2

Essential Mathematics

2.1 Scalars, Vectors, and Cartesian Tensors

Learning a discipline's language is the first step a student takes towards becoming competent in that discipline. The language of continuum mechanics is the algebra and calculus of *tensors*. Here, tensors is the generic name for those mathematical entities which are used to represent the important physical quantities of continuum mechanics. Only that category of tensors known as *Cartesian tensors* is used in this text, and definitions of these will be given in the pages that follow. The tensor equations used to develop the fundamental theory of continuum mechanics may be written in either of two distinct notations: the *symbolic notation,* or the *indicial notation.* We shall make use of both notations, employing whichever is more convenient for the derivation or analysis at hand, but taking care to establish the interrelationships between the two. However, an effort to emphasize indicial notation in most of the text has been made. This is because an introductory course must teach indicial notation to students who may have little prior exposure to the topic.

As it happens, a considerable variety of physical and geometrical quantities have important roles in continuum mechanics, and fortunately, each of these may be represented by some form of tensor. For example, such quantities as *density* and *temperature* may be specified completely by giving their magnitude, i.e., by stating a numerical value. These quantities are represented mathematically by *scalars,* which are referred to as *zeroth-order tensors.* It should be emphasized that scalars are not constants, but may actually be functions of position and/or time. Also, the exact numerical value of a scalar will depend upon the units in which it is expressed. Thus, the temperature may be given by either 68°F or 20°C at a certain location. As a general rule, lowercase Greek letters in italic print such as α, β, λ, etc. will be used as symbols for scalars in both the indicial and symbolic notations.

Several physical quantities of mechanics such as *force* and *velocity* require not only an assignment of magnitude, but also a specification of direction for their complete characterization. As a trivial example, a 20-Newton force acting vertically at a point is substantially different than a 20-Newton force acting horizontally at the point. Quantities possessing such directional properties are represented by *vectors,* which are *first-order tensors.* Geometrically, vectors are generally displayed as *arrows,* having a definite length (the magnitude), a specified orientation (the direction), and also a sense of action as indicated by the head and the tail of the arrow. Certain quantities in mechanics which are not truly vectors are also portrayed by arrows, for example, finite rotations. Consequently, in addition to the magnitude and direction characterization, the complete definition of a vector requires this further statement: vectors add (and subtract) in accordance with the triangle rule by which the arrow representing the vector sum of two vectors extends from the tail of the first component arrow to the head of the second when the component arrows are arranged "head-to-tail."

Although vectors are independent of any particular coordinate system, it is often useful to define a vector in terms of its coordinate components, and in this respect it is necessary to reference the vector to an appropriate set of axes. In view of our restriction to Cartesian tensors, we limit ourselves to consideration of Cartesian coordinate systems for designating the components of a vector.

A significant number of physical quantities having important status in continuum mechanics require mathematical entities of higher order than vectors for their representation in the hierarchy of tensors. As we shall see, among the best known of these are the *stress tensor* and the *strain tensors.* These particular tensors are *second-order tensors,* and are said to have a rank of *two.* Third-order and fourth-order tensors are not uncommon in continuum mechanics, but they are not nearly as plentiful as second-order tensors. Accordingly, the unqualified use of the word *tensor* in this text will be interpreted to mean *second-order tensor.* With only a few exceptions, primarily those representing the stress and strain tensors, we shall denote second-order tensors by uppercase Latin letters in boldfaced print, a typical example being the tensor $\mathbf{T}$.

Tensors, like vectors, are independent of any coordinate system, but just as with vectors, when we wish to specify a tensor by its components we are obliged to refer to a suitable set of reference axes. The precise definitions of tensors of various order will be given subsequently in terms of the transformation properties of their components between two related sets of Cartesian coordinate axes.

2.2 Tensor Algebra in Symbolic Notation — Summation Convention

The three-dimensional physical space of everyday life is the space in which many of the events of continuum mechanics occur. Mathematically, this space is known as a Euclidean three-space, and its geometry can be referenced to a system of Cartesian coordinate axes. In some instances, higher

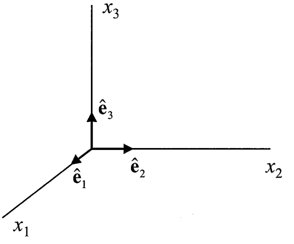


FIGURE 2.1A
Unit vectors in the coordinate directions x_1, x_2, and x_3.

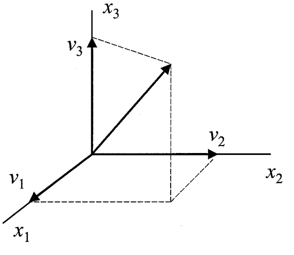


FIGURE 2.1B
Rectangular components of the vector $\mathbf{v}$.

order dimension spaces play integral roles in continuum topics. Because a scalar has only a single component, it will have the same value in every system of axes, but the components of vectors and tensors will have different component values, in general, for each set of axes.

In order to represent vectors and tensors in component form, we introduce in our physical space a right-handed system of rectangular Cartesian axes $Ox_1x_2x_3$, and identify with these axes the triad of unit base vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ shown in Figure 2.1A. All unit vectors in this text will be written with a caret placed above the boldfaced symbol. Due to the mutual perpendicularity of these base vectors, they form an orthogonal basis; furthermore, because they are unit vectors, the basis is said to be orthonormal. In terms of this basis, an arbitrary vector $\mathbf{v}$ is given in component form by

$$\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i \tag{2.2-1}$$

This vector and its coordinate components are pictured in Figure 2.1B. For the symbolic description, vectors will usually be given by lowercase Latin letters in boldfaced print, with the vector magnitude denoted by the same letter. Thus v is the magnitude of $\mathbf{v}$.

At this juncture of our discussion it is helpful to introduce a notational device called the *summation convention* that will greatly simplify the writing of the equations of continuum mechanics. Stated briefly, we agree that whenever a subscript appears exactly *twice* in a given term, that subscript will take on the values 1, 2, 3 successively, and the resulting terms summed. For example, using this scheme, we may now write Eq 2.2-1 in the simple form

$$\mathbf{v} = v_i \hat{\mathbf{e}}_i \tag{2.2-2}$$

and delete entirely the summation symbol **Σ**. For Cartesian tensors, only subscripts are required on the components; for general tensors, both subscripts and superscripts are used. The summed subscripts are called *dummy indices* since it is immaterial which particular letter is used. Thus, $v_j \hat{\mathbf{e}}_j$ is completely equivalent to $v_i \hat{\mathbf{e}}_i$, or to $v_k \hat{\mathbf{e}}_k$, when the summation convention is used. A word of caution, however: no subscript may appear more than twice, but as we shall soon see, more than one pair of dummy indices may appear in a given term. Note also that the summation convention may involve subscripts from both the unit vectors and the scalar coefficients.

Example 2.2-1

Without regard for their meaning as far as mechanics is concerned, expand the following expressions according to the summation convention:

$$\text{(a)}\ \ u_i v_i w_j \hat{\mathbf{e}}_j \qquad \text{(b)}\ \ T_{ij} v_i \hat{\mathbf{e}}_j \qquad \text{(c)}\ \ T_{ii} v_j \hat{\mathbf{e}}_j$$

Solution:

(a) Summing first on i, and then on j,

$$u_i v_i w_j \hat{\mathbf{e}}_j = \left(u_1 v_1 + u_2 v_2 + u_3 v_3\right)\left(w_1 \hat{\mathbf{e}}_1 + w_2 \hat{\mathbf{e}}_2 + w_3 \hat{\mathbf{e}}_3\right)$$

(b) Summing on i, then on j and collecting terms on the unit vectors,

$$\begin{aligned} T_{ij} v_i \hat{\mathbf{e}}_j &= T_{1j} v_1 \hat{\mathbf{e}}_j + T_{2j} v_2 \hat{\mathbf{e}}_j + T_{3j} v_3 \hat{\mathbf{e}}_j \\ &= \left(T_{11} v_1 + T_{21} v_2 + T_{31} v_3\right)\hat{\mathbf{e}}_1 + \left(T_{12} v_1 + T_{22} v_2 + T_{32} v_3\right)\hat{\mathbf{e}}_2 + \left(T_{13} v_1 + T_{23} v_2 + T_{33} v_3\right)\hat{\mathbf{e}}_3 \end{aligned}$$

(c) Summing on i, then on j,

$$T_{ii} \mathrm{v}_j \hat{\mathbf{e}}_j = \left(T_{11} + T_{22} + T_{33}\right)\left(v_1 \hat{\mathbf{e}}_1 + v_2 \hat{\mathbf{e}}_2 + v_3 \hat{\mathbf{e}}_3\right)$$

Note the similarity between (a) and (c).

With the above background in place we now list, using symbolic notation, several useful definitions from vector/tensor algebra.

1. *Addition of vectors:*

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \quad \text{or} \quad w_i\hat{\mathbf{e}}_i = (u_i + v_i)\hat{\mathbf{e}}_i \tag{2.2-3}$$

2. *Multiplication:*

(a) of a vector by a scalar:

$$\lambda\mathbf{v} = \lambda v_i\hat{\mathbf{e}}_i \tag{2.2-4}$$

(b) dot (scalar) product of two vectors:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = uv\cos\theta \tag{2.2-5}$$

where θ is the smaller angle between the two vectors when drawn from a common origin.

KRONECKER DELTA

From Eq 2.2-5 for the base vectors $\hat{\mathbf{e}}_{\mathbf{i}}$ (i = 1,2,3)

$$\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \begin{cases} 1 & \text{if numerical value of } i \ = \ \text{numerical value of } j \\ 0 & \text{if numerical value of } i \ \neq \ \text{numerical value of } j \end{cases}$$

Therefore, if we introduce the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if numerical value of } i \ = \ \text{numerical value of } j \\ 0 & \text{if numerical value of } i \ \neq \ \text{numerical value of } j \end{cases}$$

we see that

$$\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \delta_{ij} \quad (i,j=1,2,3) \tag{2.2-6}$$

Also, note that by the summation convention,

$$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3$$

and, furthermore, we call attention to the substitution property of the Kronecker delta by expanding (summing on j) the expression

$$\delta_{ij}\hat{\mathbf{e}}_j = \delta_{i1}\hat{\mathbf{e}}_1 + \delta_{i2}\hat{\mathbf{e}}_2 + \delta_{i3}\hat{\mathbf{e}}_3$$

But for a given value of i in this equation, only one of the Kronecker deltas on the right-hand side is non-zero, and it has the value one. Therefore,

$$\delta_{ij}\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i$$

and the Kronecker delta in $\delta_{ij}\hat{\mathbf{e}}_j$ causes the summed subscript j of $\hat{\mathbf{e}}_j$ to be replaced by i, and reduces the expression to simply $\hat{\mathbf{e}}_i$.

From the definition of δ_{ij} and its substitution property the dot product $\mathbf{u} \cdot \mathbf{v}$ may be written as

$$\mathbf{u} \cdot \mathbf{v} = u_i\hat{\mathbf{e}}_\mathbf{i} \cdot v_j\hat{\mathbf{e}}_\mathbf{j} = u_i v_j \hat{\mathbf{e}}_\mathbf{i} \cdot \hat{\mathbf{e}}_\mathbf{j} = u_i v_j \delta_{ij} = u_i v_i \qquad (2.2\text{-}7)$$

Note that scalar components pass through the dot product since it is a vector operator.

(c) cross (vector) product of two vectors:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = (uv \sin\theta)\hat{\mathbf{e}}$$

where $0 \leq \theta \leq \pi$, between the two vectors when drawn from a common origin, and where $\hat{\mathbf{e}}$ is a unit vector perpendicular to their plane such that a right-handed rotation about $\hat{\mathbf{e}}$ through the angle θ carries $\mathbf{u}$ into $\mathbf{v}$.

PERMUTATION SYMBOL

By introducing the permutation symbol ε_{ijk} defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if numerical values of } ijk \text{ appear as in the sequence } 12312 \\ -1 & \text{if numerical values of } ijk \text{ appear as in the sequence } 32132 \\ 0 & \text{if numerical values of } ijk \text{ appear in any other sequence} \end{cases} \qquad (2.2\text{-}8)$$

we may express the cross products of the base vectors $\hat{\mathbf{e}}_i$ (i = 1,2,3) by the use of Eq 2.2-8 as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk}\hat{\mathbf{e}}_k \qquad (i,j,k = 1,2,3) \qquad (2.2\text{-}9)$$

Also, note from its definition that the interchange of any two subscripts in ε_{ijk} causes a sign change so that, for example,

$$\varepsilon_{ijk} = -\varepsilon_{kji} = \varepsilon_{kij} = -\varepsilon_{ikj}$$

and, furthermore, that for repeated subscripts ε_{ijk} is zero as in

$$\varepsilon_{113} = \varepsilon_{212} = \varepsilon_{133} = \varepsilon_{222} = 0$$

Therefore, now the vector cross product above becomes

$$\mathbf{u} \times \mathbf{v} = u_i \hat{\mathbf{e}}_i \times v_j \hat{\mathbf{e}}_j = u_i v_j \left(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \right) = \varepsilon_{ijk} u_i v_j \hat{\mathbf{e}}_k \tag{2.2-10}$$

Again, notice how the scalar components pass through the vector cross product operator.

(d) triple scalar product (box product):

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$$

or

$$\begin{aligned} [\mathbf{u}, \mathbf{v}, \mathbf{w}] &= u_i \hat{\mathbf{e}}_i \cdot \left(v_j \hat{\mathbf{e}}_j \times w_k \hat{\mathbf{e}}_k \right) = u_i \hat{\mathbf{e}}_i \cdot \varepsilon_{jkq} v_j w_k \hat{\mathbf{e}}_q \\ &= \varepsilon_{jkq} u_i v_j w_k \delta_{iq} = \varepsilon_{ijk} u_i v_j w_k \end{aligned} \tag{2.2-11}$$

where in the final step we have used both the substitution property of δ_{ij} and the sign-change property of ε_{ijk}.

(e) triple cross product:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= u_i \hat{\mathbf{e}}_i \times \left(v_j \hat{\mathbf{e}}_j \times w_k \hat{\mathbf{e}}_k \right) = u_i \hat{\mathbf{e}}_i \times \left(\varepsilon_{jkq} v_j w_k \hat{\mathbf{e}}_q \right) \\ &= \varepsilon_{iqm} \varepsilon_{jkq} u_i v_j w_k \hat{\mathbf{e}}_m = \varepsilon_{miq} \varepsilon_{jkq} u_i v_j w_k \hat{\mathbf{e}}_m \end{aligned} \tag{2.2-12}$$

$\varepsilon - \delta$ IDENTITY

The product of permutation symbols $\varepsilon_{miq} \varepsilon_{jkq}$ in Eq 2.2-12 may be expressed in terms of Kronecker deltas by the identity

$$\varepsilon_{miq} \varepsilon_{jkq} = \delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij} \tag{2.2-13}$$

as may be proven by direct expansion. This is a *most important formula* used throughout this text and is worth memorizing. Also, by the sign-change property of ε_{ijk},

$$\varepsilon_{miq} \varepsilon_{jkq} = \varepsilon_{miq} \varepsilon_{qjk} = \varepsilon_{qmi} \varepsilon_{qjk} = \varepsilon_{qmi} \varepsilon_{jkq}$$

Additionally, it is easy to show from Eq 2.2-13 that

$$\varepsilon_{jkq}\varepsilon_{mkq} = 2\delta_{jm}$$

and

$$\varepsilon_{jkq}\varepsilon_{jkq} = 6$$

Therefore, now Eq 2.2-12 becomes

$$\begin{aligned} \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \left(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}\right)u_i v_j w_k \hat{\mathbf{e}}_m \\ &= \left(u_i v_m w_i - u_i v_i w_m\right)\hat{\mathbf{e}}_m = u_i w_i v_m \hat{\mathbf{e}}_m - u_i v_i w_m \hat{\mathbf{e}}_m \end{aligned} \tag{2.2-14}$$

which may be transcribed into the form

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

a well-known identity from vector algebra.

(f) tensor product of two vectors (dyad):

$$\mathbf{u}\,\mathbf{v} = u_i\hat{\mathbf{e}}_i v_j\hat{\mathbf{e}}_j = u_i v_j \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{2.2-15}$$

which in expanded form, summing first on i, yields

$$u_i v_j \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = u_1 v_j \hat{\mathbf{e}}_1\hat{\mathbf{e}}_j + u_2 v_j \hat{\mathbf{e}}_2\hat{\mathbf{e}}_j + u_3 v_j \hat{\mathbf{e}}_3\hat{\mathbf{e}}_j$$

and then summing on j

$$\begin{aligned} u_i v_j \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = {} & u_1 v_1 \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + u_1 v_2 \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + u_1 v_3 \hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ & + u_2 v_1 \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + u_2 v_2 \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + u_2 v_3 \hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ & + u_3 v_1 \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + u_3 v_2 \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + u_3 v_3 \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 \end{aligned} \tag{2.2-16}$$

This nine-term sum is called the *nonion* form of the *dyad*, **uv**. An alternative notation frequently used for the dyad product is

$$\mathbf{u}\otimes\mathbf{v} = u_i\hat{\mathbf{e}}_i \otimes v_j\hat{\mathbf{e}}_j = u_i v_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.2-17}$$

A sum of dyads such as

$$\mathbf{u}_1\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2 + \ldots + \mathbf{u}_\mathrm{N}\mathbf{v}_\mathrm{N} \tag{2.2-18}$$

is called a *dyadic*.

(g) vector-dyad products:

$$1.\ \ \mathbf{u} \cdot (\mathbf{vw}) = u_i\hat{\mathbf{e}}_i \cdot \left(v_j\hat{\mathbf{e}}_j w_k\hat{\mathbf{e}}_k\right) = u_i v_i w_k \hat{\mathbf{e}}_k \tag{2.2-19}$$

$$2.\ \ (\mathbf{uv}) \cdot \mathbf{w} = \left(u_i\hat{\mathbf{e}}_i v_j\hat{\mathbf{e}}_j\right) \cdot w_k\hat{\mathbf{e}}_k = u_i v_j w_j \hat{\mathbf{e}}_i \tag{2.2-20}$$

$$3.\ \ \mathbf{u} \times (\mathbf{vw}) = \left(u_i\hat{\mathbf{e}}_i \times v_j\hat{\mathbf{e}}_j\right) w_k\hat{\mathbf{e}}_k = \varepsilon_{ijq} u_i v_j w_k \hat{\mathbf{e}}_q \hat{\mathbf{e}}_k \tag{2.2-21}$$

$$4.\ \ (\mathbf{uv}) \times \mathbf{w} = u_i\hat{\mathbf{e}}_i \left(v_j\hat{\mathbf{e}}_j \times w_k\hat{\mathbf{e}}_k\right) = \varepsilon_{jkq} u_i v_j w_k \hat{\mathbf{e}}_i \hat{\mathbf{e}}_q \tag{2.2-22}$$

(Note that in products 3 and 4 the order of the base vectors $\hat{\mathbf{e}}_i$ is important.)

(h) dyad-dyad product:

$$(\mathbf{uv}) \cdot (\mathbf{ws}) = u_i\hat{\mathbf{e}}_i \left(v_j\hat{\mathbf{e}}_j \cdot w_k\hat{\mathbf{e}}_k\right) s_q\hat{\mathbf{e}}_q = u_i v_j w_j s_q \hat{\mathbf{e}}_i \hat{\mathbf{e}}_q \tag{2.2-23}$$

(i) vector-tensor products:

$$1.\ \ \mathbf{v} \cdot \mathbf{T} = v_i\hat{\mathbf{e}}_i \cdot T_{jk}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k = v_i T_{jk} \delta_{ij} \hat{\mathbf{e}}_k = v_i T_{ik} \hat{\mathbf{e}}_k \tag{2.2-24}$$

$$2.\ \ \mathbf{T} \cdot \mathbf{v} = T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \cdot v_k\hat{\mathbf{e}}_k = T_{ij}\hat{\mathbf{e}}_i \delta_{jk} v_k = T_{ij} v_j \hat{\mathbf{e}}_i \tag{2.2-25}$$

(Note that these products are also written as simply **vT** and **Tv**.)

(j) tensor-tensor product:

$$\mathbf{T} \cdot \mathbf{S} = T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \cdot S_{pq}\hat{\mathbf{e}}_p\hat{\mathbf{e}}_q = T_{ij} S_{jq} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_q \tag{2.2-26}$$

Example 2.2-2

Let the vector **v** be given by $\mathbf{v} = (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \hat{\mathbf{n}} \times (\mathbf{a} \times \hat{\mathbf{n}})$ where **a** is an arbitrary vector, and $\hat{\mathbf{n}}$ is a unit vector. Express **v** in terms of the base vectors $\hat{\mathbf{e}}_i$, expand, and simplify. (Note that $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = n_i\hat{\mathbf{e}}_i \cdot n_j\hat{\mathbf{e}}_j = n_i n_j \delta_{ij} = n_i n_i = 1$.)

Solution

In terms of the base vectors $\hat{\mathbf{e}}_i$, the given vector $\mathbf{v}$ is expressed by the equation

$$\mathbf{v} = \left(a_i \hat{\mathbf{e}}_i \cdot n_j \hat{\mathbf{e}}_j\right) n_k \hat{\mathbf{e}}_k + n_i \hat{\mathbf{e}}_i \times \left(a_j \hat{\mathbf{e}}_j \times n_k \hat{\mathbf{e}}_k\right)$$

We note here that indices *i*, *j*, and *k* appear four times in this line; however, the summation convention has not been violated. Terms that are separated by a plus or a minus sign are considered different terms, each having summation convention rules applicable within them. Vectors joined by a dot or cross product are not distinct terms, and the summation convention must be adhered to in that case. Carrying out the indicated multiplications, we see that

$$\begin{aligned}
\mathbf{v} &= \left(a_i n_j \delta_{ij}\right) n_k \hat{\mathbf{e}}_k + n_i \hat{\mathbf{e}}_i \times \left(\varepsilon_{jkq} a_j n_k \hat{\mathbf{e}}_q\right) \\
&= a_i n_i n_k \hat{\mathbf{e}}_k + \varepsilon_{iqm} \varepsilon_{jkq} n_i a_j n_k \hat{\mathbf{e}}_m \\
&= a_i n_i n_k \hat{\mathbf{e}}_k + \varepsilon_{miq} \varepsilon_{jkq} n_i a_j n_k \hat{\mathbf{e}}_m \\
&= a_i n_i n_k \hat{\mathbf{e}}_k + \left(\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}\right) n_i a_j n_k \hat{\mathbf{e}}_m \\
&= a_i n_i n_k \hat{\mathbf{e}}_k + n_i a_j n_i \hat{\mathbf{e}}_j - n_i a_i n_k \hat{\mathbf{e}}_k \\
&= n_i n_i a_j \hat{\mathbf{e}}_j = a_j \hat{\mathbf{e}}_j = \mathbf{a}
\end{aligned}$$

Since $\mathbf{a}$ must equal $\mathbf{v}$, this example demonstrates that the vector $\mathbf{v}$ may be resolved into a component $(\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$ in the direction of $\hat{\mathbf{n}}$, and a component $\hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}})$ perpendicular to $\hat{\mathbf{n}}$.

Example 2.2-3

Using Eq 2.2-13, show that (a) $\varepsilon_{mkq}\varepsilon_{jkq} = 2\delta_{mj}$ and that (b) $\varepsilon_{jkq}\varepsilon_{jkq} = 6$. (Recall that $\delta_{kk} = 3$ and $\delta_{mk}\delta_{kj} = \delta_{mj}$.)

Solution

(a) Write out Eq 2.2-13 with indice *i* replaced by *k* to get

$$\begin{aligned}
\varepsilon_{mkq}\varepsilon_{jkq} &= \delta_{mj}\delta_{kk} - \delta_{mk}\delta_{kj} \\
&= 3\delta_{mj} - \delta_{mj} = 2\delta_{mj}
\end{aligned}$$

(b) Start with the first equation in Part (a) and replace the index m with j, giving

$$\begin{aligned}\varepsilon_{jkq}\varepsilon_{jkq} &= \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk} \\ &= (3)(3) - \delta_{jj} = 9 - 3 = 6.\end{aligned}$$

Example 2.2-4

Double-dot products of dyads are defined by

$$\text{(a) } (\mathbf{uv}) \cdot \cdot (\mathbf{ws}) = (\mathbf{v} \cdot \mathbf{w})\,(\mathbf{u} \cdot \mathbf{s})$$

$$\text{(b) } (\mathbf{uv}) : (\mathbf{ws}) = (\mathbf{u} \cdot \mathbf{w})\,(\mathbf{v} \cdot \mathbf{s})$$

Expand these products and compare the component forms.

Solution

$$\text{(a) } (\mathbf{uv}) \cdot \cdot (\mathbf{ws}) = \left(v_i\hat{\mathbf{e}}_i \cdot w_j\hat{\mathbf{e}}_j\right)\left(u_k\hat{\mathbf{e}}_k \cdot s_q\hat{\mathbf{e}}_q\right) = v_i w_i u_k s_k$$

$$\text{(b) } (\mathbf{uv}) : (\mathbf{ws}) = \left(u_i\hat{\mathbf{e}}_i \cdot w_j\hat{\mathbf{e}}_j\right)\left(v_k\hat{\mathbf{e}}_k \cdot s_q\hat{\mathbf{e}}_q\right) = u_i w_i v_k s_k$$

2.3 Indicial Notation

By assigning special meaning to the subscripts, the *indicial notation* permits us to carry out the tensor operations of addition, multiplication, differentiation, etc. without the use, or even the appearance of the base vectors $\hat{\mathbf{e}}_i$ in the equations. We simply agree that the tensor rank (order) of a term is indicated by the number of "free," that is, unrepeated, subscripts appearing in that term. Accordingly, a term with no free indices represents a scalar, a term with one free index a vector, a term having two free indices a second-order tensor, and so on. Specifically, the symbol

λ = *scalar* (zeroth-order tensor) λ
v_i = *vector* (first-order tensor) $\mathbf{v}$, or equivalently, its 3 components
$u_i v_j$ = *dyad* (second-order tensor) $\mathbf{uv}$, or its 9 components
T_{ij} = *dyadic* (second-order tensor) $\mathbf{T}$, or its 9 components
Q_{ijk} = *triadic* (third-order tensor) $\mathbf{Q}$ or its 27 components
C_{ijkm} = *tetradic* (fourth-order tensor) $\mathbf{C}$, or its 81 components

For tensors defined in a three-dimensional space, the free indices take on the values 1,2,3 successively, and we say that these indices have a *range* of three. If N is the number of free indices in a tensor, that tensor has 3^N components in three space.

We must emphasize that in the indicial notation exactly *two types* of subscripts appear:

1. "free" indices, which are represented by letters that occur only *once* in a given term, and
2. "summed" or "dummy" indices which are represented by letters that appear *twice* in a given term.

Furthermore, every term in a valid equation must have the same letter subscripts for the free indices. No letter subscript may appear more than twice in any given term.

Mathematical operations among tensors are readily carried out using the indicial notation. Thus addition (and subtraction) among tensors of equal rank follows according to the typical equations; $u_i + v_i - w_i = s_i$ for vectors, and $T_{ij} - V_{ij} + S_{ij} = Q_{ij}$ for second-order tensors. Multiplication of two tensors to produce an *outer tensor* product is accomplished by simply setting down the tensor symbols side by side with no dummy indices appearing in the expression. As a typical example, the outer product of the vector v_i and tensor T_{jk} is the third-order tensor $v_i T_{jk}$. *Contraction* is the process of *identifying* (that is, setting equal to one another) any two indices of a tensor term. An *inner tensor product* is formed from an outer tensor product by one or more contractions involving indices from separate tensors in the outer product. We note that the rank of a given tensor is reduced by *two* for each contraction. Some outer products, which contract, form well-known inner products listed below.

Outer Products:	**Contraction(s):**	**Inner Products:**
$u_i\, v_j$	$i = j$	$u_i\, v_i$ (vector dot product)
$\varepsilon_{ijk} u_q v_m$	$j = q,\ k = m$	$\varepsilon_{ijk} u_j v_k$ (vector cross product)
$\varepsilon_{ijk} u_q v_m w_n$	$i = q,\ j = m,\ k = n$	$\varepsilon_{ijk} u_i v_j w_k$ (box product)

A tensor is *symmetric* in any two indices if interchange of those indices leaves the tensor value unchanged. For example, if $S_{ij} = S_{ji}$ and $C_{ijm} = C_{jim}$, both of these tensors are said to be symmetric in the indices i and j. A tensor is *anti-symmetric* (or *skew-symmetric*) in any two indices if interchange of those indices causes a sign change in the value of the tensor. Thus, if $A_{ij} = -A_{ji}$, it is anti-symmetric in i and j. Also, recall that by definition, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$, etc., and hence the permutation symbol is anti-symmetric in all indices.

Example 2.3-1

Show that the inner product $S_{ij}\, A_{ij}$ of a symmetric tensor $S_{ij} = S_{ji}$, and an anti-symmetric tensor $A_{ij} = -A_{ji}$ is zero.

Solution

By definition of symmetric tensor A_{ij} and skew-symmetric tensor $S_{ij},$ we have

$$S_{ij}\,A_{ij} = -S_{ji}\,A_{ji} = -S_{mn}\,A_{mn} = -S_{ij}\,A_{ij}$$

where the last two steps are the result of all indices being dummy indices. Therefore, $2S_{ij}A_{ij} = 0$, or $S_{ij}\,A_{ij} = 0$.

One of the most important advantages of the indicial notation is the compactness it provides in expressing equations in three dimensions. A brief listing of typical equations of continuum mechanics is presented below to illustrate this feature.

1. $\phi\ = S_{ij}T_{ij} - S_{ii}T_{jj}$ (1 equation, 18 terms on RHS)
2. $t_i\ = Q_{ij}n_j$ (3 equations, 3 terms on RHS of each)
3. $T_{ij}\ =\ \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$ (9 equations, 4 terms on RHS of each)

Example 2.3-2

By direct expansion of the expression $v_i = \varepsilon_{ijk}W_{jk}$ determine the components of the vector v_i in terms of the components of the tensor W_{jk}.

Solution

By summing first on j and then on k and then omitting the zero terms, we find that

$$\begin{aligned} v_i &= \varepsilon_{i1k}W_{1k} + \varepsilon_{i2k}W_{2k} + \varepsilon_{i3k}W_{3k} \\ &= \varepsilon_{i12}W_{12} + \varepsilon_{i13}W_{13} + \varepsilon_{i21}W_{21} + \varepsilon_{i23}W_{23} + \varepsilon_{i31}W_{31} + \varepsilon_{i32}W_{32} \end{aligned}$$

Therefore,

$$v_1 = \varepsilon_{123}W_{23} + \varepsilon_{132}W_{32} = W_{23} - W_{32}$$

$$v_2 = \varepsilon_{213}W_{13} + \varepsilon_{231}W_{31} = W_{31} - W_{13}$$

$$v_3 = \varepsilon_{312}W_{12} + \varepsilon_{321}W_{21} = W_{12} - W_{21}$$

Note that if the tensor W_{jk} were symmetric, the vector v_i would be a null (zero) vector.

2.4 Matrices and Determinants

For computational purposes it is often expedient to use the *matrix representation* of vectors and tensors. Accordingly, we review here several definitions and operations of elementary matrix theory.

A *matrix* is an ordered rectangular array of elements enclosed by square brackets and subjected to certain operational rules. The typical element A_{ij} of the matrix is located in the i^{th} (horizontal) row, and in the j^{th} (vertical) column of the array. A matrix having elements A_{ij}, which may be numbers, variables, functions, or any of several mathematical entities, is designated by $[A_{ij}]$, or symbolically by the kernel letter $\mathcal{A}$. The vector or tensor which the matrix represents is denoted by the kernel symbol in boldfaced print. An M by N matrix (written $M \times N$) has M rows and N columns, and may be displayed as

$$\mathcal{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix} \tag{2.4-1}$$

If $M = N$, the matrix is a *square matrix*. A $1 \times N$ matrix $[A_{1N}]$ is an *row matrix,* and an $M \times 1$ matrix $[A_{M1}]$ is a *column matrix.* Row and column matrices represent vectors, whereas a 3×3 square matrix represents a second-order tensor. A scalar is represented by a 1×1 matrix (a single element). The unqualified use of the word *matrix* in this text is understood to mean a 3×3 square matrix, that is, the matrix representation of a second-order tensor.

A *zero*, or *null* matrix has all elements equal to zero. A *diagonal matrix* is a square matrix whose elements not on the *principal diagonal,* which extends from A_{11} to A_{NN}, are all zeros. Thus for a diagonal matrix, $A_{ij} = 0$ for $i \neq j$. The *unit* or *identity matrix* **I**, which, incidentally, is the matrix representation of the Kronecker delta, is a diagonal matrix whose diagonal elements all have the value *one*.

The $N \times M$ matrix formed by interchanging the rows and columns of the $M \times N$ matrix $\mathcal{A}$ is called the *transpose* of $\mathcal{A}$, and is written as $\mathcal{A}^{\mathrm{T}}$, or $[A_{ij}]^{\mathrm{T}}$. By definition, the elements of a matrix $\mathcal{A}$ and its transpose are related by the equation $A_{ij}^{\mathrm{T}} = A_{ji}$. A square matrix for which $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$, or in element form, $A_{ij} = A_{ij}^{\mathrm{T}}$ is called a symmetric matrix; one for which $\mathcal{A} = -\mathcal{A}^{\mathrm{T}}$, or $A_{ij}^{\mathrm{T}} = -A_{ij}^{\mathrm{T}}$ is called an anti-symmetric, or skew-symmetric matrix. The elements of the principal diagonal of a skew-symmetric matrix are all zeros. Two matrices are *equal* if they are identical element by element. Matrices having the same number of rows and columns may be added (or subtracted) element by element. Thus if $\mathcal{A} = \mathcal{B} + \mathcal{C}$, the elements of $\mathcal{A}$ are given by

$$A_{ij} = B_{ij} + C_{ij} \tag{2.4-2}$$

Addition of matrices is commutative, $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$, and associative, $\mathcal{A} + (\mathcal{B} + \mathcal{C}) = (\mathcal{A} + \mathcal{B}) + \mathcal{C}$.

Example 2.4-1

Show that the square matrix $\mathcal{A}$ can be expressed as the sum of a symmetric and a skew-symmetric matrix by the decomposition

$$\mathcal{A} = \frac{\mathcal{A} + \mathcal{A}^{\mathrm{T}}}{2} + \frac{\mathcal{A} - \mathcal{A}^{\mathrm{T}}}{2}$$

Solution

Let the decomposition be written as $\mathcal{A} = \mathcal{B} + \mathcal{C}$ where $\mathcal{B} = \frac{1}{2}\left(\mathcal{A} + \mathcal{A}^{\mathrm{T}}\right)$ and $\mathcal{C} = \frac{1}{2}\left(\mathcal{A} - \mathcal{A}^{\mathrm{T}}\right)$. Then writing $\mathcal{B}$ and $\mathcal{C}$ in element form,

$$B_{ij} = \frac{A_{ij} + A_{ij}^{\mathrm{T}}}{2} = \frac{A_{ij} + A_{ji}}{2} = \frac{A_{ji} + A_{ji}^{\mathrm{T}}}{2} = B_{ji} = B_{ij}^{\mathrm{T}} \qquad \text{(symmetric)}$$

$$C_{ij} = \frac{A_{ij} - A_{ij}^{\mathrm{T}}}{2} = \frac{A_{ij} - A_{ji}}{2} = -\frac{A_{ji} - A_{ji}^{\mathrm{T}}}{2} = -C_{ji} = -C_{ij}^{\mathrm{T}} \qquad \text{(skew-symmetric)}$$

Therefore, $\mathcal{B}$ is symmetric, and $\mathcal{C}$ skew-symmetric.

Multiplication of the matrix $\mathcal{A}$ by the scalar λ results in the matrix $\lambda\mathcal{A}$, or $[\lambda\mathcal{A}_{ij}]$. The product of two matrices $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{AB}$, is defined only if the matrices are *conformable*, that is, if the *prefactor* matrix $\mathcal{A}$ has the same number of columns as the *postfactor* matrix $\mathcal{B}$ has rows. Thus, the product of an $M \times Q$ matrix multiplied by a $Q \times N$ matrix is an $M \times N$ matrix. The product matrix $\mathcal{C} = \mathcal{AB}$ has elements given by

$$C_{ij} = A_{ik}B_{kj} \tag{2.4-3}$$

in which k is, of course, a summed index. Therefore, each element C_{ij} of the product matrix is an inner product of the i^{th} row of the prefactor matrix with the j^{th} column of the postfactor matrix. In general, matrix multiplication is not commutative, $\mathcal{AB} \neq \mathcal{BA}$, but the associative and distributive laws of multiplication do hold for matrices. The product of a matrix with itself is the *square* of the matrix, and is written $\mathcal{AA} = \mathcal{A}^2$. Likewise, the *cube* of the matrix is $\mathcal{AAA} = \mathcal{A}^3$, and in general, matrix products obey the exponent rule

$$\mathcal{A}^m\mathcal{A}^n = \mathcal{A}^n\mathcal{A}^m = \mathcal{A}^{m+n} \tag{2.4-4}$$

where m and n are positive integers, or zero. Also, we note that

$$\left(\mathcal{A}^n\right)^{\mathrm{T}} = \left(\mathcal{A}^{\mathrm{T}}\right)^n \tag{2.4-5}$$

and if $\mathcal{B}\mathcal{B} = \mathcal{A}$ then

$$\mathcal{B} = \sqrt{\mathcal{A}} = (\mathcal{A})^{1/2} \tag{2.4-6}$$

but the square root is not unique.

Example 2.4-2

Show that for arbitrary matrices $\mathcal{A}$ and $\mathcal{B}$:

(a) $(\mathcal{A} + \mathcal{B})^{\mathrm{T}} = \mathcal{A}^{\mathrm{T}} + \mathcal{B}^{\mathrm{T}}$,

(b) $(\mathcal{A}\mathcal{B})^{\mathrm{T}} = \mathcal{B}^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}$ and

(c) $\mathbf{I}\mathcal{B} = \mathcal{B}\mathbf{I} = \mathcal{B}$ where $\mathbf{I}$ is the identity matrix.

Solution

(a) Let $\mathcal{A} + \mathcal{B} = \mathcal{C}$, then in element form $C_{ij} = A_{ij} + B_{ij}$ and therefore $\mathcal{C}^{\mathrm{T}}$ is given by

$$C_{ij}^{\mathrm{T}} = C_{ji} = A_{ji} + B_{ji} = A_{ij}^{\mathrm{T}} + B_{ij}^{\mathrm{T}} \quad \text{or} \quad \mathcal{C}^{\mathrm{T}} = (\mathcal{A} + \mathcal{B})^{\mathrm{T}} = \mathcal{A}^{\mathrm{T}} + \mathcal{B}^{\mathrm{T}}$$

(b) Let $\mathcal{A}\mathcal{B} = \mathcal{C}$, then in element form

$$C_{ij} = A_{ik}B_{kj} = A_{ki}^{\mathrm{T}}B_{jk}^{\mathrm{T}} = B_{jk}^{\mathrm{T}}A_{ki}^{\mathrm{T}} = C_{ji}^{\mathrm{T}}$$

Hence, $(\mathcal{A}\mathcal{B})^{\mathrm{T}} = \mathcal{B}^{\mathrm{T}}\,\mathcal{A}^{\mathrm{T}}$.

(c) Let $\mathbf{I}\mathcal{B} = \mathcal{C}$, then in element form

$$C_{ij} = \delta_{ik}B_{kj} = B_{ij} = B_{ik}\delta_{kj}$$

by the substitution property. Thus, $\mathbf{I}\mathcal{B} = \mathcal{B}\mathbf{I} = \mathcal{B}$.

The *determinant* of a square matrix is formed from the square array of elements of the matrix, and this array evaluated according to established mathematical rules. The determinant of the matrix $\mathcal{A}$ is designated by either det $\mathcal{A}$, or by $\left|A_{ij}\right|$, and for a 3×3 matrix $\mathcal{A}$,

$$\det \mathcal{A} = |A_{ij}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \tag{2.4-7}$$

A *minor* of det $\mathcal{A}$ is another determinant $|M_{ij}|$ formed by deleting the i^{th} row and j^{th} column of $|A_{ij}|$. The *cofactor* of the element A_{ij} (sometimes referred to as the *signed minor*) is defined by

$$A_{ij}^{(c)} = (-1)^{i+j} |M_{ij}| \tag{2.4-8}$$

where superscript *(c)* denotes cofactor of matrix $\mathcal{A}$.

Evaluation of a determinant may be carried out by a standard method called *expansion by cofactors*. In this method, any row (or column) of the determinant is chosen, and each element in that row (or column) is multiplied by its cofactor. The sum of these products gives the value of the determinant. For example, expansion of the determinant of Eq 2.4-7 by the first row becomes

$$\det \mathcal{A} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \tag{2.4-9}$$

which upon complete expansion gives

$$\det \mathcal{A} = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31})$$

$$+ A_{13}(A_{21}A_{32} - A_{22}A_{31}) \tag{2.4-10}$$

Several interesting properties of determinants are worth mentioning at this point. To begin with, the interchange of any two rows (or columns) of a determinant causes a sign change in its value. Because of this property and because of the sign-change property of the permutation symbol, the det $\mathcal{A}$ of Eq 2.4-7 may be expressed in the indicial notation by the alternative forms (see Prob. 2.11)

$$\det \mathcal{A} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} \tag{2.4-11}$$

Furthermore, following an arbitrary number of column interchanges with the accompanying sign change of the determinant for each, it can be shown from the first form of Eq 2.4-11 that, (see Prob 2.12)

$$\varepsilon_{qmn} \det \mathcal{A} = \varepsilon_{ijk} A_{iq} A_{jm} A_{kn} \tag{2.4-12}$$

Finally, we note that if the det $\mathcal{A} = 0$, the matrix is said to be *singular*. It may be easily shown that every 3×3 skew-symmetric matrix is singular. Also, the determinant of the diagonal matrix, $\mathcal{D}$, is simply the product of its diagonal elements: det $\mathcal{D} = D_{11}D_{22}\dots D_{NN}$.

Example 2.4-3

Show that for matrices $\mathcal{A}$ and $\mathcal{B}$, det $\mathcal{AB}$ = det $\mathcal{BA}$ = det $\mathcal{A}$ det $\mathcal{B}$.

Solution

Let $\mathcal{C} = \mathcal{AB}$, then $C_{ij} = A_{ik}B_{kj}$ and from Eq. (2.4-11)

$$\begin{aligned}\det \mathcal{C} &= \varepsilon_{ijk}C_{i1}C_{j2}C_{k3}\\ &= \varepsilon_{ijk}A_{iq}B_{q1}A_{jm}B_{m2}A_{kn}B_{n3}\\ &= \varepsilon_{ijk}A_{iq}A_{jm}A_{kn}B_{q1}B_{m2}B_{n3}\end{aligned}$$

but from Eq 2.4-12,

$$\varepsilon_{ijk}A_{iq}A_{jm}A_{kn} = \varepsilon_{qmn}\det \mathcal{A}$$

so now

$$\det \mathcal{C} = \det \mathcal{AB} = \varepsilon_{qmn}B_{q1}B_{m2}B_{n3}\det \mathcal{A} = \det \mathcal{B} \det \mathcal{A}$$

By a direct interchange of $\mathcal{A}$ and $\mathcal{B}$, det $\mathcal{AB}$ = det $\mathcal{BA}$.

Example 2.4-4

Use Eq 2.4-9 and Eq 2.4-10 to show that det $\mathcal{A}$ = det $\mathcal{A}^{\mathrm{T}}$.

Solution

Since

$$\left|A^T\right| = \begin{vmatrix} A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{vmatrix}$$

cofactor expansion by the first column here yields

$$\left|\mathcal{A}^{\mathrm{T}}\right| = A_{11}\begin{vmatrix} A_{22} & A_{32}\\ A_{23} & A_{33}\end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{31}\\ A_{23} & A_{33}\end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{31}\\ A_{22} & A_{32}\end{vmatrix}$$

which is identical to Eq 2.4-9 and hence equal to Eq 2.4-10.

The *inverse* of the matrix $\mathcal{A}$ is written $\mathcal{A}^{-1}$, and is defined by

$$\mathcal{A}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A} = \mathbf{I} \tag{2.4-13}$$

where $\mathbf{I}$ is the identity matrix. Thus, if $\mathcal{A}\mathcal{B} = \mathbf{I}$, then $\mathcal{B} = \mathcal{A}^{-1}$, and $\mathcal{A} = \mathcal{B}^{-1}$. The adjoint matrix $\mathcal{A}^{*}$ is defined as the transpose of the cofactor matrix

$$\mathcal{A}^{*} = [\mathcal{A}^{(c)}]^{\mathrm{T}} \tag{2.4-14}$$

In terms of the adjoint matrix the inverse matrix is expressed by

$$\mathcal{A}^{-1} = \frac{\mathcal{A}^{*}}{\det\mathcal{A}} \tag{2.4-15}$$

which is actually a working formula by which an inverse matrix may be calculated. This formula shows that the inverse matrix exists only if $\det\mathcal{A} \neq 0$, i.e., only if the matrix $\mathcal{A}$ is non-singular. In particular, a 3×3 skew-symmetric matrix has no inverse.

Example 2.4-5

Show from the definition of the inverse, Eq 2.4-13 that

$$\text{(a) } (\mathcal{A}\mathcal{B})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$$

$$\text{(b) } (\mathcal{A}^{\mathrm{T}})^{-1} = (\mathcal{A}^{-1})^{\mathrm{T}}$$

Solution

(a) By premultiplying the matrix product $\mathcal{A}\mathcal{B}$ by $\mathcal{B}^{-1}\mathcal{A}^{-1}$, we have (using Eq 2.4-13),

$$\mathcal{B}^{-1}\mathcal{A}^{-1}\,\mathcal{A}\mathcal{B} = \mathcal{B}^{-1}\mathbf{I}\,\mathcal{B} = \mathcal{B}^{-1}\,\mathcal{B} = \mathbf{I}$$

and therefore $\mathcal{B}^{-1}\mathcal{A}^{-1} = (\mathcal{A}\mathcal{B})^{-1}$.

(b) Taking the transpose of both sides of Eq 2.4-13 and using the result of Example 2.4-2 (b) we have

$$(\mathcal{A}\mathcal{A}^{-1})^{\mathrm{T}} = (\mathcal{A}^{-1})^{\mathrm{T}}\mathcal{A}^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$$

Hence, $(\mathcal{A}^{-1})^{\mathrm{T}}$ must be the inverse of $\mathcal{A}^{\mathrm{T}}$, or $(\mathcal{A}^{-1})^{\mathrm{T}} = (\mathcal{A}^{\mathrm{T}})^{-1}$.

An *orthogonal* matrix, call it $\mathbf{Q}$, is a square matrix for which $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. From this definition we note that a *symmetric orthogonal* matrix is its own inverse, since in this case

$$\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}} = \mathbf{Q} \tag{2.4-16}$$

Also, if $\mathcal{A}$ and $\mathcal{B}$ are orthogonal matrices.

$$(\mathcal{A}\mathcal{B})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1} = \mathcal{B}^{\mathrm{T}}\mathcal{A}^{\mathrm{T}} = (\mathcal{A}\mathcal{B})^{\mathrm{T}} \tag{2.4-17}$$

so that the product matrix is likewise orthogonal. Furthermore, if $\mathcal{A}$ is orthogonal it may be shown (see Prob. 2.16) that

$$\det \mathcal{A} = \pm 1 \tag{2.4-18}$$

As mentioned near the beginning of this section, a vector may be represented by a row or column matrix, a second-order tensor by a square 3×3 matrix. For computational purposes, it is frequently advantageous to transcribe vector/tensor equations into their matrix form. As a very simple example, the vector-tensor product, $\mathbf{u} = \mathbf{T} \cdot \mathbf{v}$ (symbolic notation) or $u_i = T_{ij}v_j$ (indicial notation) appears in matrix form as

$$\left[u_{i1}\right] = \left[T_{ij}\right]\left[v_{j1}\right] \quad \text{or} \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.4-19}$$

In much the same way the product

$$\mathbf{w} = \mathbf{v} \cdot \mathbf{T}, \quad \text{or} \quad w_i = v_j T_{ji}$$

appears as

$$\left[w_{1i}\right] = \left[v_{1j}\right]\left[T_{ji}\right] \quad \text{or} \quad \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}^{\mathrm{T}} \tag{2.4-20}$$

2.5 Transformations of Cartesian Tensors

As already mentioned, although vectors and tensors have an identity independent of any particular reference or coordinate system, the relative values of their respective components do depend upon the specific axes to which they are referred. The relationships among these various components when given with respect to two separate sets of coordinate axes are known as the *transformation equations*. In developing these transformation equations for Cartesian tensors we consider two sets of rectangular Cartesian axes, $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$, sharing a common origin, and oriented relative to one

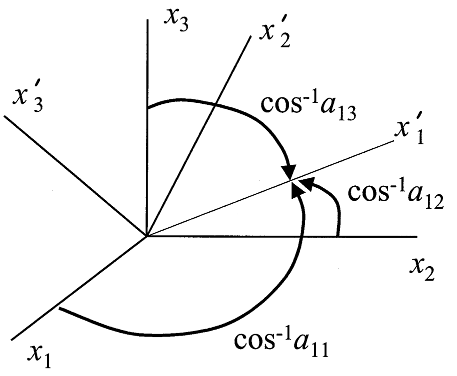


FIGURE 2.2A
Rectangular coordinate system $Ox'_1x'_2x'_3$ relative to $Ox_1x_2x_3$. Direction cosines shown for coordinate x'_1 relative to unprimed coordinates. Similar direction cosines are defined for x'_2 and x'_3 coordinates.

		$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
		x_1	x_2	x_3
$\hat{\mathbf{e}}'_1$	x'_1	a_{11}	a_{12}	a_{13}
$\hat{\mathbf{e}}'_2$	x'_2	a_{21}	a_{22}	a_{23}
$\hat{\mathbf{e}}'_3$	x'_3	a_{31}	a_{32}	a_{33}

FIGURE 2.2B
Transformation table between $Ox_1x_2x_3$. and $Ox'_1x'_2x'_3$ axes.

another so that the direction cosines between the primed and unprimed axes are given by $a_{ij} = \cos(x'_i, x_j)$ as shown in Figure 2.2A.

The square array of the nine direction cosines displayed in Figure 2.2B is useful in relating the unit base vectors $\hat{\mathbf{e}}'_i$ and $\hat{\mathbf{e}}_i$ to one another, as well as relating the primed and unprimed coordinates x'_i and x_i of a point. Thus, the primed base vectors $\hat{\mathbf{e}}'_i$ are given in terms of the unprimed vectors $\hat{\mathbf{e}}_i$ by the equations (as is also easily verified from the geometry of the vectors in the diagram of Figure 2.2A),

$$\hat{\mathbf{e}}'_1 = a_{11}\hat{\mathbf{e}}_1 + a_{12}\hat{\mathbf{e}}_2 + a_{13}\hat{\mathbf{e}}_3 = a_{1j}\hat{\mathbf{e}}_j \tag{2.5-1a}$$

$$\hat{\mathbf{e}}'_2 = a_{21}\hat{\mathbf{e}}'_1 + a_{22}\hat{\mathbf{e}}'_2 + a_{23}\hat{\mathbf{e}}'_3 = a_{2j}\hat{\mathbf{e}}_j \tag{2.5-1b}$$

$$\hat{\mathbf{e}}'_3 = a_{31}\hat{\mathbf{e}}_1 + a_{32}\hat{\mathbf{e}}_2 + a_{33}\hat{\mathbf{e}}_3 = a_{3j}\hat{\mathbf{e}}_j \tag{2.5-1c}$$

or in compact indicial form

$$\hat{\mathbf{e}}'_i = a_{ij}\hat{\mathbf{e}}_j \tag{2.5-2}$$

By defining the matrix $\mathcal{A}$ whose elements are the direction cosines a_{ij}, Eq 2.5-2 can be written in matrix form as

$$\left[\hat{\mathbf{e}}'_{i1}\right]=\left[a_{ij}\right]\left[\hat{\mathbf{e}}_{j1}\right] \quad or \quad \begin{bmatrix}\hat{\mathbf{e}}'_1\\ \hat{\mathbf{e}}'_2\\ \hat{\mathbf{e}}'_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{bmatrix} \tag{2.5-3}$$

where the elements of the column matrices are unit vectors. The matrix $\mathcal{A}$ is called the *transformation matrix* because, as we shall see, of its role in transforming the components of a vector (or tensor) referred to one set of axes into the components of the same vector (or tensor) in a rotated set.

Because of the perpendicularity of the primed axes, $\hat{\mathbf{e}}'_i\cdot\hat{\mathbf{e}}'_j=\delta_{ij}$. But also, in view of Eq 2.5-2, $\hat{\mathbf{e}}'_i\cdot\hat{\mathbf{e}}'_j=a_{iq}\hat{\mathbf{e}}_q\cdot a_{jm}\hat{\mathbf{e}}_m=a_{iq}a_{jm}\delta_{qm}=a_{iq}a_{jq}=\delta_{ij}$, from which we extract the *orthogonality condition* on the direction cosines (given here in both indicial and matrix form),

$$a_{iq}a_{jq}=\delta_{ij} \quad \text{or} \quad \mathcal{A}\mathcal{A}^{\mathrm{T}}=\mathbf{I} \tag{2.5-4}$$

Note that this is simply the inner product of the i^{th} row with the j^{th} row of the matrix $\mathcal{A}$. By an analogous derivation to that leading to Eq 2.5-2, but using the columns of $\mathcal{A}$, we obtain

$$\hat{\mathbf{e}}_i=a_{ji}\hat{\mathbf{e}}'_j \tag{2.5-5}$$

which in matrix form is

$$\left[\hat{\mathbf{e}}_{1i}\right]=\left[\hat{\mathbf{e}}'_{1j}\right]\left[a_{ji}\right] \quad \text{or} \quad \left[\hat{\mathbf{e}}_1 \quad \hat{\mathbf{e}}_2 \quad \hat{\mathbf{e}}_3\right]=\left[\hat{\mathbf{e}}'_1 \quad \hat{\mathbf{e}}'_2 \quad \hat{\mathbf{e}}'_3\right]\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{2.5-6}$$

Note that using the transpose $\mathcal{A}^{\mathrm{T}}$, Eq 2.5-6 may also be written

$$\left[\hat{\mathbf{e}}_{i1}\right]=\left[a_{ij}\right]^{\mathrm{T}}\left[\hat{\mathbf{e}}'_{j1}\right] \quad \text{or} \quad \begin{bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}'_1\\ \hat{\mathbf{e}}'_2\\ \hat{\mathbf{e}}'_3\end{bmatrix} \tag{2.5-7}$$

in which column matrices are used for the vectors $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$. By a consideration of the dot product $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\delta_{ij}$ and Eq 2.5-5 we obtain a second orthogonality condition

$$a_{ij}a_{ik}=\delta_{jk} \quad \text{or} \quad \mathcal{A}^{\mathrm{T}}\mathcal{A}=\mathbf{I} \tag{2.5-8}$$

which is the inner product of the j^{th} column with the k^{th} column of $\mathcal{A}$.

Consider next an arbitrary vector $\mathbf{v}$ having components v_i in the unprimed system, and v_i' in the primed system. Then using Eq 2.5-5,

$$\mathbf{v} = v_j'\hat{\mathbf{e}}_j' = v_i\hat{\mathbf{e}}_i = v_i a_{ji}\hat{\mathbf{e}}_j'$$

from which by matching coefficients on $\hat{\mathbf{e}}_j'$ we have (in both the indicial and matrix forms),

$$v_j' = a_{ji}v_i \quad \text{or} \quad \mathbf{v}' = \mathbf{A}\mathbf{v} = \mathbf{v}\mathbf{A}^{\mathrm{T}} \tag{2.5-9}$$

which is the *transformation law* expressing the primed components of an arbitrary vector in terms of its unprimed components. Although the elements of the transformation matrix are written as a_{ij} we must emphasize that they are not the components of a second-order Cartesian tensor as it would appear.

Multiplication of Eq 2.5-9 by a_{jk} and use of the orthogonality condition Eq 2.5-8 obtains the inverse law

$$v_k = a_{jk}v_j' \quad \text{or} \quad \mathbf{v} = \mathbf{v}'\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{v}' \tag{2.5-10}$$

giving the unprimed components in terms of the primed.

By a direct application of Eq 2.5-10 to the dyad $\mathbf{uv}$ we have

$$u_i v_j = a_{qi}u_q' a_{mj}v_m' = a_{qi}a_{mj}u_q'v_m' \tag{2.5-11}$$

But a dyad is, after all, one form of a second-order tensor, and so by an obvious adaptation of Eq 2.5-11 we obtain the transformation law for a second-order tensor, $\mathbf{T}$, as

$$T_{ij} = a_{qi}a_{mj}T_{qm}' \quad \text{or} \quad \mathbf{T} = \mathbf{A}^{\mathrm{T}}\mathbf{T}'\mathbf{A} \tag{2.5-12}$$

which may be readily inverted with the help of the orthogonality conditions to yield

$$T_{ij}' = a_{iq}a_{jm}T_{qm} \quad \text{or} \quad \mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^{\mathrm{T}} \tag{2.5-13}$$

Note carefully the location of the summed indices q and m in Equations 2.5-12 and 2.5-13. Finally, by a logical generalization of the pattern of the transformation rules developed thus far, we state that for an arbitrary Cartesian tensor of any order

$$R_{ij\ldots k}' = a_{iq}a_{jm}\cdots a_{kn}R_{qm\ldots n} \tag{2.5-14}$$

The primed axes may be related to the unprimed axes through either a *rotation* about an axis through the origin, or by a *reflection* of the axes in one of the coordinate planes (or by a combination of such changes). As a simple example, conside a 90° counterclockwise rotation about the x_2 axis shown in Figure 2.3*a*. The matrix of direction cosines for this rotation is

$$\left[a_{ij}\right] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

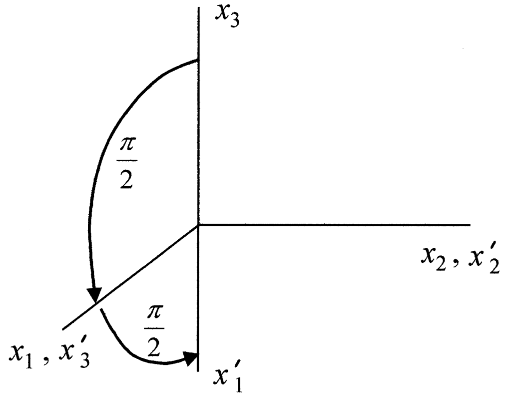


FIGURE 2.3A
$Ox'_1x'_2x'_3$ axes relative to $Ox_1x_2x_3$ axes following a 90° rotation counterclockwise about the x_2 axis.

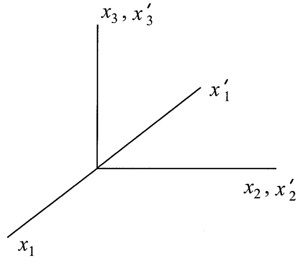


FIGURE 2.3B
$Ox'_1x'_2x'_3$ axes relative to $Ox_1x_2x_3$ axes following a reflection in the x_2x_3-plane.

and det $\mathcal{A} = 1$. The transformation of tensor components in this case is called a *proper orthogonal transformation*. For a reflection of axes in the x_2x_3 plane shown in Figure 2.3B the transformation matrix is

$$\left[a_{ij}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where det $\mathcal{A} = -1$, and we have an *improper orthogonal transformation*. It may be shown that *true* (*polar*) *vectors* transform by the rules $v'_i = a_{ij}v_j$ and $v_j = a_{ij}v'_i$ regardless of whether the axes transformation is proper or improper. However, *pseudo* (*axial*) *vectors* transform correctly only according to $v'_i = (\det \mathcal{A})\, a_{ij}v_j$ and $v_j = (\det \mathcal{A})\, a_{ij}\, v'_i$ under an improper transformation of axes.

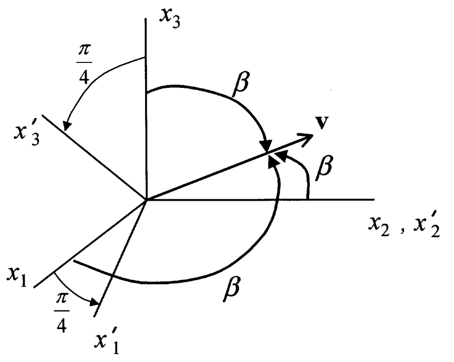


FIGURE E2.5-1
Vector $\mathbf{v}$ with respect to axes $Ox'_1x'_2x'_3$ and $Ox_1x_2x_3$.

Example 2.5-1

Let the primed axes $Ox'_1x'_2x'_3$ be given with respect to the unprimed axes by a 45° counterclockwise rotation about the x_2 axis as shown. Determine the primed components of the vector given by $\mathbf{v} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$.

Solution

Here the transformation matrix is

$$\left[a_{ij}\right] = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

and from Eq 2.5-9 in matrix form

$$\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

Example 2.5-2

Determine the primed components of the tensor

$$\left[T_{ij}\right] = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 8 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

under the rotation of axes described in Example 2.5-1.

Solution

Here Eq 2.5-13 may be used. Thus in matrix form

$$\left[T'_{ij}\right]=\begin{bmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2 & 6 & 4\\ 0 & 8 & 0\\ 4 & 2 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ -1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}$$

$$=\begin{bmatrix}-3 & 4/\sqrt{2} & 1\\ 0 & 8 & 0\\ 1 & 8/\sqrt{2} & 5\end{bmatrix}$$

2.6 Principal Values and Principal Directions of Symmetric Second-Order Tensors

First, let us note that in view of the form of the inner product of a second-order tensor **T** with the arbitrary vector **u** (which we write here in both the indicial and symbolic notation),

$$T_{ij}u_j = v_i \quad \text{or} \quad \mathbf{T}\cdot\mathbf{u}=\mathbf{v} \tag{2.6-1}$$

any second-order tensor may be thought of as a *linear transformation* which transforms the *antecedent* vector **u** into the *image* vector **v** in a Euclidean three-space. In particular, for every symmetric tensor **T** having real components T_{ij}, and defined at some point in physical space, there is associated with each direction at that point (identified by the unit vector n_i), an image vector v_i given by

$$T_{ij}n_j = v_i \quad \text{or} \quad \mathbf{T}\cdot\hat{\mathbf{n}}=\mathbf{v} \tag{2.6-2}$$

If the vector v_i determined by Eq 2.6-2 happens to be a scalar multiple of n_i, that is, if

$$T_{ij}n_j = \lambda n_i \quad \text{or} \quad \mathbf{T}\cdot\hat{\mathbf{n}}=\lambda\hat{\mathbf{n}} \tag{2.6-3}$$

the direction defined by n_i is called a *principal direction,* or *eigenvector,* of **T**, and the scalar λ is called a *principal value,* or *eigenvalue* of **T**. Using the substitution property of the Kronecker delta, Eq 2.6-3 may be rewritten as

$$\left(T_{ij} - \lambda \delta_{ij}\right) n_j = 0 \quad \text{or} \quad (\mathbf{T} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = 0 \tag{2.6-4}$$

or in expanded form

$$\left(T_{11} - \lambda\right) n_1 + T_{12} n_2 + T_{13} n_3 = 0 \tag{2.6-5a}$$

$$T_{21} n_1 + \left(T_{22} - \lambda\right) n_2 + T_{23} n_3 = 0 \tag{2.6-5b}$$

$$T_{31} n_1 + T_{32} n_2 + \left(T_{33} - \lambda\right) n_3 = 0 \tag{2.6-5c}$$

This system of homogeneous equations for the unknown direction n_i and the unknown λ's will have non-trivial solutions only if the determinant of coefficients vanishes. Thus,

$$\left| T_{ij} - \lambda \delta_{ij} \right| = 0 \tag{2.6-6}$$

which upon expansion leads to the cubic in λ (called the *characteristic equation*)

$$\lambda^3 - I_{\mathbf{T}} \lambda^2 + II_{\mathbf{T}} \lambda - III_{\mathbf{T}} = 0 \tag{2.6-7}$$

where the coefficients here are expressed in terms of the known components of **T** by

$$I_{\mathbf{T}} = T_{ii} = \operatorname{tr} \mathbf{T} \tag{2.6-8a}$$

$$II_{\mathbf{T}} = \frac{1}{2}\left(T_{ii} T_{jj} - T_{ij} T_{ji}\right) = \frac{1}{2}\left[(\operatorname{tr} \mathbf{T})^2 - \operatorname{tr}\left(\mathbf{T}^2\right)\right] \tag{2.6-8b}$$

$$III_{\mathbf{T}} = \varepsilon_{ijk} T_{1i} T_{2j} T_{3k} = \det \mathbf{T} \tag{2.6-8c}$$

and are known as the *first, second,* and *third invariants,* respectively, of the tensor **T**. The sum of the elements on the principal diagonal of the matrix form of any tensor is called the *trace* of that tensor, and for the tensor **T** is written tr **T** as in Eq 2.6-8.

The roots $\lambda_{(1)}$, $\lambda_{(2)}$, and $\lambda_{(3)}$ of Eq 2.6-7 are all real for a symmetric tensor **T** having real components. With each of these roots $\lambda_{(q)} \quad (q = 1, 2, 3)$ we can determine a principal direction $n_i^{(q)} \quad (q = 1, 2, 3)$ by solving Eq 2.6-4 together with the normalizing condition $n_i n_i = 1$. Thus, Eq 2.6-4 is satisfied by

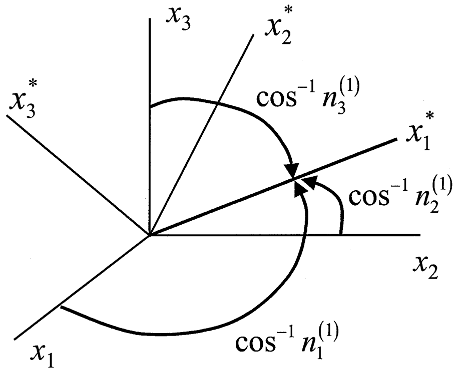


FIGURE 2.4
Principal axes $Ox_1^* x_2^* x_3^*$ relative to axes $Ox_1x_2x_3$.

$$\left[T_{ij} - \lambda_{(q)}\delta_{ij}\right]n_i^{(q)} = 0\,,\ (q = 1,2,3) \tag{2.6-9}$$

with

$$n_i^{(q)} n_i^{(q)} = 1\,,\ (q = 1,2,3) \tag{2.6-10}$$

If the $\lambda_{(q)}$'s are distinct the principal directions are unique and mutually perpendicular. If, however, there is a pair of equal roots, say $\lambda_{(1)} = \lambda_{(2)}$, then only the direction associated with $\lambda_{(3)}$ will be unique. In this case any other two directions which are orthogonal to $n_i^{(3)}$, and to one another so as to form a right-handed system, may be taken as principal directions. If $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$, every set of right-handed orthogonal axes qualifies as principal axes, and every direction is said to be a principal direction.

In order to reinforce the concept of principal directions, let the components of the tensor **T** be given initially with respect to arbitrary Cartesian axes $Ox_1x_2x_3$, and let the principal axes of **T** be designated by $Ox_1^* x_2^* x_3^*$, as shown in Figure 2.4. The transformation matrix $\mathcal{A}$ between these two sets of axes is established by taking the direction cosines $n_i^{(q)}$ as calculated from Eq 2.6-9 and Eq 2.6-10 as the elements of the q^{th} row of $\mathcal{A}$. Therefore, by definition, $a_{ij} \equiv n_j^{(i)}$ as detailed in the table below.

	x_1 *or* $\hat{\mathbf{e}}_1$	x_2 *or* $\hat{\mathbf{e}}_2$	x_3 *or* $\hat{\mathbf{e}}_3$	
x_1^* *or* $\hat{\mathbf{e}}_1^*$	$a_{11} = n_1^{(1)}$	$a_{12} = n_2^{(1)}$	$a_{13} = n_3^{(1)}$	
x_2^* *or* $\hat{\mathbf{e}}_2^*$	$a_{21} = n_1^{(2)}$	$a_{22} = n_2^{(2)}$	$a_{23} = n_3^{(2)}$	
x_3^* *or* $\hat{\mathbf{e}}_3^*$	$a_{31} = n_1^{(3)}$	$a_{32} = n_2^{(3)}$	$a_{33} = n_3^{(3)}$	(2.6-11)

The transformation matrix here is orthogonal and in accordance with the transformation law for second-order tensors

$$T^*_{ij} = a_{iq} a_{jm} T_{qm} \quad \text{or} \quad \mathbf{T}^* = \mathbf{A} \mathbf{T} \mathbf{A}^{\mathrm{T}} \tag{2.6-12}$$

where T^* is a diagonal matrix whose elements are the principal values $\lambda_{(q)}$.

Example 2.6-1

Determine the principal values and principal directions of the second-order tensor **T** whose matrix representation is

$$\left[T_{ij}\right] = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution

Here Eq 2.6-6 is given by

$$\begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

which upon expansion by the third row becomes

$$(3-\lambda)\left(10 - 7\lambda + \lambda^2 - 4\right) = 0$$

or

$$(3-\lambda)(6-\lambda)(1-\lambda) = 0$$

Hence, $\lambda_{(1)} = 3, \lambda_{(2)} = 6, \lambda_{(3)} = 1$ are the principal values of **T**. For $\lambda_{(1)} = 3$, Eq 2.6-5 yields the equations

$$2n_1 + 2n_2 = 0$$

$$2n_1 - n_2 = 0$$

which are satisfied only if $n_1 = n_2 = 0$, and so from $n_i n_i = 1$ we have $n_3 = \pm 1$. For $\lambda_{(2)} = 6$, Eq 2.6-5 yields

$$-n_1 + 2n_2 = 0$$

$$2n_1 - 4n_2 = 0$$

$$-3n_3 = 0$$

so that $n_1 = 2n_2$ and since $n_3 = 0$, we have $(2n_2)^2 + n_2^2 = 1$, or $n_2 = \pm 1/\sqrt{5}$, and $n_1 = \pm 2/\sqrt{5}$. For $\lambda_{(3)} = 1$, Eq 2.6-5 yields

$$4n_1 + 2n_2 = 0$$

$$2n_1 + \quad n_2 = 0$$

together with $2n_3 = 0$. Again $n_3 = 0$, and here $n_1^2 + (-2n_1)^2 = 1$ so that $n_1 = \pm 1/\sqrt{5}$ and $n_1 = \pm 2/\sqrt{5}$. From these results the transformation matrix $\mathcal{A}$ is given by

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm 2/\sqrt{5} & \pm 1/\sqrt{5} & 0 \\ \mp 1/\sqrt{5} & \pm 2/\sqrt{5} & 0 \end{bmatrix}$$

which identifies two sets of principal direction axes, one a reflection of the other with respect to the origin. Also, it may be easily verified that $\mathcal{A}$ is orthogonal by multiplying it with its transpose $\mathcal{A}^{\mathrm{T}}$ to obtain the identity matrix. Finally, from Eq 2.6-12 we see that using the upper set of the ± signs,

$$\begin{bmatrix} 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2/\sqrt{5} & -1/\sqrt{5} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2.6-2

Show that the principal values for the tensor having the matrix

$$[T_{ij}] = \begin{bmatrix} 5 & 1 & \sqrt{2} \\ 1 & 5 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 6 \end{bmatrix}$$

have a multiplicity of two, and determine the principal directions.

Solution
Here Eq 2.6-6 is given by

$$\begin{vmatrix} 5-\lambda & 1 & \sqrt{2} \\ 1 & 5-\lambda & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 6-\lambda \end{vmatrix} = 0$$

for which the characteristic equation becomes

$$\lambda^3 - 16\lambda^2 + 80\lambda - 128 = 0$$

or

$$(\lambda - 8)(\lambda - 4)^2 = 0$$

For $\lambda_{(1)} = 8$, Eq 2.6-5 yields

$$-3n_1 + n_2 + \sqrt{2}n_3 = 0$$

$$n_1 - 3n_2 + \sqrt{2}n_3 = 0$$

$$\sqrt{2}n_1 + \sqrt{2}n_2 - 2n_3 = 0$$

From the first two of these equations $n_1 = n_2$, and from the second and third equations $n_3 = \sqrt{2}n_2$. Therefore, using, $n_i n_i = 1$, we have

$$(n_2)^2 + (n_2)^2 + \left(\sqrt{2}n_2\right)^2 = 1$$

and so $n_1 = n_2 = \pm 1/2$ and $n_3 = \pm 1/\sqrt{2}$, from which the unit vector in the principal direction associated with $\lambda_{(1)} = 8$ (the so-called normalized eigen-vector) is

$$\hat{\mathbf{n}}^{(1)} = \frac{1}{2}\left(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \sqrt{2}\hat{\mathbf{e}}_3\right) = \hat{\mathbf{e}}_3^*$$

For $\hat{\mathbf{n}}^{(2)}$, we choose any unit vector perpendicular to $\hat{\mathbf{n}}^{(1)}$; an obvious choice being

$$\hat{\mathbf{n}}^{(2)} = \frac{-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{2}} = \hat{\mathbf{e}}_2^*$$

Then $\hat{\mathbf{n}}^{(3)}$ is constructed from $\hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \times \hat{\mathbf{n}}^{(2)}$, so that

$$\hat{\mathbf{n}}^{(3)} = \frac{1}{2}\left(-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \sqrt{2}\hat{\mathbf{e}}_3\right) = \hat{\mathbf{e}}_3^*$$

Thus, the transformation matrix $\mathcal{A}$ is given by Eq 2.6-11 as

$$\left[a_{ij}\right] = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

In concluding this section, we mention several interesting properties of symmetric second-order tensors. (1) The principal values and principal directions of $\mathbf{T}$ and $\mathbf{T}^{\mathrm{T}}$ are the same. (2) The principal values of $\mathbf{T}^{-1}$ are reciprocals of the principal values of $\mathbf{T}$, and both have the same principal directions. (3) The product tensors $\mathbf{TQ}$ and $\mathbf{QT}$ have the same principal values. (4) A symmetric tensor is said to be *positive* (*negative*) *definite* if all of its principal values are positive (negative); and *positive* (*negative*) *semi-definite* if one principal value is zero and the others positive (negative).

2.7 Tensor Fields, Tensor Calculus

A *tensor field* assigns to every location $\mathbf{x}$, at every instant of time t, a tensor $T_{ij\ldots k}(\mathbf{x},t)$, for which $\mathbf{x}$ ranges over a finite region of space, and t varies over some interval of time. The field is continuous and hence differentiable if the components $T_{ij\ldots k}(\mathbf{x},t)$ are continuous functions of $\mathbf{x}$ and t. Tensor fields may be of any order. For example, we may denote typical scalar, vector, and tensor fields by the notations $\phi(\mathbf{x},t)$, $v_i(\mathbf{x},t)$, and $T_{ij}(\mathbf{x},t)$, respectively.

Partial differentiation of a tensor field with respect to the variable t is symbolized by the operator $\partial/\partial t$ and follows the usual rules of calculus. On the other hand, partial differentiation with respect to the coordinate x_q will be indicated by the operator $\partial/\partial x_q$, which may be abbreviated as simply ∂_q. Likewise, the second partial $\partial^2/\partial x_q \partial x_m$ may be written ∂_{qm}, and so on. As an additional measure in notational compactness it is customary in continuum mechanics to introduce the *subscript comma* to denote partial differentiation with respect to the coordinate variables. For example, we write $\phi_{,i}$ for $\partial\phi / \partial x_i$; $v_{i,j}$ for $\partial v_i / \partial x_j$;$T_{ij,k}$ for $\partial T_{ij} / \partial x_k$; and $u_{i,jk}$ for $\partial^2 u_i / \partial x_j \partial x_k$. We note from these examples that differentiation with respect to a coordinate produces a tensor of one order higher. Also, a useful identity results from the derivative $\partial x_i / \partial x_j$, viz.,

$$\partial x_i / \partial x_j = x_{i,j} = \delta_{ij} \tag{2.7-1}$$

In the notation adopted here the operator $\boldsymbol{\nabla}$ (del) of vector calculus, which in symbolic notation appears as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3 = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \tag{2.7-2}$$

takes on the simple form ∂_i. Therefore, we may write the *scalar gradient* $\boldsymbol{\nabla}\phi$ = **grad** ϕ as

$$\partial_i \phi = \phi_{,i} \tag{2.7-3}$$

the vector gradient $\boldsymbol{\nabla}\,\mathbf{v}$ as

$$\partial_i v_j = v_{j,i} \tag{2.7-4}$$

the divergence of **v**, $\boldsymbol{\nabla}\cdot\mathbf{v}$ as

$$\partial_i v_i = v_{i,i} \tag{2.7-5}$$

and the curl of **v**, $\boldsymbol{\nabla}\times\mathbf{v}$ as

$$\varepsilon_{ijk}\partial_j v_k = \varepsilon_{ijk} v_{k,j} \tag{2.7-6}$$

Note in passing that many of the identities of vector analysis can be verified with relative ease by manipulations using the indicial notation. For example, to show that div (**curl v**) = 0 for any vector **v** we write from Eqs 2.7-6 and Eq 2.7-5

$$\partial_i\left(\varepsilon_{ijk} v_{k,j}\right) = \varepsilon_{ijk} v_{k,ji} = 0$$

and because the first term of this inner product is skew-symmetric in *i* and *j*, whereas the second term is symmetric in the same indices, (since v_k is assumed to have continuous spatial gradients), their product is zero.

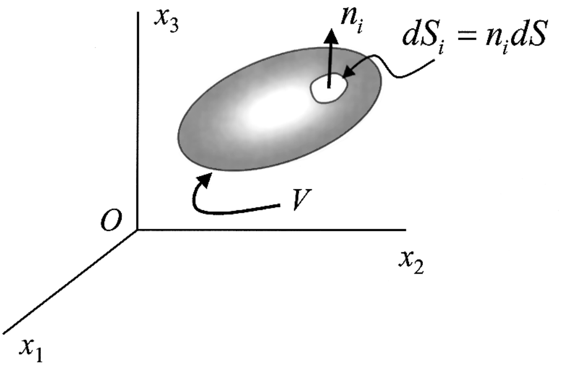


FIGURE 2.5A
Volume V with infinitesimal surface element dS_i having a unit normal vector n_i.

2.8 Integral Theorems of Gauss and Stokes

Consider an arbitrary continuously differentiable tensor field $T_{ij\ldots k}$ defined on some finite region of physical space. Let V be a volume in this space with a closed surface S bounding the volume, and let the outward normal to this bounding surface be n_i as shown in Figure 2.5A so that the element of surface is given by $dS_i = n_i dS$. The *divergence theorem of Gauss* establishes a relationship between the surface integral having $T_{ij\ldots k}$ as integrand to the volume integral for which a coordinate derivative of $T_{ij\ldots k}$ is the integrand. Specifically,

$$\int_S T_{ij\ldots k} n_q dS = \int_V T_{ij\ldots k,q} dV \tag{2.8-1}$$

Several important special cases of this theorem for scalar and vector fields are worth noting, and are given here in both indicial and symbolic notation.

$$\int_S \lambda n_q dS = \int_V \lambda_{,q} dV \quad \text{or} \quad \int_S \lambda \hat{\mathbf{n}} dS = \int_V \mathbf{grad}\lambda dV \tag{2.8-2}$$

$$\int_S v_q n_q dS = \int_V v_{q,q} dV \quad \text{or} \quad \int_S \mathbf{v} \cdot \hat{\mathbf{n}} dS = \int_V \mathbf{div}\ \mathbf{v}\ dV \tag{2.8-3}$$

$$\int_S \varepsilon_{ijk} n_j v_k dS = \int_V \varepsilon_{ijk} v_{k,j} dV \quad \text{or} \quad \int_S \hat{\mathbf{n}} \times \mathbf{v} dS = \int_V \mathbf{curl}\ \mathbf{v}\ dV \tag{2.8-4}$$

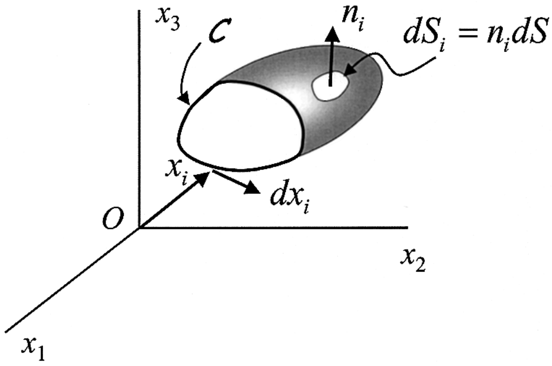


FIGURE 2.5B
Bounding space curve C with tangential vector dx_i and surface element dS_i for partial volume.

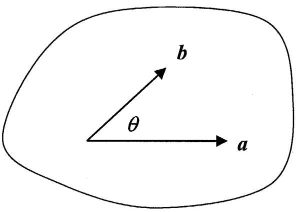
Called *Gauss's divergence theorem*, Eq 2.8-3 is the one presented in a traditional vector calculus course.

Whereas Gauss's theorem relates an integral over a closed volume to an integral over its bounding surface, *Stokes' theorem* relates an integral over an open surface (a so-called *cap*) to a line integral around the bounding curve of the surface. Therefore, let C be the bounding space curve to the surface S, and let dx_i be the *differential tangent vector* to C as shown in Figure 2.5B. (A hemispherical surface having a circular bounding curve C is a classic example). If n_i is the outward normal to the surface S, and v_i is any vector field defined on S and C, Stokes' theorem asserts that

$$\int_S \varepsilon_{ijk} n_i v_{k,j} dS = \int_C v_k dx_k \quad \text{or} \quad \int_S \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) dS = \int_C \mathbf{v} \cdot d\mathbf{x} \tag{2.8-5}$$

The integral on the right-hand side of this equation is often referred to as the *circulation* when the vector $\mathbf{v}$ is the velocity vector.

Problems



2.1 Let $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, or in indicial notation,

$$v_i\hat{\mathbf{e}}_i = a_j\hat{\mathbf{e}}_j \times b_k\hat{\mathbf{e}}_k = \varepsilon_{ijk}a_jb_k\hat{\mathbf{e}}_i$$

Using indicial notation, show that,

(a) $\mathbf{v} \cdot \mathbf{v} = a^2b^2\sin^2\theta$

(b) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$

(c) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0$

2.2 With respect to the triad of base vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (not necessarily unit vectors), the triad $\mathbf{u}^1$, $\mathbf{u}^2$, and $\mathbf{u}^3$ is said to be a *reciprocal basis* if $\mathbf{u}_i \cdot \mathbf{u}^j = \delta_{ij}$ (i, j = 1, 2, 3). Show that to satisfy these conditions,

$$\mathbf{u}^1 = \frac{\mathbf{u}_2 \times \mathbf{u}_3}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}\ ;\ \mathbf{u}^2 = \frac{\mathbf{u}_3 \times \mathbf{u}_1}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}\ ;\ \mathbf{u}^3 = \frac{\mathbf{u}_1 \times \mathbf{u}_2}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}$$

and determine the reciprocal basis for the specific base vectors

$$\mathbf{u}_1 = 2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$$

$$\mathbf{u}_2 = 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$$

$$\mathbf{u}_3 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

Answer:
$$\mathbf{u}^1 = \frac{1}{5}\left(3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3\right)$$

$$\mathbf{u}^2 = \frac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\right)$$

$$\mathbf{u}^3 = \frac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3\right)$$

2.3 Let the position vector of an arbitrary point $P(x_1x_2x_3)$ be $\mathbf{x} = x_i\hat{e}_i$, and let $\mathbf{b} = b_i\hat{e}_i$ be a *constant vector*. Show that $(\mathbf{x} - \mathbf{b}) \cdot \mathbf{x} = 0$ is the vector equation of a spherical surface having its center at $\mathbf{x} = \frac{1}{2}\mathbf{b}$ with a radius of $\frac{1}{2}b$.

2.4 Using the notations $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$ and $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$ show that

(a) the tensor $\mathbf{A}$ having components A_{ij} can always be decomposed into a sum of its symmetric $A_{(ij)}$ and skew-symmetric $A_{[ij]}$ parts, respectively, by the decomposition,

$$A_{ij} = A_{(ij)} + A_{[ij]}$$

(b) the trace of **A** is expressed in terms of $A_{(ij)}$ by

$$A_{ii} = A_{(ii)}$$

(c) for arbitrary tensors **A** and **B**,

$$A_{ij}\, B_{ij} = A_{(ij)}\, B_{(ij)} + A_{[ij]}\, B_{[ij]}$$

2.5 Expand the following expressions involving Kronecker deltas, and simplify where possible.

(a) $\delta_{ij}\,\delta_{ij}$, (b) $\delta_{ij}\delta_{jk}\delta_{ki}$, (c) $\delta_{ij}\delta_{jk}$, (d) $\delta_{ij}\,A_{ik}$

Answer: (a) 3, (b) 3, (c) δ_{ik}, (d) A_{jk}

2.6 If $a_i = \varepsilon_{ijk} b_j c_k$ and $b_i = \varepsilon_{ijk}\, g_j\, h_k$, substitute b_j into the expression for a_i to show that

$$a_i = g_k\, c_k\, h_i - h_k\, c_k\, g_i$$

or, in symbolic notation, $\mathbf{a} = (\mathbf{c} \cdot \mathbf{g})\mathbf{h} - (\mathbf{c} \cdot \mathbf{h})\mathbf{g}$.

2.7 By summing on the repeated subscripts determine the simplest form of

(a) $\varepsilon_{3jk} a_j a_k$ (b) $\varepsilon_{ijk}\,\delta_{kj}$ (c) $\varepsilon_{1jk}\, a_2 T_{kj}$ (d) $\varepsilon_{1jk}\delta_{3j} v_k$

Answer: (a) 0, (b) 0, (c) $a_2(T_{32} - T_{23})$, (d) $-v_2$

2.8 (a) Show that the tensor $B_{ik} = \varepsilon_{ijk}\, v_j$ is skew-symmetric.

(b) Let B_{ij} be skew-symmetric, and consider the vector defined by $v_i = \varepsilon_{ijk} B_{jk}$ (often called the *dual* vector of the tensor **B**). Show that $B_{mq} = \frac{1}{2}\,\varepsilon_{mqi} v_i$.

2.9 If $A_{ij} = \delta_{ij}\, B_{kk} + 3\, B_{ij}$, determine B_{kk} and using that solve for B_{ij} in terms of A_{ij} and its first invariant, A_{ii}.

Answer: $B_{kk} = \frac{1}{6} A_{kk}$; $B_{ij} = \frac{1}{3} A_{ij} - \frac{1}{18}\,\delta_{ij} A_{kk}$

2.10 Show that the value of the quadratic form $T_{ij} x_i x_j$ is unchanged if T_{ij} is replaced by its symmetric part, $\frac{1}{2}(T_{ij} + T_{ji})$.

2.11 Show by direct expansion (or otherwise) that the box product $\lambda = \varepsilon_{ijk}\, a_i b_j c_k$ is equal to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Thus, by substituting A_{1i} for a_i, A_{2j} for b_j and A_{3k} for c_k, derive Eq 2.4-11 in the form $\det \mathcal{A} = \varepsilon_{ijk}\, A_{1i}\, A_{2j}\, A_{3k}$ where A_{ij} are the elements of $\mathcal{A}$.

2.12 Starting with Eq 2.4-11 of the text in the form

$$\det \mathcal{A} = \varepsilon_{ijk}\, A_{i1} A_{j2}\, A_{k3}$$

show that by an arbitrary number of interchanges of columns of A_{ij} we obtain

$$\varepsilon_{qmn} \det \mathcal{A} = \varepsilon_{ijk} A_{iq} A_{jm} A_{kn}$$

which is Eq 2.4-12. Further, multiply this equation by the appropriate permutation symbol to derive the formula,

$$6 \det \mathcal{A} = \varepsilon_{qmn} \varepsilon_{ijk} A_{iq} A_{jm} A_{kn}$$

2.13 Let the determinant of the tensor A_{ij} be given by

$$\det \mathcal{A} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

Since the interchange of any two rows or any two columns causes a sign change in the value of the determinant, show that after an arbitrary number of row and column interchanges

$$\begin{vmatrix} A_{mq} & A_{mr} & A_{ms} \\ A_{nq} & A_{nr} & A_{ns} \\ A_{pq} & A_{pr} & A_{ps} \end{vmatrix} = \varepsilon_{mnp} \varepsilon_{qrs} \det \mathcal{A}$$

Now let $A_{ij} \equiv \delta_{ij}$ in the above determinant which results in $\det \mathcal{A} = 1$ and, upon expansion, yields

$$\varepsilon_{mnp} \varepsilon_{qrs} = \delta_{mq}(\delta_{nr}\delta_{ps} - \delta_{ns}\delta_{pr}) - \delta_{mr}\,(\delta_{nq}\delta_{ps} - \delta_{ns}\delta_{pq}) + \delta_{ms}(\delta_{nq}\delta_{pr} - \delta_{nr}\delta_{pq})$$

Thus, by setting $p = q$, establish Eq 2.2-13 in the form

$$\varepsilon_{mnq} \varepsilon_{qrs} = \delta_{mr}\delta_{ns} - \delta_{ms}\delta_{nr}$$

2.14 Show that the square matrices

$$\left[b_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \left[c_{ij}\right] = \begin{bmatrix} 5 & 2 \\ -12 & -5 \end{bmatrix}$$

are both square roots of the identity matrix.

2.15 Using the square matrices below, demonstrate

(a) that the transpose of the square of a matrix is equal to the square of its transpose (Eq 2.4-5 with $n = 2$).

(b) that $(\mathcal{AB})^T = \mathcal{B}^T\mathcal{A}^T$ as was proven in Example 2.4-2.

$$\left[a_{ij}\right] = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 5 & 1 & 2 \end{bmatrix}, \quad \left[b_{ij}\right] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix}$$

2.16 Let $\mathcal{A}$ be any orthogonal matrix, i.e., $\mathcal{AA}^T = \mathcal{AA}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Thus, by using the results in Examples 2.4-3 and 2.4-4, show that det $\mathcal{A} = \pm 1$.

2.17 A tensor is called *isotropic* if its components have the same set of values in every Cartesian coordinate system at a point. Assume that $\mathbf{T}$ is an isotropic tensor of rank two with components T_{ij} relative to axes $Ox_1x_2x_3$. Let axes $Ox_1'x_2'x_3'$ be obtained with respect to $Ox_1x_2x_3$ by a right-hand rotation of 120° about the axis along $\hat{\mathbf{n}} = \left(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right)/\sqrt{3}$. Show by the transformation between these axes that $T_{11} = T_{22} = T_{33}$, as well as other relationships. Further, let axes $Ox_1''x_2''x_3''$ be obtained with respect to $Ox_1x_2x_3$ by a right-hand rotation of 90° about x_3. Thus, show by the additional considerations of this transformation that if $\mathbf{T}$ is any isotropic tensor of second order, it can be written as $\lambda\mathbf{I}$ where λ is a scalar and $\mathbf{I}$ is the identity tensor.

2.18 For a proper orthogonal transformation between axes $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ show the invariance of δ_{ij} and ε_{ijk}. That is, show that

(a) $\quad \delta'_{ij} = \delta_{ij}$

(b) $\quad \varepsilon'_{ijk} = \varepsilon_{ijk}$

Hint: For part (b) let $\varepsilon'_{ijk} = a_{iq}\, a_{jm}\, a_{kn}\, \varepsilon_{qmn}$ and make use of Eq 2.4-12.

2.19 The angles between the respective axes of the $Ox_1'x_2'x_3'$ and the $Ox_1x_2x_3$ Cartesian systems are given by the table below

	x_1	x_2	x_3
x_1'	45°	90°	45°
x_2'	60°	45°	120°
x_3'	120°	45°	60°

Determine

(a) the transformation matrix between the two sets of axes, and show that it is a proper orthogonal transformation.

(b) the equation of the plane $x_1 + x_2 + x_3 = 1/\sqrt{2}$ in its primed axes form, that is, in the form $b_1 x_1' + b_2 x_2' + b_3 x_3' = b$.

Answer:

$$\text{(a)}\quad \left[a_{ij}\right] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & -1/2 \\ -1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix},$$

$$\text{(b)}\quad 2x_1' + x_2' + x_3' = 1$$

2.20 Making use of Eq 2.4-11 of the text in the form $\det \mathcal{A} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$ write Eq 2.6-6 as

$$\left|T_{ij} - \lambda\delta_{ij}\right| = \varepsilon_{ijk}\left(T_{1i} - \lambda\delta_{1i}\right)\left(T_{2j} - \lambda\delta_{2j}\right)\left(T_{3k} - \lambda\delta_{3k}\right) = 0$$

and show by expansion of this equation that

$$\lambda^3 - T_{ii}\lambda^2 + \left[\frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)\right]\lambda - \varepsilon_{ijk}T_{1i}T_{2j}T_{3k} = 0$$

to verify Eq 2.6-8 of the text.

2.21 For the matrix representation of tensor **B** shown below,

$$\left[B_{ij}\right] = \begin{bmatrix} 17 & 0 & 0 \\ 0 & -23 & 28 \\ 0 & 28 & 10 \end{bmatrix}$$

determine the principal values (eigenvalues) and the principal directions (eigenvectors) of the tensor.

Answer: $\lambda_1 = 17, \qquad \lambda_2 = 26, \qquad \lambda_3 = -39$

$$\hat{\mathbf{n}}^{(1)} = \hat{\mathbf{e}}_1, \quad \hat{\mathbf{n}}^{(2)} = \left(4\hat{\mathbf{e}}_2 + 7\hat{\mathbf{e}}_3\right)/\sqrt{65}, \quad \hat{\mathbf{n}}^{(3)} = \left(-7\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3\right)/\sqrt{65}$$

2.22 Consider the symmetrical matrix

$$\left[B_{ij}\right] = \begin{bmatrix} \frac{5}{2} & 0 & \frac{3}{2} \\ 0 & 4 & 0 \\ \frac{3}{2} & 0 & \frac{5}{2} \end{bmatrix}$$

(a) Show that a multiplicity of two occurs among the principal values of this matrix.

(b) Let λ_1 be the unique principal value and show that the transformation matrix

$$\left[a_{ij}\right] = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

gives **B*** according to $B^*_{ij} = a_{iq} a_{jm} B_{qm}$.

(c) Taking the square root of $\left[B^*_{ij}\right]$ and transforming back to $Ox_1x_2x_3$ axes show that

$$\left[\sqrt{B_{ij}}\right] = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$$

(d) Verify that the matrix

$$\left[C_{ij}\right] = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 2 & 0 \\ -\frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

is also a square root of $\left[B_{ij}\right]$.

2.23 Determine the principal values of the matrix

$$\left[K_{ij}\right] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 11 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{bmatrix}$$

and show that the principal axes $Ox_1^* x_2^* x_3^*$ are obtained from $Ox_1x_2x_3$ by a rotation of 60° about the x_1 axis.

Answer: $\lambda_1 = 4$, $\lambda_2 = 8$, $\lambda_3 = 12$.

2.24 Determine the principal values $\lambda_{(q)}$ $(q = 1,2,3)$ and principal directions $\hat{\mathbf{n}}^{(\mathbf{q})}$ $(q = 1,2,3)$ for the symmetric matrix

$$\left[T_{ij}\right] = \frac{1}{2}\begin{bmatrix} 3 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 9/2 & 3/2 \\ 1/\sqrt{2} & 3/2 & 9/2 \end{bmatrix}$$

Answer: $\lambda_{(1)} = 1,\ \lambda_{(2)} = 2,\ \lambda_{(3)} = 3$

$$\hat{\mathbf{n}}^{(1)} = \tfrac{1}{2}\left(\sqrt{2}\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\right),\ \hat{\mathbf{n}}^{(2)} = \tfrac{1}{2}\left(\sqrt{2}\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right),\ \hat{\mathbf{n}}^{(3)} = -\left(\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right)/\sqrt{2}$$

2.25 Let $\mathbf{D}$ be a constant tensor whose components do not depend upon the coordinates. Show that

$$\boldsymbol{\nabla}\left(\mathbf{x}\cdot\mathbf{D}\right) = \mathbf{D}$$

where $\mathbf{x} = x_i\hat{\mathbf{e}}_i$ is the position vector.

2.26 Consider the vector $\mathbf{x} = x_i\hat{\mathbf{e}}_i$ having a magnitude squared $x^2 = x_1^2 + x_2^2 + x_3^2$. Determine

(a) **grad** x (d) $\operatorname{div}(x^n\mathbf{x})$

(b) **grad** (x^{-n}) (e) $\mathbf{curl}(x^n\mathbf{x})$, where n is a positive integer

(c) $\boldsymbol{\nabla}^2(1/x)$

Answer: (a) x_i/x, (b) $-\,nx_i/x^{(n+2)}$, (c) 0, (d) $x^n(n+3)$, (e) 0.

2.27 If λ and ϕ are scalar functions of the coordinates x_i, verify the following vector identities. Transcribe the left-hand side of the equations into indicial notation and, following the indicated operations, show that the result is the right-hand side.

(a) $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \tfrac{1}{2}\,\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$

(b) $\mathbf{v}\cdot\mathbf{u}\times\mathbf{w} = \mathbf{v}\times\mathbf{u}\cdot\mathbf{w}$

(c) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla}^2\mathbf{v}$

(d) $\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\phi) = \lambda\boldsymbol{\nabla}^2\phi + \boldsymbol{\nabla}\lambda\cdot\boldsymbol{\nabla}\phi$

(e) $\boldsymbol{\nabla}^2(\lambda\phi) = \lambda\boldsymbol{\nabla}^2\phi + 2(\boldsymbol{\nabla}\lambda)\cdot(\boldsymbol{\nabla}\phi) + \phi\boldsymbol{\nabla}^2\lambda$

(f) $\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = (\boldsymbol{\nabla}\times\mathbf{u})\cdot\mathbf{v} - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$

2.28 Let the vector $\mathbf{v} = \mathbf{b}\times\mathbf{x}$ be one for which $\mathbf{b}$ does not depend upon the coordinates. Use indicial notation to show that

(a) **curl** $\mathbf{v} = 2\mathbf{b}$

(b) div $\mathbf{v} = 0$

2.29 Transcribe the left-hand side of the following equations into indicial notation and verify that the indicated operations result in the expressions on the right-hand side of the equations for the scalar ϕ, and vectors **u** and **v**.

(a) $\text{div}(\phi\mathbf{v}) = \phi \text{ div } \mathbf{v} + \mathbf{v} \cdot \mathbf{grad}\phi$

(b) $\mathbf{u} \times \mathbf{curl\ v} + \mathbf{v} \times \mathbf{curl\ u} = -(\mathbf{u} \cdot \mathbf{grad})\ \mathbf{v} - (\mathbf{v} \cdot \mathbf{grad})\ \mathbf{u} + \mathbf{grad}\ (\mathbf{u} \cdot \mathbf{v})$

(c) $\text{div } (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \mathbf{curl\ u} - \mathbf{u} \cdot \mathbf{curl\ v}$

(d) $\mathbf{curl}\ (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \mathbf{grad})\mathbf{u} - (\mathbf{u} \cdot \mathbf{grad})\ \mathbf{v} + \mathbf{u} \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \mathbf{u}$

(e) $\mathbf{curl}\ (\mathbf{curl\ u}) = \mathbf{grad}\ (\text{div } \mathbf{u}) - \boldsymbol{\nabla}^2\mathbf{u}$

2.30 Let the volume V have a bounding surface S with an outward unit normal n_i. Let x_i be the position vector to any point in the volume or on its surface. Show that

(a) $$\int_S x_i n_j dS = \delta_{ij} V$$

(b) $$\int_S \boldsymbol{\nabla}(\mathbf{x} \cdot \mathbf{x}) \cdot \hat{\mathbf{n}} dS = 6V$$

(c) $$\int_S \lambda \mathbf{w} \cdot \hat{\mathbf{n}} dS = \int_V \mathbf{w} \cdot \mathbf{grad}\ \lambda\, dV$$, where $\mathbf{w} = \mathbf{curl\ v}$ and $\lambda = \lambda(\mathbf{x})$.

(d) $\int_S \left[\hat{\mathbf{e}}_i \times \mathbf{x}, \hat{\mathbf{e}}_j, \hat{\mathbf{n}}\right] dS = 2V\delta_{ij}$ where $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ are coordinate base vectors.

Hint: Write the box product

$$\left[\hat{\mathbf{e}}_i \times \mathbf{x}, \hat{\mathbf{e}}_j, \hat{\mathbf{n}}\right] = \left(\hat{\mathbf{e}}_i \times \mathbf{x}\right) \cdot \left(\hat{\mathbf{e}}_j \times \hat{\mathbf{n}}\right)$$

and transcribe into indicial notation.

2.31 Use Stokes' theorem to show that upon integrating around the space curve C having a differential tangential vector dx_i that for $\phi(\mathbf{x})$.

$$\oint_C \phi_{,i} dx_i = 0$$

2.32 For the position vector x_i having a magnitude x, show that $x_{,j} = x_j/x$ and therefore,

(a) $$x_{,ij} = \frac{\delta_{ij}}{x} - \frac{x_i x_j}{x^3}$$

(b) $$(x^{-1})_{,ij} = \frac{3\mathrm{x}_i\mathrm{x}_j}{\mathrm{x}^5} - \frac{\delta_{ij}}{\mathrm{x}^3}$$

(c) $$x_{,ii} = \frac{2}{x}$$

2.33 Show that for arbitrary tensors **A** and **B**, and arbitrary vectors **a** and **b**,

(a) $(\mathbf{A}\cdot\mathbf{a})\cdot(\mathbf{B}\cdot\mathbf{b}) = \mathbf{a}\cdot(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})\cdot\mathbf{b}$

(b) $\mathbf{b}\times\mathbf{a} = \frac{1}{2}(\mathbf{B}-\mathbf{B}^{\mathrm{T}})\cdot\mathbf{a}$, if $2b_i = \varepsilon_{ijk}B_{kj}$

(c) $\mathbf{a}\cdot\mathbf{A}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{A}^{\mathrm{T}}\cdot\mathbf{a}$

2.34 Use Eqs 2.4-11 and 2.4-12 as necessary to prove the identities

(a) $[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}] = (\det\mathcal{A})\,[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

(b) $\mathbf{A}^{\mathrm{T}}\cdot(\mathbf{Aa}\times\mathbf{Ab}) = (\det\mathcal{A})\,(\mathbf{a}\times\mathbf{b})$

for arbitrary vectors **a**, **b**, **c**, and tensor **A**.

2.35 Let $\phi = \phi(x_i)$ and $\psi = \psi(x_i)$ be scalar functions of the coordinates. Recall that in the indicial notation $\phi_{,i}$ represents $\boldsymbol{\nabla}\phi$ and $\phi_{,ii}$ represents $\boldsymbol{\nabla}^2\phi$. Now apply the divergence theorem, Eq 2.8-1, to the field $\phi\psi_{,i}$ to obtain

$$\int_S \phi\psi_{,i}n_i\,dS = \int_V \left(\phi_{,i}\psi_{,i} + \phi\psi_{,ii}\right)dV$$

Transcribe this result into symbolic notation as

$$\int_S \phi\boldsymbol{\nabla}\psi\cdot\hat{\mathbf{n}}\,dS\ \boldsymbol{\nabla}\int_S \phi\frac{\partial\psi}{\partial n}\,dS = \int_V \left(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi + \phi\boldsymbol{\nabla}^2\psi\right)dV$$

which is known as *Green's first identity*. Show also by the divergence theorem that

$$\int_S \left(\phi\psi_{,i} - \psi\phi_{,i}\right)n_i\,dS = \int_V \left(\phi\psi_{,ii} - \psi\phi_{,ii}\right)dV$$

and transcribe into symbolic notation as

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \int_V \left(\phi\,\boldsymbol{\nabla}^2\psi - \psi\,\boldsymbol{\nabla}^2\phi\right)dV$$

which is known as *Green's second identity.*