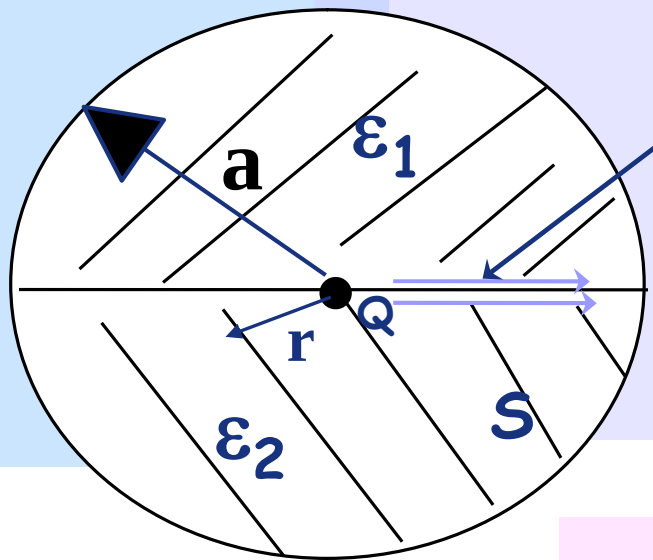
$$\vec{E}_1 = E_1(r)\hat{r}, \quad \vec{E}_2 = E_2(r)\hat{r},$$





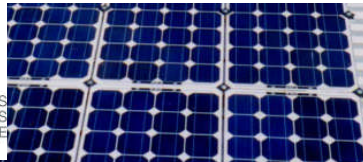
Consideraciones sobre Simetría

I. Caso dos medios con carga puntual Q en el centro



Condición
de Borde

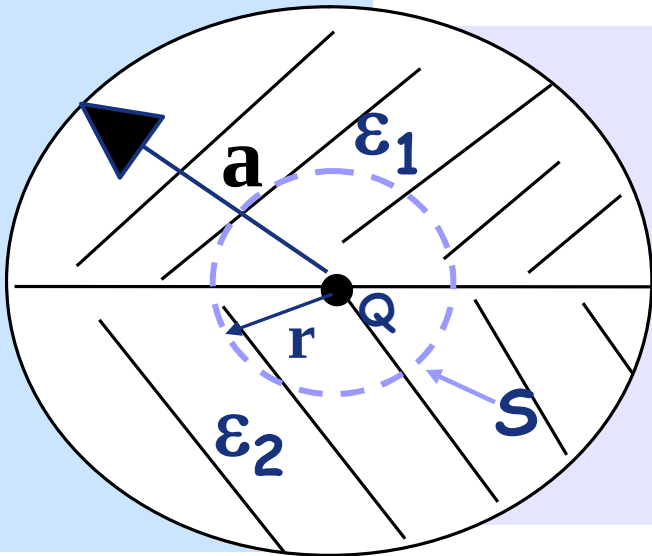
$$E_{1t} = E_{2t} \Rightarrow \begin{cases} E_1(r) = E_2(r) \\ \frac{D_1(r)}{\epsilon_1} = \frac{D_2(r)}{\epsilon_2} \end{cases}$$



Consideraciones sobre Simetría

I. Caso dos medios con carga puntual Q en el centro

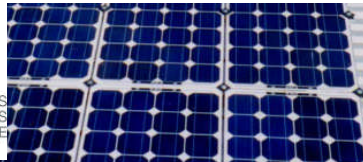
y aplicando la Ley de Gauss



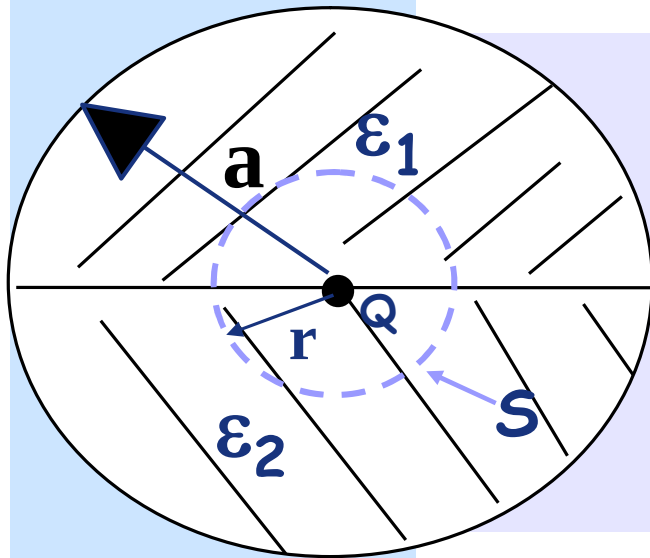
$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \Rightarrow$$

$$\iint_{\text{ZONAI}} \vec{D}_1 \cdot d\vec{s} + \iint_{\text{ZONAII}} \vec{D}_2 \cdot d\vec{s} = Q \Rightarrow$$

$$D_1 2\pi r^2 + D_2 2\pi r^2 = Q$$



Consideraciones sobre Simetría

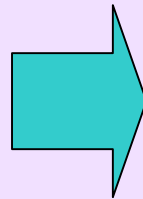


$$\left. \begin{aligned} D_1 2\pi r^2 + D_2 2\pi r^2 &= Q \\ \epsilon_1 D_2 &= \epsilon_1 D_2 \end{aligned} \right\} \Rightarrow$$

$$\vec{D}_1 = \frac{\epsilon_1 Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}, \quad \vec{D}_2 = \frac{\epsilon_2 Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$

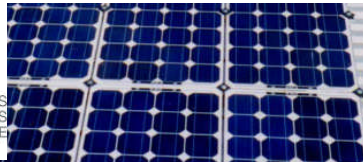
$$\vec{E}_1 = \vec{E}_2 = \frac{Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = (\epsilon - \epsilon_0) \vec{E}$$



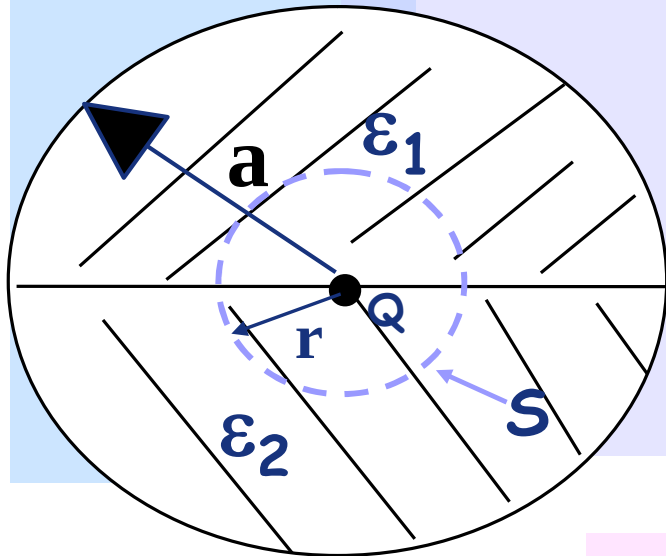
$$\vec{P}_1 = \frac{(\epsilon_1 - \epsilon_0)Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$

$$\vec{P}_2 = \frac{(\epsilon_2 - \epsilon_0)Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$



Consideraciones sobre Simetría

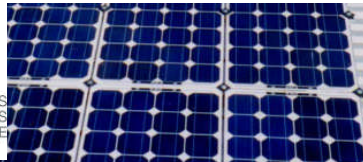
$$\rho_P = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cdot P_\theta) - \frac{1}{r \sin \theta} \frac{\partial P_\phi}{\partial \phi}$$



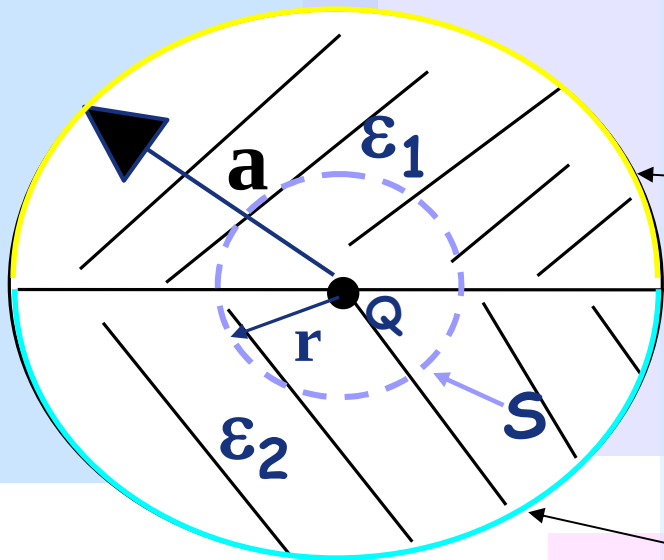
$$\vec{P} = \frac{(\varepsilon - \varepsilon_0)Q}{2\pi(\varepsilon_1 + \varepsilon_2)r^2} \hat{r}$$

$$\Rightarrow \rho_P = -\nabla \cdot \vec{P} = 0$$

$$\sigma_{Pi} = \vec{P} \cdot \hat{n} = \left. \frac{(\varepsilon_i - \varepsilon_0)Q}{2\pi(\varepsilon_1 + \varepsilon_2)r^2} \right|_{r=a} \quad \hat{r} \cdot \hat{r} = \frac{(\varepsilon_i - \varepsilon_0)Q}{2\pi(\varepsilon_1 + \varepsilon_2)a^2}$$



Consideraciones sobre Simetría



$$\vec{P}_i = \frac{(\epsilon_i - \epsilon_0)Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$

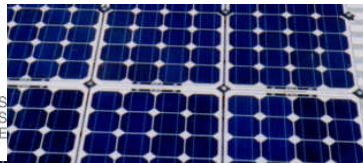
$$\sigma_{P1} = \vec{P}_1 \cdot \hat{n} = \frac{(\epsilon_1 - \epsilon_0)Q}{2\pi(\epsilon_1 + \epsilon_2)a^2}$$

$$\sigma_{P2} = \vec{P}_2 \cdot \hat{n} = \frac{(\epsilon_2 - \epsilon_0)Q}{2\pi(\epsilon_1 + \epsilon_2)a^2}$$



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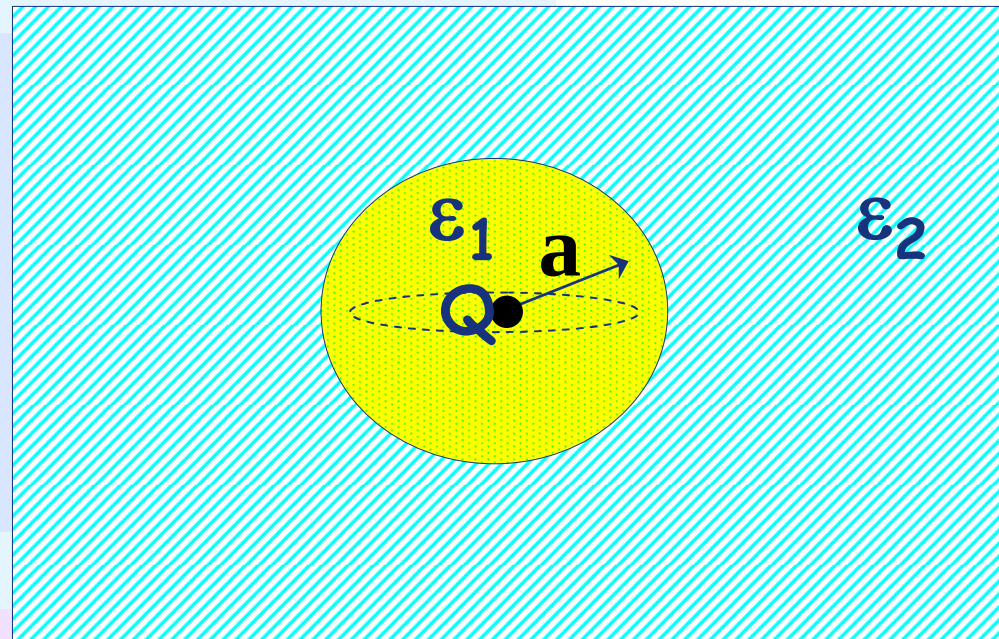
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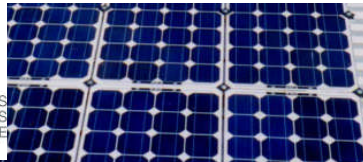


Consideraciones sobre Simetría

II. Caso dos medios con carga puntual Q en el centro

Calcular E y D en
todo el espacio



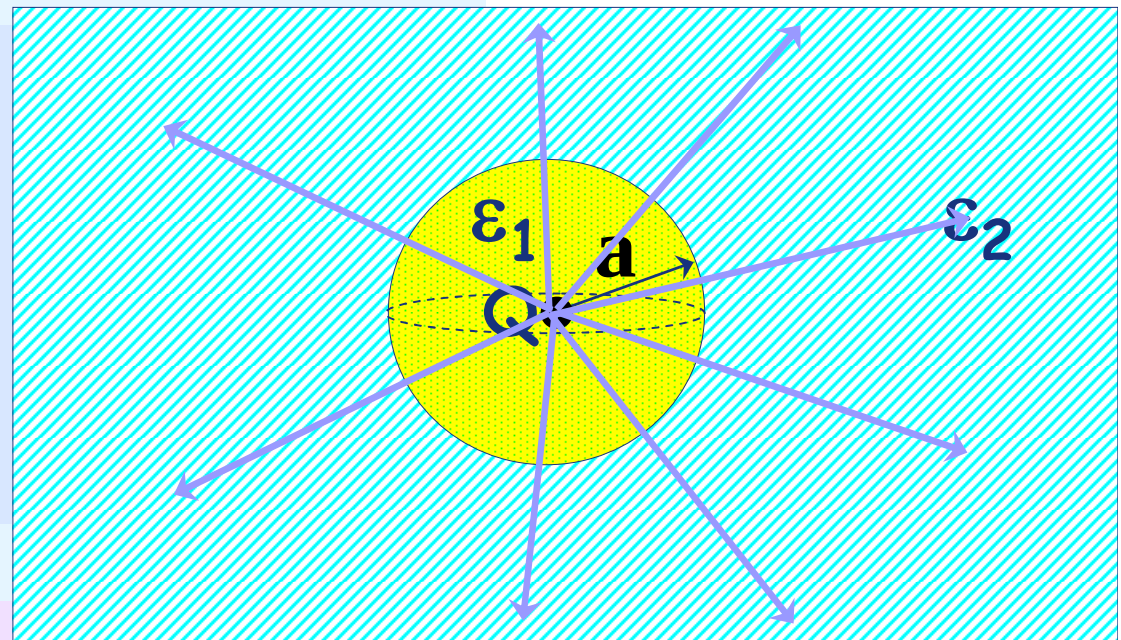


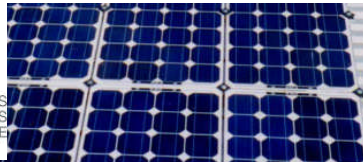
Consideraciones sobre Simetría

II. Caso dos medios con carga puntual Q en el centro

Campos son radiales

$$\begin{aligned}\vec{D}_1 &= D_1(r)\vec{r}, & \vec{D}_2 &= D_2(r)\vec{r}, \\ \vec{E}_1 &= E_1(r)\vec{r}, & \vec{E}_2 &= E_2(r)\vec{r},\end{aligned}$$





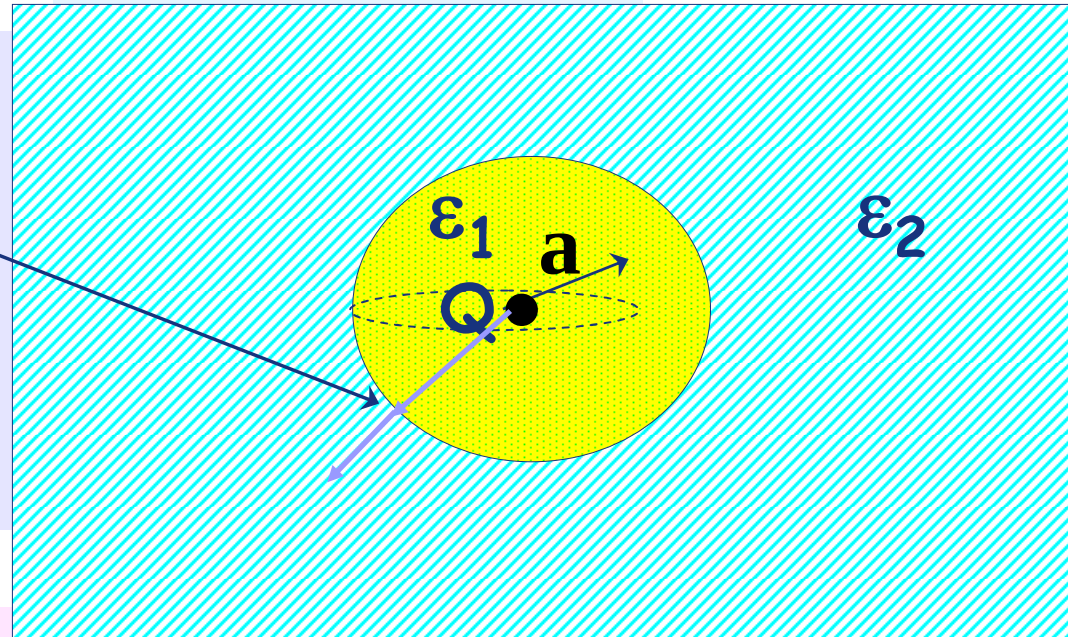
Consideraciones sobre Simetría

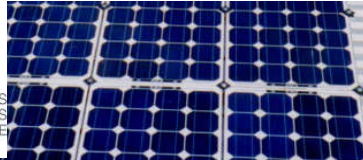
II. Caso dos medios con carga puntual Q en el centro

Condición de
Borde en $r = a$

$$\vec{D}_1(r=a) = \vec{D}_2(r=a)$$

NO hay carga libre
en la interfaz





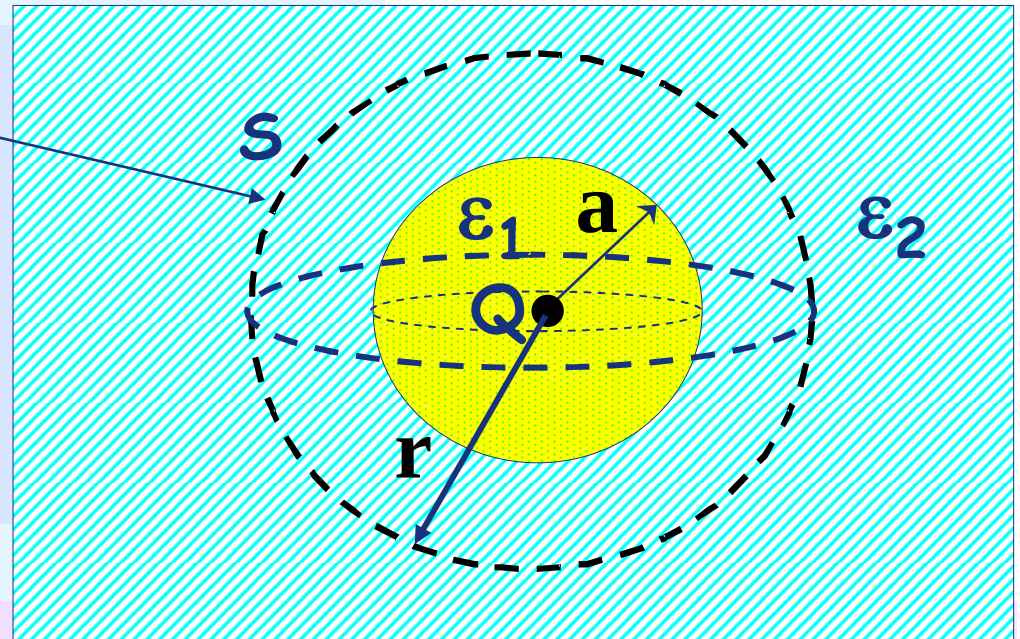
Consideraciones sobre Simetría

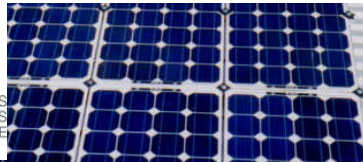
II. Caso dos medios con carga puntual Q en el centro

Aplicando la ley de Gauss en S

$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \Rightarrow 4\pi r^2 D(r) = Q$$

$$\Rightarrow \vec{D}_2(r) = \frac{Q}{4\pi r^2} \hat{r}, \quad \vec{E}_2(r) = \frac{Q}{4\pi \epsilon_2 r^2} \hat{r}$$





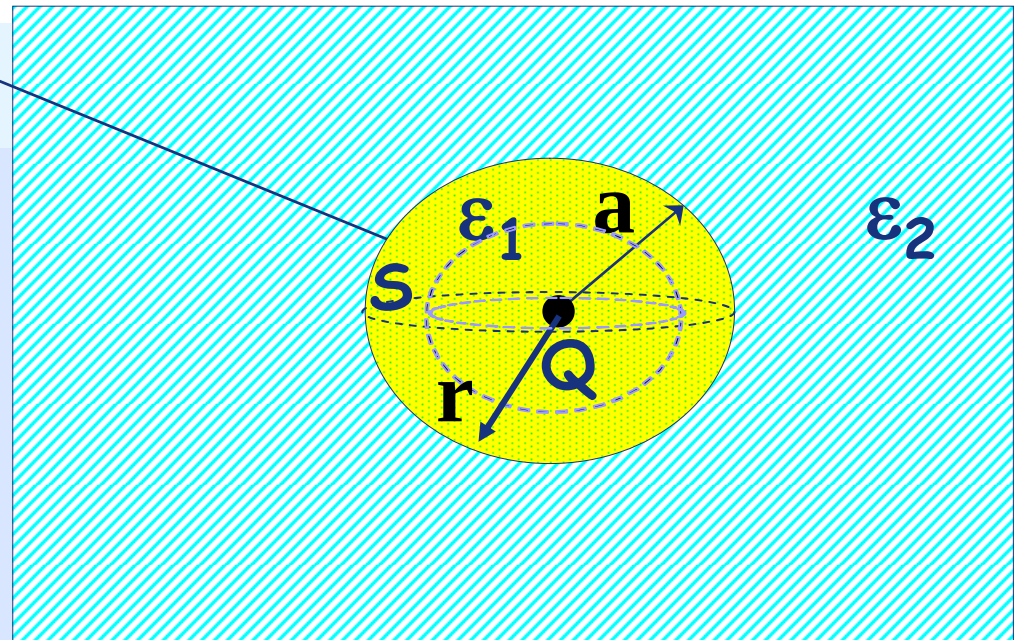
Consideraciones sobre Simetría

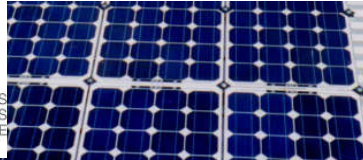
Aplicando la ley de Gauss en S

$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \Rightarrow 4\pi r^2 D(r) = Q$$

$$\Rightarrow \vec{D}_1(\vec{r}) = \frac{Q}{4\pi r^2} \hat{r}, \quad \vec{E}_1(\vec{r}) = \frac{Q}{4\pi\epsilon_1 r^2} \hat{r}$$

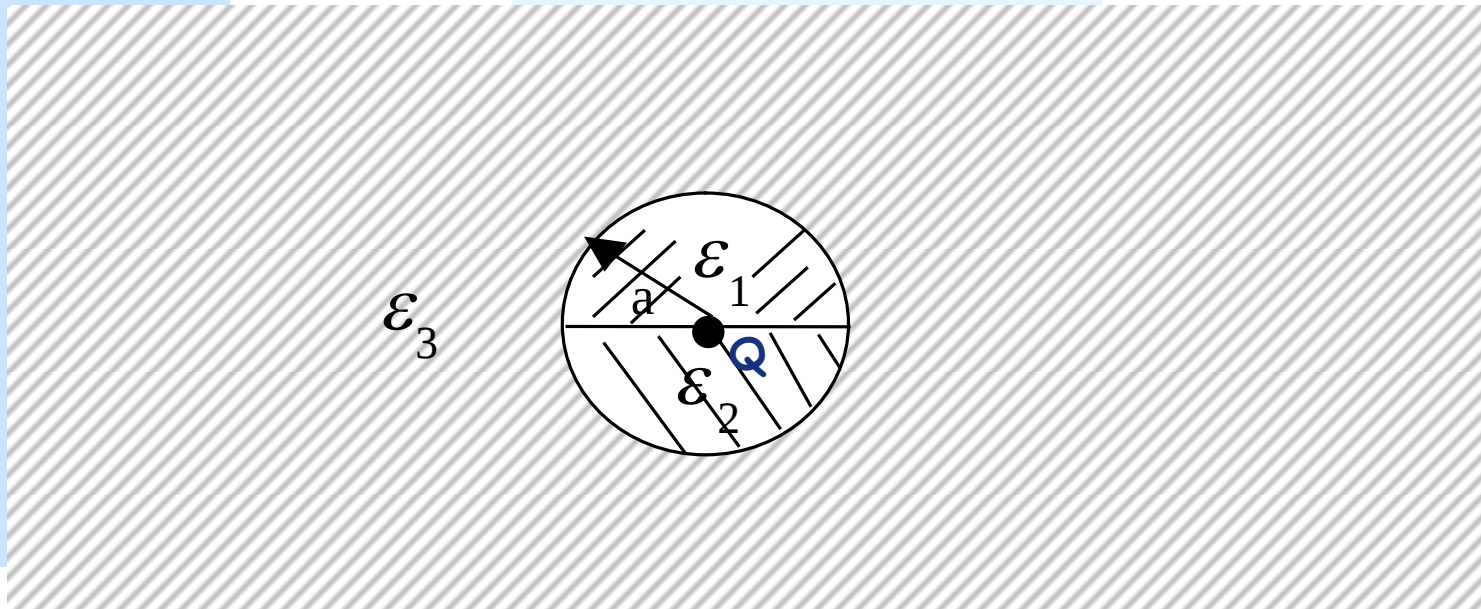
Notar que $D_1 = D_2$ en todo el espacio

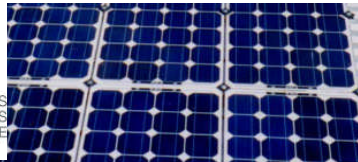




Consideraciones sobre Simetría

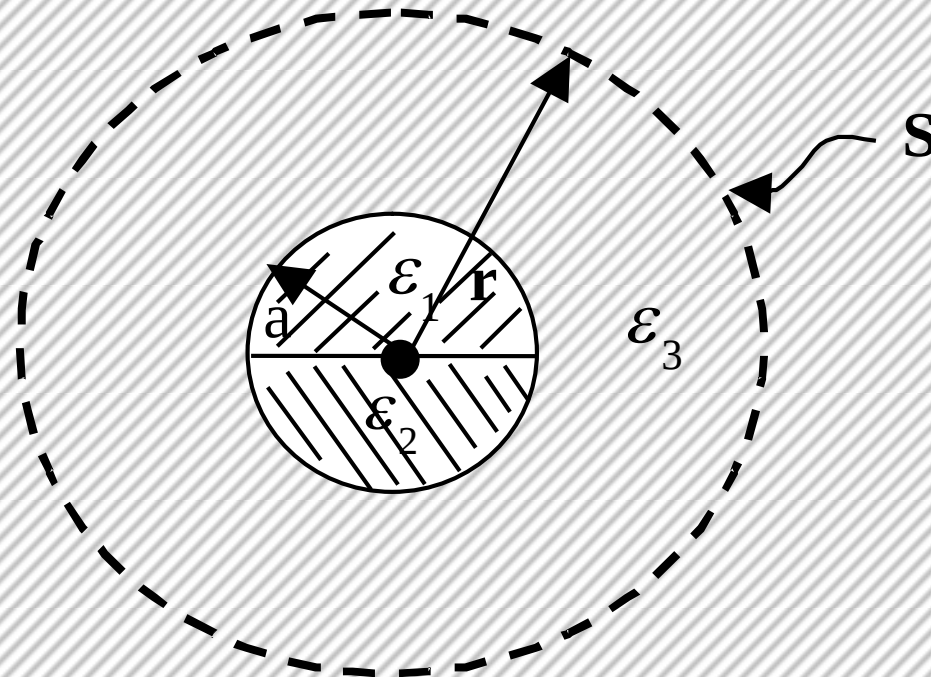
III. Caso tres medios con carga puntual Q en el centro



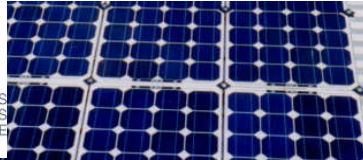


Consideraciones sobre Simetría

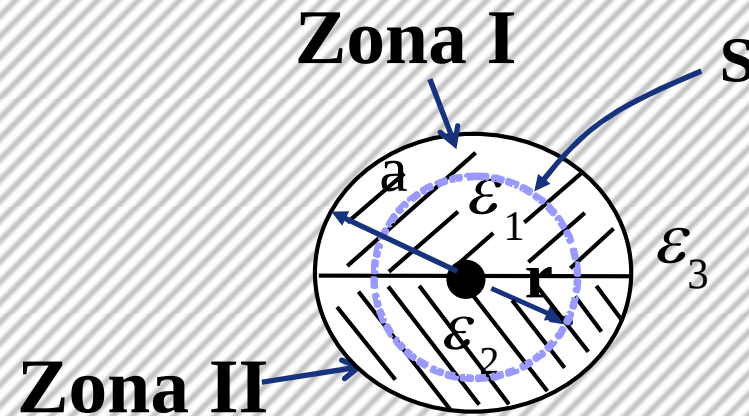
Zona III



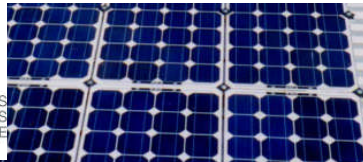
$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \Rightarrow 4\pi r^2 D(r) = Q \Rightarrow \vec{D}_3(r) = \frac{Q}{4\pi r^2} \hat{r}, \quad \vec{E}_3(r) = \frac{Q}{4\pi \epsilon_3 r^2} \hat{r}$$



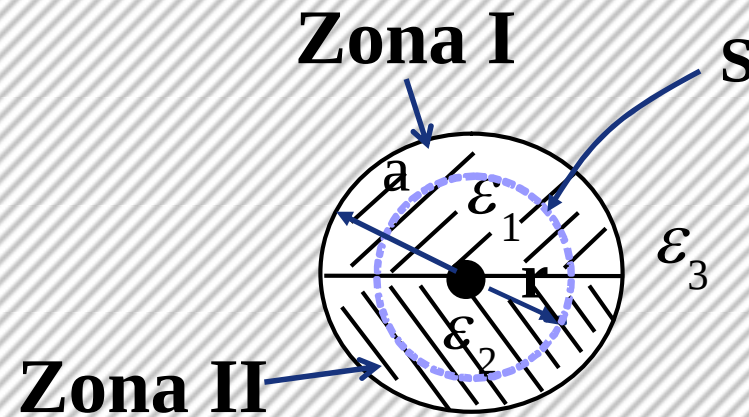
Consideraciones sobre Simetría



Para $0 < r < a$ tenemos dos medios. En la superficie de separación la componente tangencial del campo es la misma

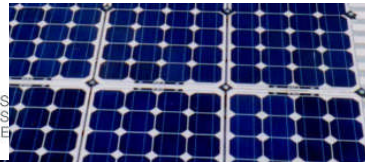


Consideraciones sobre Simetría

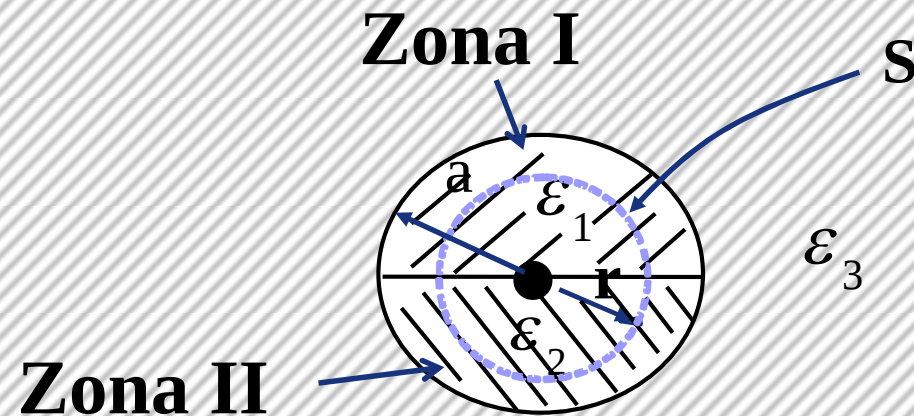


Luego dado que los campos son radiales, se debe cumplir:

$$E_{1t} = E_{2t} \Rightarrow \begin{cases} E_1(r) = E_2(r) \\ \frac{D_1(r)}{\epsilon_1} = \frac{D_2(r)}{\epsilon_2} \end{cases}$$

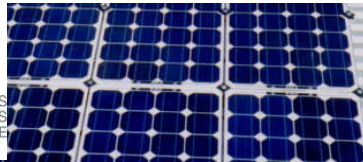


Consideraciones sobre Simetría

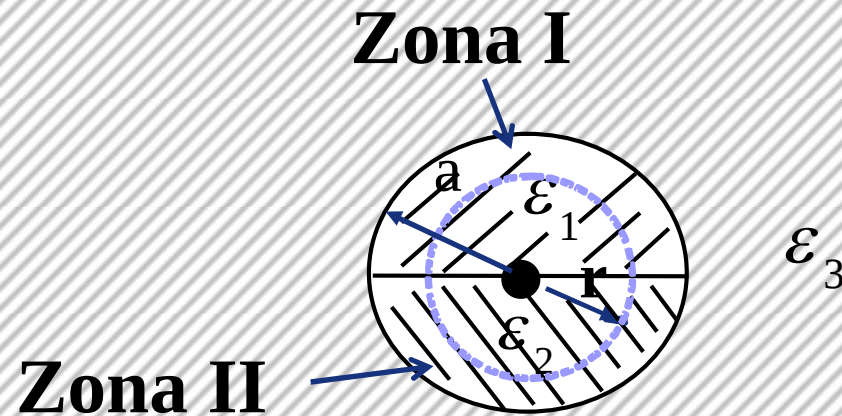


Aplicando la Ley de Gauss:

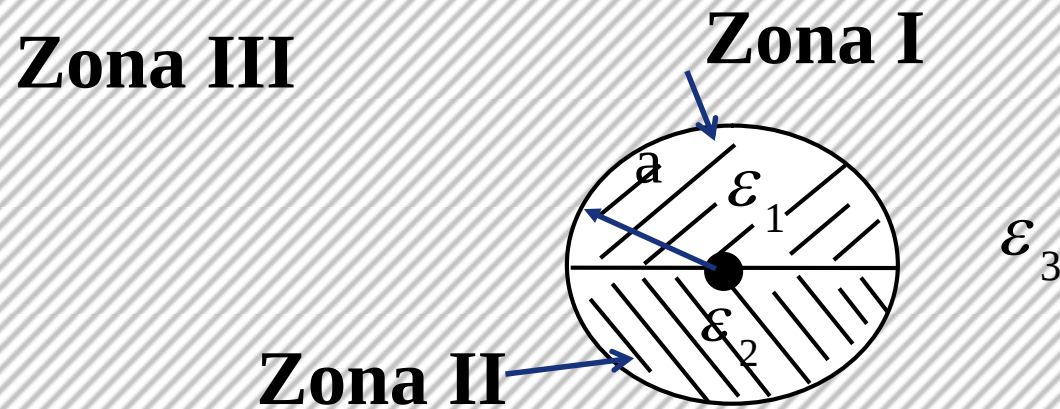
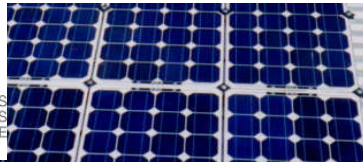
$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{libre}} \Rightarrow \iint_{\text{ZONA I}} \vec{D}_1 \cdot d\vec{S} + \iint_{\text{ZONA II}} \vec{D}_2 \cdot d\vec{S} = Q \Rightarrow D_1 2\pi r^2 + D_2 2\pi r^2 = Q$$



Consideraciones sobre Simetría

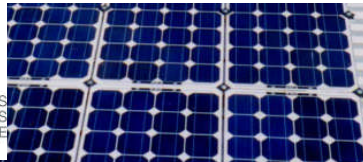


$$\left. \begin{array}{l} D_1 2\pi r^2 + D_2 2\pi r^2 = Q \\ \epsilon_1 D_2 = \epsilon_2 D_1 \end{array} \right\} \Rightarrow \bar{D}_1 = \frac{\epsilon_1 Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}, \quad \bar{D}_2 = \frac{\epsilon_2 Q}{2\pi(\epsilon_1 + \epsilon_2)r^2} \hat{r}$$



En resumen: $\vec{D}_1 = \frac{\varepsilon_1 Q}{2\pi(\varepsilon_1 + \varepsilon_2)r^2} \hat{r}, \quad \vec{D}_2 = \frac{\varepsilon_2 Q}{2\pi(\varepsilon_1 + \varepsilon_2)r^2} \hat{r}, \quad \vec{D}_3 = \frac{Q}{4\pi r^2} \hat{r}$

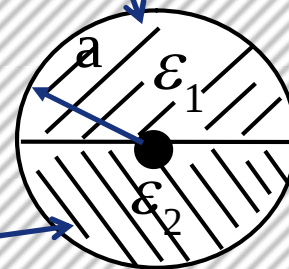
Notar que si $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \Rightarrow \vec{D}_1 = \vec{D}_2 = \vec{D}_3 = \frac{Q}{4\pi r^2} \hat{r}$



Zona III

Zona I

Zona II

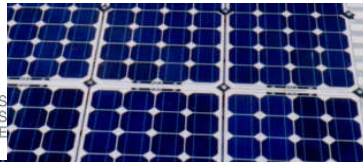


ϵ_3

Pero si aplicamos la condición de borde para D en $r=a$:

$$\bar{D}_1(r=a) = \bar{D}_3(r=a) \Rightarrow \frac{\epsilon_1 Q}{2\pi(\epsilon_1 + \epsilon_2)a^2} = \frac{Q}{4\pi a^2} \Rightarrow 2\epsilon_1 = (\epsilon_1 + \epsilon_2)$$

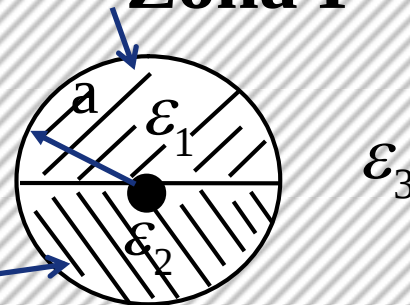
$$\bar{D}_2(r=a) = \bar{D}_3(r=a) \Rightarrow \frac{\epsilon_2 Q}{2\pi(\epsilon_1 + \epsilon_2)a^2} = \frac{Q}{4\pi a^2} \Rightarrow 2\epsilon_2 = (\epsilon_1 + \epsilon_2)$$



Zona III

Zona I

Zona II

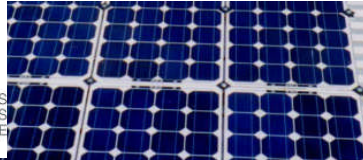


¿...?

Pero si aplicamos la condición de borde para D en r=a:

$$\bar{D}_1(r=a) = \bar{D}_3(r=a) \Rightarrow \frac{\varepsilon_1 Q}{2\pi(\varepsilon_1 + \varepsilon_2)a^2} = \frac{Q}{4\pi a^2} \Rightarrow 2\varepsilon_1 = (\varepsilon_1 + \varepsilon_2)$$

$$\bar{D}_2(r=a) = \bar{D}_3(r=a) \Rightarrow \frac{\varepsilon_2 Q}{2\pi(\varepsilon_1 + \varepsilon_2)a^2} = \frac{Q}{4\pi a^2} \Rightarrow 2\varepsilon_2 = (\varepsilon_1 + \varepsilon_2)$$

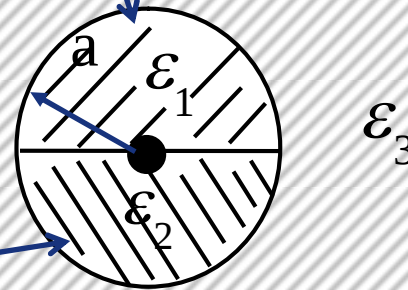


Consideraciones sobre Simetría

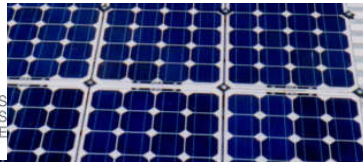
Zona III

Zona I

Zona II

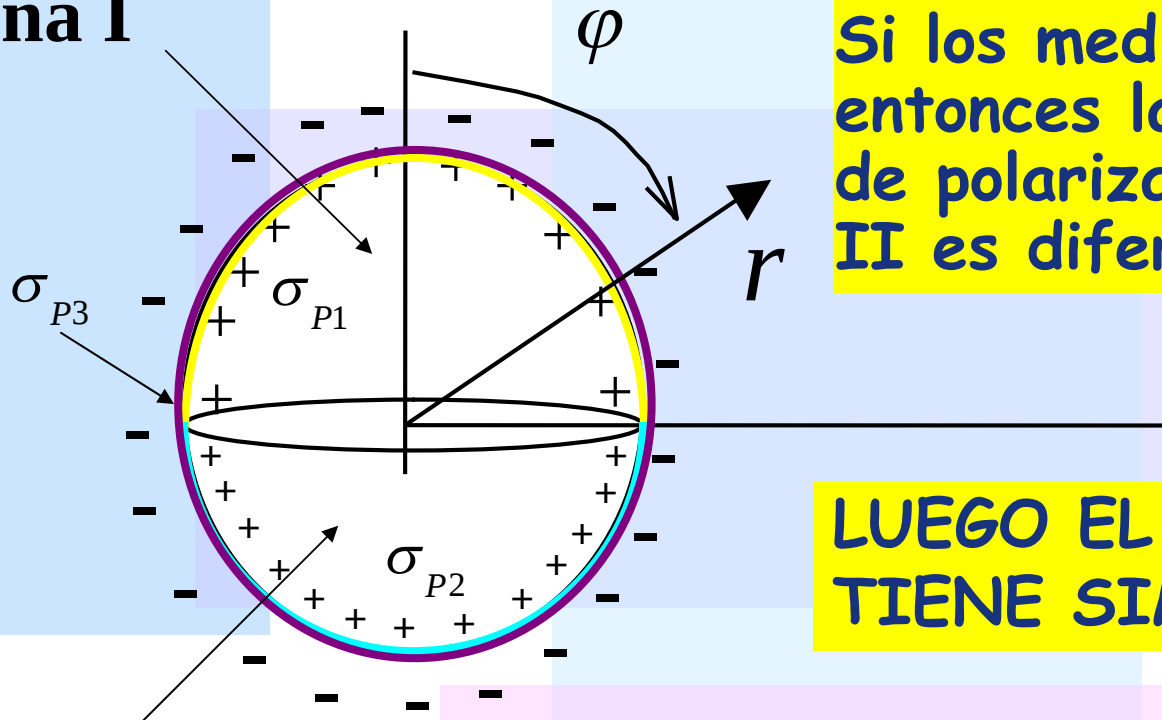


Al aplicar simetría no debemos olvidar que estamos simplificando un problema más complejo!



Consideraciones sobre Simetría

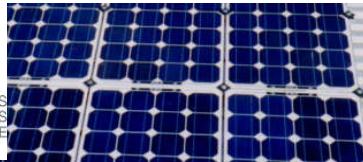
Zona I



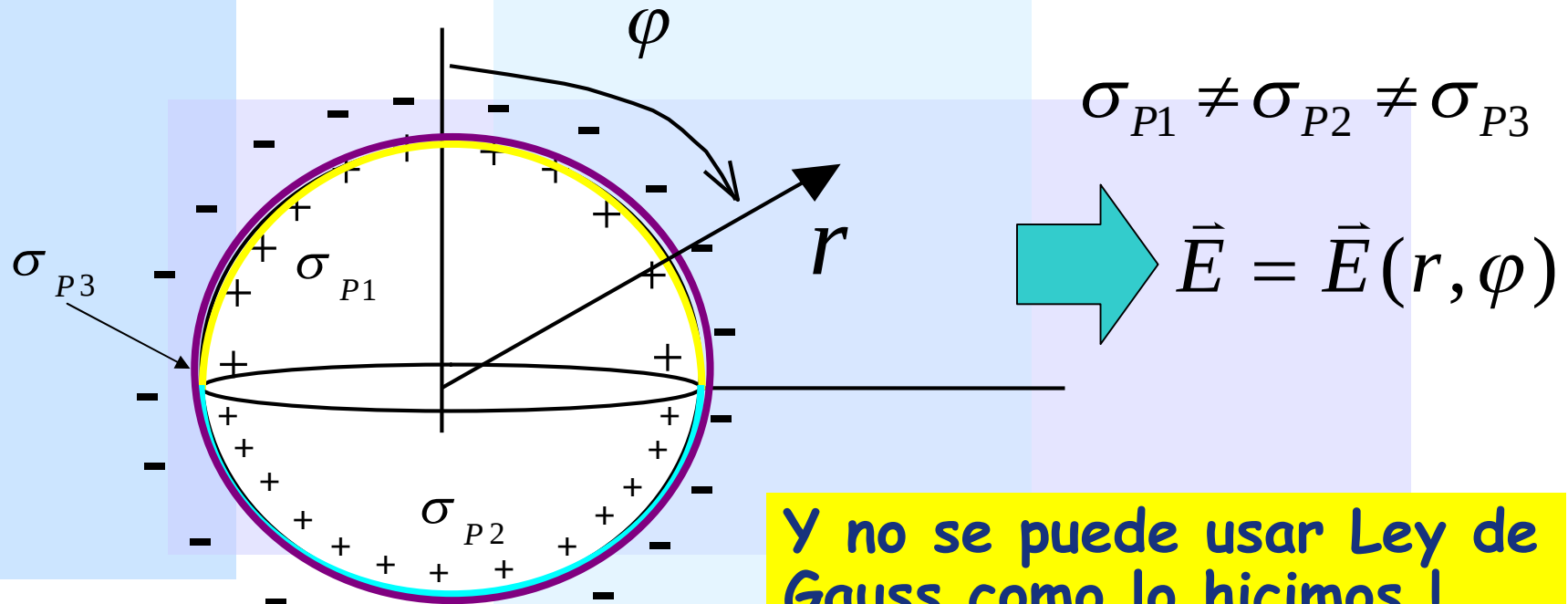
Zona II

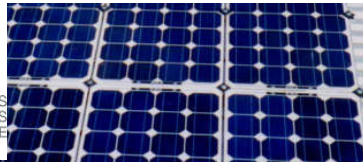
Si los medios son diferentes, entonces la carga superficial de polarización en zonas I y II es diferente

LUEGO EL SISTEMA NO TIENE SIMETRÍA SEGÚN ϕ

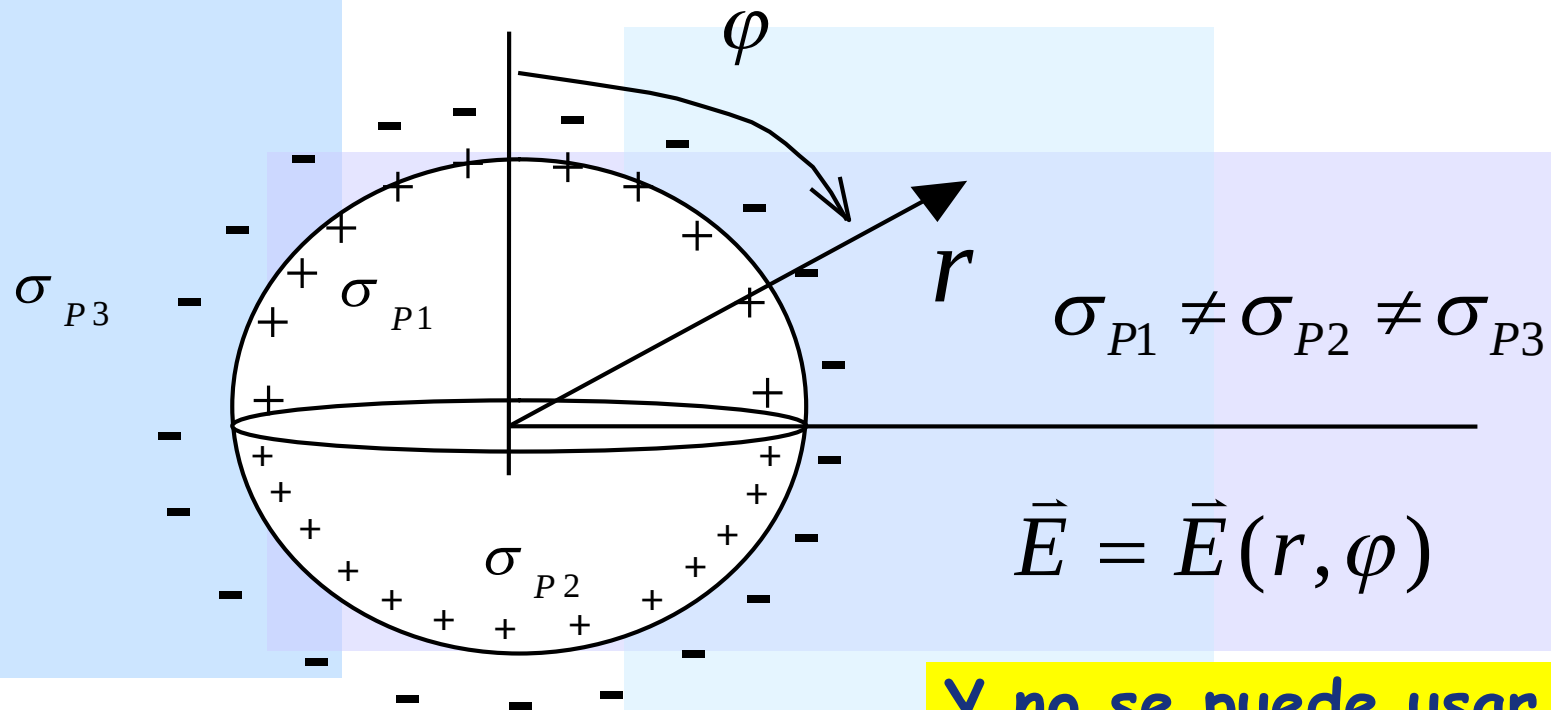


Consideraciones sobre Simetría





Consideraciones sobre Simetría



Y no se puede usar Ley de Gauss como lo hicimos !