0.0.1 PAUTA CONTROL 1. Un punto base más los indicados. Los puntajes son referenciales para un camino de solución...

- Problema 1

Si alumno utiliza otros pasos, evalue usted más o menos de acuerdo a este método...

Problema 1 Tenemos

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\phi}^2) - mgR\cos\theta.$$

De allí se escriben las ecuaciones de Lagrange

$$\begin{aligned} 0 &= \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{R}\sin\theta \\ 0 &= \dot{p}_\phi = \frac{d}{dt}(mR^2\sin^2\theta\dot{\phi}) \qquad \text{(a) 1p)} \end{aligned}$$

El Hamiltoniano con $\dot{\phi} = \frac{p_\phi}{mR^2\sin^2\theta}$ y $\dot{\theta} = \frac{p_\theta}{mR^2}$ es

$$H = \frac{1}{2}\frac{p_\theta^2}{mR^2} + \frac{1}{2}\frac{p_\phi^2}{mR^2\sin^2\theta} + mgR\cos\theta$$

Las ecuaciones de Hamilton serán

$$\begin{aligned} \dot{\phi} &= \frac{p_\phi}{mR^2\sin^2\theta} \\ \dot{\theta} &= \frac{p_\theta}{mR^2} \\ \dot{p}_\phi &= 0 \\ \dot{p}_\theta &= \frac{p_\phi^2}{mR^2\sin^3\theta}\cos\theta + mgR\sin\theta \qquad \text{(b) 1p)} \end{aligned}$$

Las cantidades conservadas son

$$\begin{aligned} p_\phi &= mR^2\sin^2\theta\dot{\phi} = \frac{3}{4}mR^2\Omega \\ H &= E = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\phi}^2) + mgR\cos\theta \\ &= \frac{1}{2}m(\frac{3}{4}R^2\Omega^2) + \frac{1}{2}mgR \qquad \text{(c) 1p)} \end{aligned}$$

Inicio letra (d). Para averiguar los extremos de θ, eliminemos $\dot{\phi} = \frac{3}{4}\frac{\Omega}{\sin^2\theta}$ de la ecuación de conservación de energía

$$\begin{aligned} &\frac{1}{2}(R^2\dot{\theta}^2 + \frac{9}{16}\frac{R^2\Omega^2}{\sin^2\theta}) + gR\cos\theta \\ = &\frac{1}{2}(\frac{3}{4}R^2\Omega^2) + \frac{1}{2}gR \end{aligned}$$

$$\dot\theta^2\sin^2\theta = (\frac{3}{4}\Omega^2+\frac{g}{R}(1-2\cos\theta))(1-\cos^2\theta)-\frac{9}{16}\Omega^2$$

colocando $u=\cos\theta$

$$\begin{aligned}\dot u^2 &= f(u) = (\frac{3}{4}\Omega^2+\frac{g}{R}(1-2u))(1-u^2)-\frac{9}{16}\Omega^2\\ &= \frac{2g}{R}\left(u-\frac{1}{2}\right)(u^2-\frac{3R\Omega^2}{8g}u-1-\frac{3R\Omega^2}{16g})\end{aligned}$$

Una raiz de $f(u)=0$ es $u_1=\frac{1}{2}\Rightarrow\theta_1=60^o$. La otra es

$$\begin{aligned}u_2 &= \frac{3\Omega^2R}{16g}-\sqrt{(\frac{3\Omega^2R}{16g})^2+1+\frac{3\Omega^2R}{16g}} && \text{(d) 1p}\\ &= p-\sqrt{p^2+1+p}\ <0\ \text{ con } p=\frac{3\Omega^2R}{16g}\\ \theta_2 &> \frac{\pi}{2}\end{aligned}$$

Como sistema de tres grados de libertad (Inicio letra (f))

$$L=\frac{1}{2}m(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2)-mgr\cos\theta$$

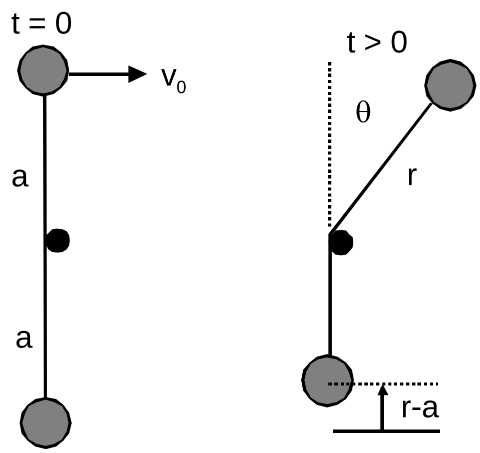
$$\begin{aligned}0 &= r\ddot\theta-r\sin\theta\cos\theta\dot\phi^2-g\sin\theta\\ 0 &= \dot p_\phi=\frac{d}{dt}(mr^2\sin^2\theta\dot\phi)\\ m\ddot r-(mr\dot\theta^2+mr\sin^2\theta\dot\phi^2-mg\cos\theta) &= \lambda_1\\ r &= R\end{aligned}$$

$$\begin{aligned}0 &= \ddot\theta-\sin\theta\cos\theta\dot\phi^2-\frac{g}{R}\sin\theta\\ 0 &= \dot p_\phi=\frac{d}{dt}(mR^2\sin^2\theta\dot\phi)\\ mg\cos\theta-mR\dot\theta^2-mR\sin^2\theta\dot\phi^2 &= \lambda_1=N\\ r &= R && \text{(f) 1p}\end{aligned}$$

recuerde que

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial\dot r}\frac{1}{2}(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2)-\frac{\partial}{\partial r}\frac{1}{2}(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2) &= \vec a\cdot\frac{\partial\vec r}{\partial r}=a_r\\ \ddot r-r\dot\theta^2-r\sin^2\theta\dot\phi^2 &= \vec a\cdot\frac{\partial\vec r}{\partial r}=a_r\end{aligned}$$

- Problema 2



El Lagrangeano es la energía cinética

$$L = K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

de donde las ecuaciones de Lagrange son

$$\begin{aligned} 2m\ddot{r} - mr\dot{\theta}^2 &= 0 \\ \dot{p}_\theta &= \frac{d}{dt}mr^2\dot{\theta} = 0. \end{aligned} \qquad \text{(a) 1.5p)}$$

Cantidades conservadas

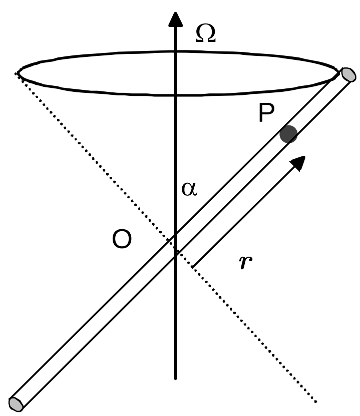
$$\begin{aligned} p_\theta &= mr^2\dot{\theta} = \text{ constante} = maV_0 \\ H &= K = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}mV_0^2 \end{aligned} \qquad \text{(b) 1.5p)}$$

luego eliminando $\dot{\theta} = a\frac{V_0}{r^2}$

$$0 = 2\ddot{r} - r\dot{\theta}^2 = 2\ddot{r} - r(a\frac{V_0}{r^2})^2$$

$$
\begin{aligned}
2\ddot{r} &= \frac{d}{dr}\dot{r}^2 = a^2\frac{V_0^2}{r^3} \\
\dot{r}^2 &= \int_a^r a^2\frac{V_0^2}{r^3}dr = \frac{1}{2}V_0^2\frac{r^2-a^2}{r^2} \\
\dot{r} &= \frac{1}{\sqrt{2}}\frac{V_0}{r}\sqrt{r^2-a^2} \\
\int_a^r \frac{rdr}{\sqrt{r^2-a^2}} &= \frac{1}{\sqrt{2}}V_0 t = \sqrt{r^2-a^2} \\
r &= \sqrt{a^2+\frac{1}{2}V_0^2t^2} && \text{(c) 1.5 p)} \\
\dot{\theta} &= a\frac{V_0}{r^2} = a\frac{V_0}{a^2+\frac{1}{2}V_0^2t^2} \\
\theta &= \int_0^t a\frac{V_0}{a^2+\frac{1}{2}V_0^2t^2}dt = \sqrt{2}\arctan\frac{\sqrt{2}V_0t}{2a} && \text{(d) 1.5 p)}
\end{aligned}
$$

- Problema 3



Tenemos

$$
\begin{aligned}
L &= \frac{1}{2}m(\dot{r}^2+r^2\Omega^2\sin^2\alpha) - mgr\cos\alpha \\
m\ddot{r} - mr\Omega^2\sin^2\alpha + mg\cos\alpha &= 0 \\
\ddot{r} &= r\Omega^2\sin^2\alpha - g\cos\alpha && \text{(a) 2p)}
\end{aligned}
$$

Hamiltoniano

$$
\begin{aligned}
\dot{r} &= \frac{p_r}{m} \\
H &= p_r\dot{r} - L = \frac{1}{2}\frac{p_r^2}{m} - \frac{1}{2}mr^2\Omega^2\sin^2\alpha + mgr\cos\alpha
\end{aligned}
$$

Punto de equilibrio $\ddot{r} = 0 \Rightarrow$

$$
r_0 = \frac{g\cos\alpha}{\Omega^2\sin^2\alpha}
$$

linealización

$$r = r_0 + \epsilon$$

$$\ddot{\epsilon} = (\frac{g \cos \alpha}{\Omega^2 \sin^2 \alpha} + \epsilon)\Omega^2 \sin^2 \alpha - g \cos \alpha = \epsilon \Omega^2 \sin^2 \alpha \qquad \text{(c) 2p)}$$

inestable porque $\ddot{\epsilon}$ y ϵ tienen el mismo signo. Otra forma es ver el potencial radial efectivo

$$\begin{aligned} U^{ef} &= -\frac{1}{2} m r^2 \Omega^2 \sin^2 \alpha + m g r \cos \alpha \\ \frac{d}{dr} U^{ef} &= -m r \Omega^2 \sin^2 \alpha + m g \cos \alpha = 0 \text{ en } r = r_0 \end{aligned}$$

pero es un máximo porque

$$\frac{d^2}{dr^2} U^{ef} = -m \Omega^2 \sin^2 \alpha < 0.$$