Ejemplo: Una nube de tormenta con q = 20 C

Modelo simple: Esfera de radio R = 5 km = 5×10^3 m

Potencial en la superficie: $V = \dfrac{q}{4\pi\varepsilon_0 R} = \dfrac{20\cdot 8.988\cdot 10^9}{5\cdot 10^3} = 36\cdot 10^6\ \ \mathrm{V}$

Energía: $W = \dfrac{3}{5}\cdot\dfrac{q^2}{4\pi\varepsilon_0 R} = \dfrac{3}{5}\cdot qV$

$$W = \frac{3\cdot 20\cdot 36\cdot 10^6}{5} = 12\cdot 36\cdot 10^6\ \ \mathrm{J}\ \ = 4.32\cdot 10^8\ \ \mathrm{J}$$

Comparación: 1 KWh = $10^3 \times 3.6 \times 10^3 = 3.6 \times 10^6$ J

Campo eléctrico en la superficie de la esfera : $E = \dfrac{q}{4\pi\varepsilon_0 R^2}$

$$E = \frac{20 \cdot 9 \cdot 10^9}{25 \cdot 10^6} = 7.2 \cdot 10^3 \quad \text{V/m}$$

Ejemplo: Supong. que se tiene cierta distribución de carga entre $r = 0$ y $r = R$ con simetría esférica, de modo que el potencial toma la forma:

$V(r) = \alpha r^3 \qquad r < R$

Calcular el campo eléctrico y la densidad de carga ρ.

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} \qquad\qquad \vec{E} = -3\alpha r^2 \hat{r}$$

$$\rho = -\varepsilon_0 \nabla^2 V \qquad \text{(Ecuac. de Poisson)}$$

$$\rho = \varepsilon_0 \nabla \cdot \vec{E} \qquad \text{(Ley de Gauss dif.)}$$

Divergencia en coordenadas esféricas:

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\,\mathrm{sen}(\theta)}\frac{\partial}{\partial \theta}\left(\mathrm{sen}\theta E_\theta\right) + \frac{1}{r\,\mathrm{sen}(\theta)}\frac{\partial E_\varphi}{\partial \varphi}$$

$$\rho = \varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) \quad = \frac{\varepsilon_0}{r^2}\frac{\partial}{\partial r}\left(-3\alpha r^4\right) = -\frac{\varepsilon_0}{r^2}\cdot 12\alpha r^3 = -12\varepsilon_0 \alpha r$$

Estudiar para r > R. V(r) = ? ρ(r) = ? Completar datos.

$$\text{Ecuac. de Laplace:} \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \qquad \Rightarrow \qquad V(r) = -\frac{a}{r} + b$$

$$\text{Si} \quad r \to \infty, \qquad V(r) \to 0 \quad \therefore \quad b = 0$$

$$\text{Continuidad:} \quad \alpha R^3 = -\frac{a}{R} \quad \therefore \quad a = -\alpha R^4$$

$$V(r) = \frac{\alpha R^4}{r} \qquad r \geq R$$

Carga total dentro de la esfera : $q_1 = 4\pi \int_0^R \rho(r)\, r^2 dr$

$$q_1 = 4\pi \int_0^R \left(-\varepsilon_0 \cdot 12\alpha r\right) r^2 dr = -4\pi\varepsilon_0 \cdot 3\alpha R^4$$

Potencial causado por q_1 : $V_1(r) = \dfrac{q_1}{4\pi\varepsilon_0 r}$ $\quad r \geq R$

$$V_1(r) = \frac{-4\pi\varepsilon_0 \cdot 3\alpha R^4}{4\pi\varepsilon_0 r} \qquad r \geq R$$

Pero : $V(r) = \dfrac{\alpha R^4}{r}$

Agregar q_2 (superficial): $V_2(r) = \dfrac{q_2}{4\pi\varepsilon_0 r}$ $\quad r \geq R$

$$V_1(r) + V_2(r) = \frac{\alpha R^4}{r} \qquad r \geq R$$

$$\frac{-4\pi\varepsilon_0 \cdot 3\alpha R^4}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 r} = \frac{\alpha R^4}{r} \qquad r \geq R$$

$$-4\pi\varepsilon_0 \cdot 3\alpha R^4 + q_2 = \alpha R^4 4\pi\varepsilon_0$$

$$q_2 = 4\pi\varepsilon_0\left(\alpha R^4 + 3\alpha R^4\right) \quad = 4\pi\varepsilon_0 \cdot 4\alpha R^4$$

$$\sigma = \frac{q_2}{4\pi R^2} = 4\varepsilon_0 \alpha R^2$$

Solución numérica de la Ecuación de Laplace:

$$\nabla^2 V = 0$$

En coordenadas cartesianas y dos dimensiones:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial V}{\partial x}(x+\Delta x/2) \approx \frac{V(x+\Delta x) - V(x)}{\Delta x}$$

$$\frac{\partial V}{\partial x}(x-\Delta x/2) \approx \frac{V(x) - V(x-\Delta x)}{\Delta x}$$

$$\frac{\partial^2 V}{\partial x^2}(x) \approx \frac{\frac{\partial V}{\partial x}(x+\Delta x/2) - \frac{\partial V}{\partial x}(x-\Delta x/2)}{\Delta x}$$

$$\frac{\partial^2 V}{\partial x^2}(x) \approx \frac{V(x+\Delta x) - 2V(x) + V(x-\Delta x)}{(\Delta x)^2}$$

$$V = V(x, y)$$

$$\frac{\partial^2 V}{\partial x^2}(x, y) \approx \frac{V(x + \Delta x, y) - 2V(x, y) + V(x - \Delta x, y)}{(\Delta x)^2}$$

$$\frac{\partial^2 V}{\partial y^2}(x, y) \approx \frac{V(x, y + \Delta y) - 2V(x, y) + V(x, y - \Delta y)}{(\Delta y)^2}$$

supong: $\Delta\mathbf{x} = \Delta\mathbf{y}$:

$$\frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2}(\mathbf{x},\mathbf{y}) + \frac{\partial^2 \mathbf{V}}{\partial \mathbf{y}^2}(\mathbf{x},\mathbf{y})$$

$$\approx \frac{1}{(\Delta\mathbf{x})^2}\left[\mathbf{V}(\mathbf{x}+\Delta\mathbf{x},\mathbf{y}) - 2\mathbf{V}(\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x}-\Delta\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}+\Delta\mathbf{y}) - 2\mathbf{V}(\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}-\Delta\mathbf{y})\right]$$

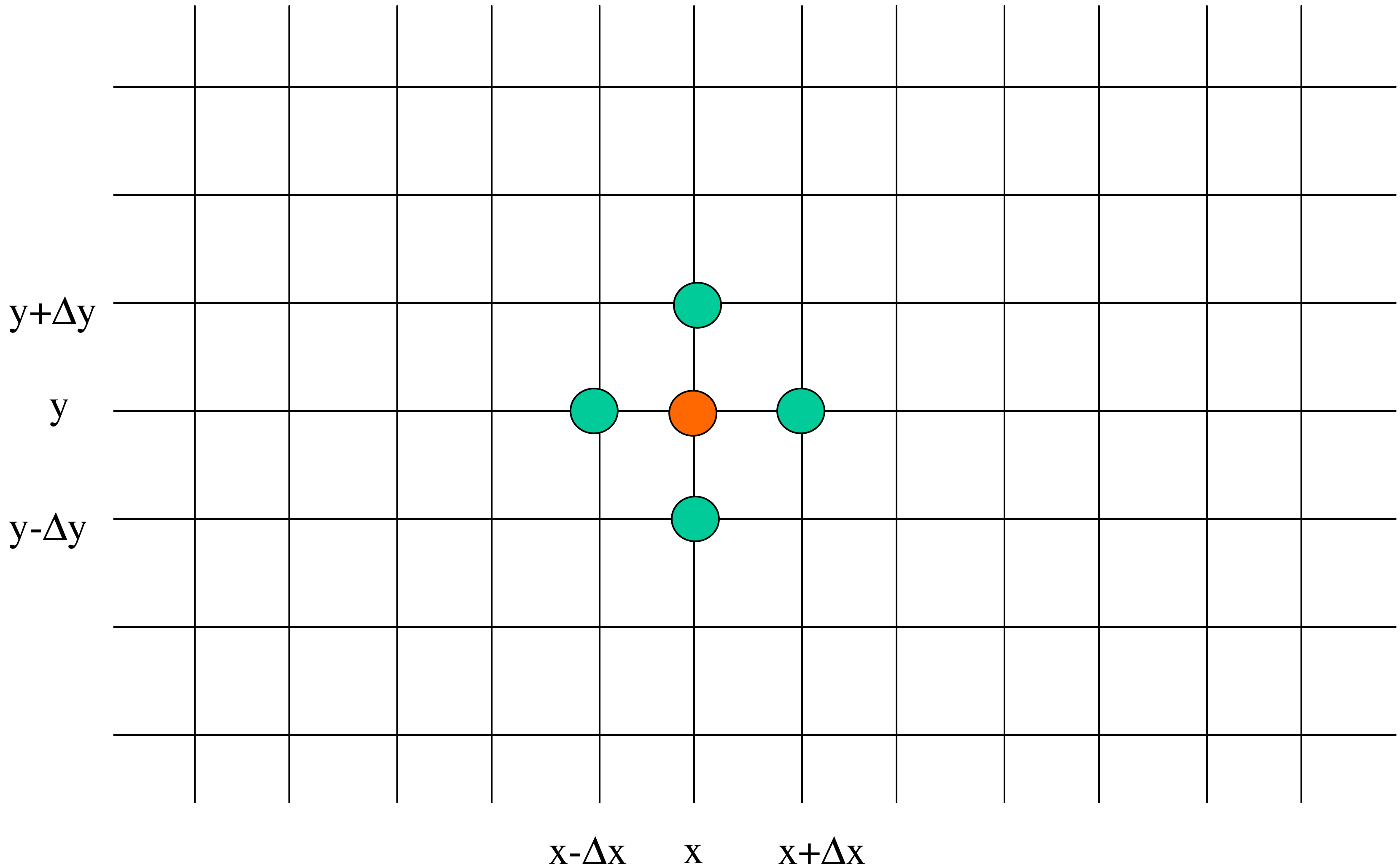
$$\frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2}(\mathbf{x},\mathbf{y}) + \frac{\partial^2 \mathbf{V}}{\partial \mathbf{y}^2}(\mathbf{x},\mathbf{y}) = 0$$

$$\frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2}(\mathbf{x},\mathbf{y}) + \frac{\partial^2 \mathbf{V}}{\partial \mathbf{y}^2}(\mathbf{x},\mathbf{y}) = 0$$

$$\mathbf{V}(\mathbf{x}+\Delta\mathbf{x},\mathbf{y}) - 4\mathbf{V}(\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x}-\Delta\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}+\Delta\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}-\Delta\mathbf{y}) = 0$$

$$\mathbf{V}(\mathbf{x},\mathbf{y}) = \frac{1}{4}\left[\mathbf{V}(\mathbf{x}+\Delta\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x}-\Delta\mathbf{x},\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}+\Delta\mathbf{y}) + \mathbf{V}(\mathbf{x},\mathbf{y}-\Delta\mathbf{y})\right]$$

Algoritmo de relajación: Ver Apuntes P.Cordero, cap. 2, pág. 45



Demostración en V.B.