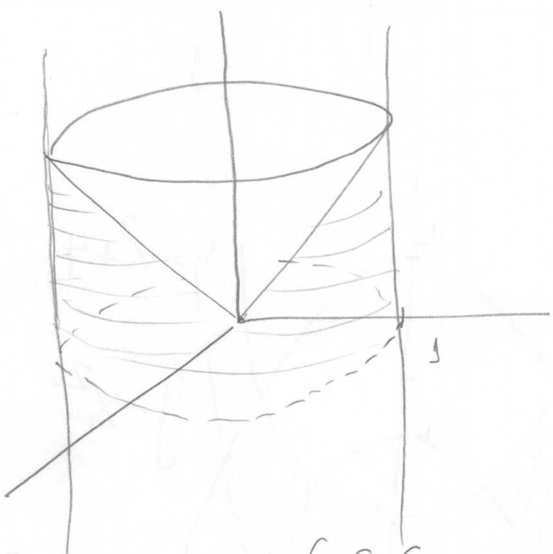
EXAMEN

P1

a.) V: · Exterior a la hoja superior del cono $z^2 = x^2 + y^2$

· Interior al cilindro $x^2+y^2=1$,

· $z \geq 0$

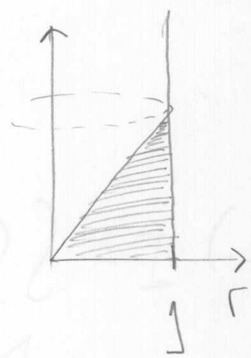


En cilíndricas

V: $0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

$0 \leq z \leq r$



$$I = \iiint_V 2z(x^2+y^2)\,dxdydz$$

$$= \int_0^{2\pi}\int_0^{1}\int_0^{r} 2zr^2 \cdot r\,dzdrd\theta$$

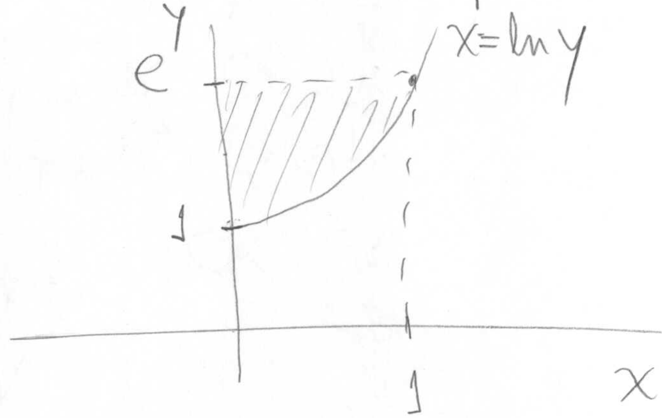
$$= 2\pi \cdot \int_0^1 r^2 \cdot r^2 \cdot r\,dr$$

$$= 2\pi \cdot \frac{1}{6} \Rightarrow \boxed{I = \frac{\pi}{3}}$$

b.) i.) $I = \int_1^e \int_0^{\ln y} f(x,y)\,dx\,dy$

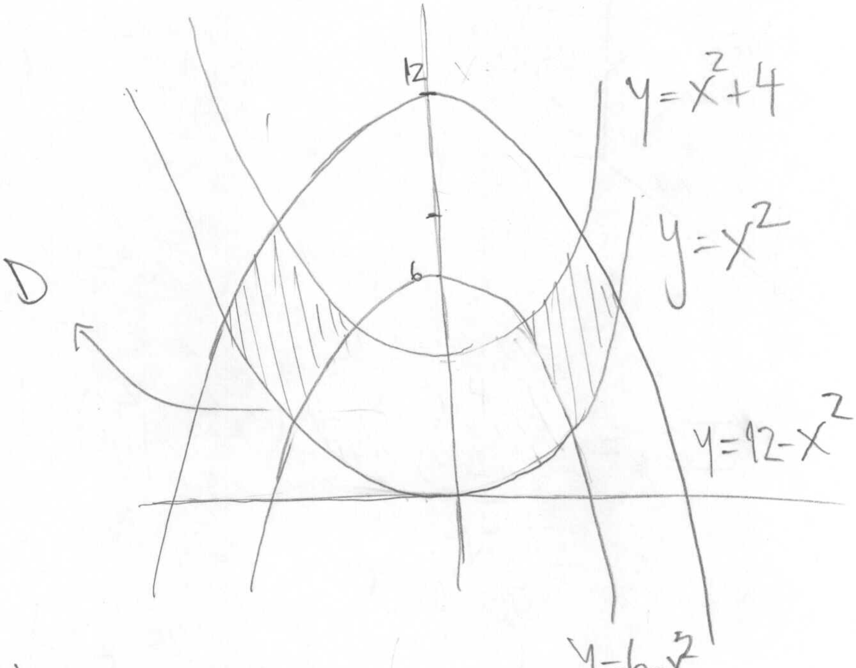
$1 \le y \le e$

$0 \le x \le \ln y$



$= \int_0^1 \int_{e^x}^{e} f(x,y)\,dx\,dy$

ii.) $I = \iint_D xy\,dx\,dy$



$u = y - x^2$

$v = y + x^2$

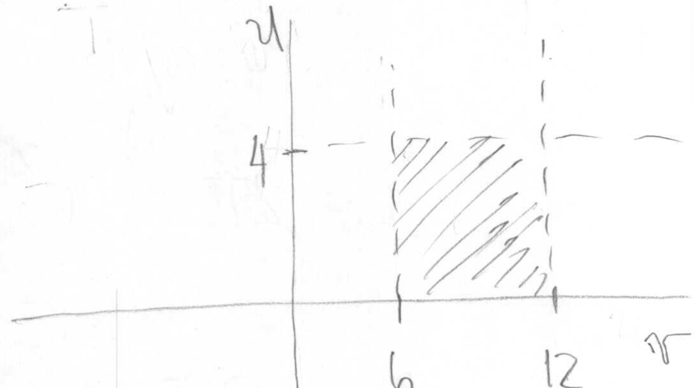
$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x^2 \\ y + x^2 \end{pmatrix}$

$\Rightarrow \operatorname{Det} DT = \begin{vmatrix} -2x & 1 \\ 2x & 1 \end{vmatrix} = -4x^2$

$\Rightarrow \operatorname{Det} DT \neq 0$

$\forall (x,y) \in D$.



$y = x^2 \xrightarrow{T} u = 0$

$y = x^2 + 4 \to u = 4$

$y = 6 - x^2 \to v = 6$

$y = 12 - x^2 \to v = 12$

$$I = \iint_D xy\,dx\,dy \qquad y = \frac{u+v}{2}$$

$$= \frac{1}{4}\iint_D \underbrace{4x^2}_{|\det DT|} y\,dx\,dy$$

$$= \frac{1}{4}\int_6^{12}\int_0^4 \frac{u+v}{2}\,du\,dv$$

$$= \frac{1}{8}\int_6^{12} \frac{u^2}{2}\Big|_0^4 + v\cdot u\Big|_0^4 \; dv = \frac{1}{8}\int_6^{12}(8+4v)\,dv$$

$$= 6 + \frac{1}{8}2\cdot v^2\Big|_6^{12} = 6 + \frac{2\cdot(12^2-6^2)}{8}$$

$$= 33$$

P2 $g : \mathbb{R} \to \mathbb{R}$ continua
$h : \mathbb{R} \to \mathbb{R}$ primitiva de g

i.) $u = x - h(y)$
$v = y$

$$f(x,y) = \tilde{f}(u(x,y), v(x,y))$$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y) = \frac{\partial \tilde{f}}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \tilde{f}}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial \tilde{f}}{\partial u} \cdot 1$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial \tilde{f}}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \tilde{f}}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial \tilde{f}}{\partial u} \cdot \left(-\frac{d}{dy} h(y)\right) + \frac{\partial \tilde{f}}{\partial v} \cdot 1$$

Pero h primitiva de g $\Rightarrow \frac{d}{dy} h(y) = g(y)$

$$\Rightarrow \frac{\partial f}{\partial y}(x,y) = -g(y) \cdot \frac{\partial \tilde{f}}{\partial u} + \frac{\partial \tilde{f}}{\partial v}$$

$$\Rightarrow g(y) \cdot \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 \Leftrightarrow g(y) \cdot \frac{\partial \tilde{f}}{\partial u} - g(y) \cdot \frac{\partial \tilde{f}}{\partial u} + \frac{\partial \tilde{f}}{\partial v} = 0$$

$$\Leftrightarrow \boxed{\frac{\partial \tilde{f}}{\partial v} = 0} \quad (*)$$

Con lo cual, las sols. de (*) sólo dependen de la variable u. Esto es, son de la forma

$$f = F(u)$$

con lo cual

$$\boxed{f(x,y) = F(x - h(y))}$$

ii.) $y \cdot \frac{\partial f}{\partial x} + (1+y^2)\frac{\partial f}{\partial y} = 0 \Leftrightarrow \frac{y}{1+y^2}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$

Luego, $g(y) = \frac{y}{1+y^2}$, la cual sabemos que una primitiva viene dada por

$$h(y) = \frac{1}{2}\ln(1+y^2) + C$$

de donde la solución es

$$\boxed{f(x,y) = F\left(x - \frac{1}{2}\ln(1+y^2) + C\right)}$$

b.) $u(x,y) = \dfrac{y^2 - x^2}{2} \qquad v(x,y) = \dfrac{y^2 + x^2}{2}$

$$F(x,y) = F(u(x,y), v(x,y))$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial u} \cdot \left(-x\right) + \frac{\partial F}{\partial v} \cdot (x)$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}(y) + \frac{\partial F}{\partial v} \cdot (y)$$

$$\frac{\partial^2 F}{\partial x^2} = \left(\frac{\partial^2 F}{\partial u^2} \cdot (-x) + \frac{\partial^2 F}{\partial v \partial u} \cdot (x)\right) \cdot -x + \frac{\partial F}{\partial u} \cdot (-1)$$

$$+ \left(\frac{\partial^2 F}{\partial u \partial v} \cdot (-x) + \frac{\partial^2 F}{\partial v^2} \cdot (x)\right) \cdot x + \frac{\partial F}{\partial v} \cdot (1)$$

$$\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2 F}{\partial u^2} \cdot (y) + \frac{\partial^2 F}{\partial v \partial u} (y)\right) \cdot y + \frac{\partial F}{\partial u} \cdot (1)$$

$$+ \left(\frac{\partial^2 F}{\partial u \partial v} (y) + \frac{\partial^2 F}{\partial v^2} (y)\right) \cdot y + \frac{\partial F}{\partial v} (1)$$

$$\Rightarrow \quad \frac{\partial^2 F}{\partial x^2} = -x^2 \cdot \frac{\partial^2 F}{\partial u^2} + x^2 \frac{\partial^2 F}{\partial v^2} - \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v} - 2x^2 \frac{\partial^2 F}{\partial u \partial v}$$

$$\frac{\partial^2 F}{\partial y^2} = y^2 \cdot \frac{\partial^2 F}{\partial u^2} + y^2 \frac{\partial^2 F}{\partial v^2} + 2y^2 \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial F}{\partial v} + \frac{\partial F}{\partial u}.$$

$$y^2 \cdot \frac{\partial^2 F}{\partial x^2} - x^2 \frac{\partial^2 F}{\partial y^2} = 0$$

$$\Longleftrightarrow \cancelto{0}{\frac{\partial^2 F}{\partial u^2}\left(x^2y^2 - x^2y^2\right)} + \cancelto{0}{\frac{\partial^2 F}{\partial v^2}\left(x^2y^2 - x^2y^2\right)}$$

$$+ \frac{\partial F}{\partial u}\left(\underbrace{-y^2 - x^2}_{2v}\right) + \frac{\partial F}{\partial v}\left(\underbrace{y^2 - x^2}_{2u}\right)$$

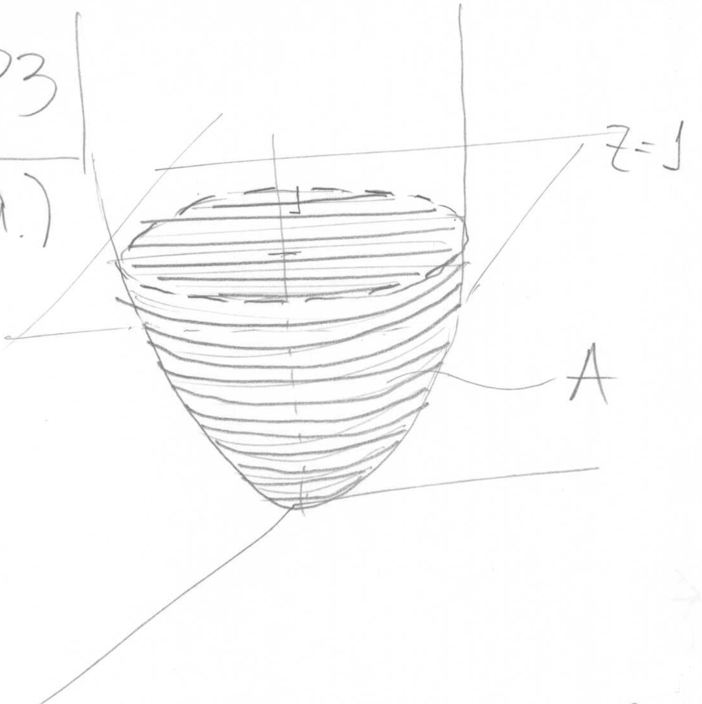
$$+ 2\frac{\partial^2 F}{\partial u \partial v}\left(-x^2y^2 - x^2y^2\right) = 0$$

$$\Longleftrightarrow 4 \underbrace{x^2 y^2}_{-(u^2-v^2)} \frac{\partial^2 F}{\partial u \partial v} = 2\underbrace{\left(\frac{y^2 - x^2}{2}\right)}_{u} \frac{\partial F}{\partial v} - 2\underbrace{\left(\frac{y^2 + x^2}{2}\right)}_{v} \frac{\partial F}{\partial u}$$

$$\Rightarrow \boxed{2(u^2 - v^2)\frac{\partial^2 F}{\partial u \partial v} = v \cdot \frac{\partial F}{\partial u} - u \frac{\partial F}{\partial v}}$$

P3

a.)



$x^2+y^2=z \Leftrightarrow r^2=z$

Sabemos que en el interior de A no hay máximos ni mínimos locales, ya que $\nabla f(x,y,z)=(1,1,1)^t \neq 0$ veamos los bordes

Paraboloide

$\mathcal{L}=f(x,y,z)-\lambda(x^2+y^2-z)$

$$\nabla\mathcal{L}=0 \Leftrightarrow \begin{array}{ll} 1-2\lambda x=0 & (1) \\ 1-2\lambda y=0 & (2) \\ 1+\lambda=0 & (3) \\ x^2+y^2=z & (4) \end{array}$$

De (3) $\Rightarrow \boxed{\lambda=-1}$

$\Rightarrow x=-\frac{1}{2}, \quad y=-\frac{1}{2}, \quad z=\frac{1}{2}$

$$\boxed{P=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)}$$

$$\Rightarrow \boxed{P_1 = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)}$$

Superficie $z=1$

$$L = x+y+z-\mu(z-1)$$

$$\frac{\partial L}{\partial x} = 0 \Leftrightarrow 1 = 0 \rightarrow$$

$$\frac{\partial L}{\partial y} = 0 \Leftrightarrow 1 = 0 \rightarrow$$

$$\frac{\partial L}{\partial z} = 0 \Leftrightarrow 1 - \mu = 0$$

Intersección

$$L = x+y+z-\lambda(x^2+y^2-z)-\mu(z-1)$$

$$\frac{\partial L}{\partial x} = 0 \Leftrightarrow 1-2\lambda x = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 0 \Leftrightarrow 1-2\lambda y = 0 \quad (2)$$

$$\frac{\partial L}{\partial z} = 0 \Leftrightarrow 1-\lambda-\mu = 0 \quad (3)$$

$$x^2+y^2 = z \quad (4)$$

$$z = 1 \quad (5)$$

$(5) \Rightarrow z = 1$

$(1) \Rightarrow \lambda \neq 0 \Rightarrow$ $(1)\wedge(2)$ $x = \frac{1}{2\lambda}, y = \frac{1}{2\lambda}$

$\overset{(4)}{\Rightarrow} \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$

$\Rightarrow \frac{1}{2\lambda^2} = 1 \Rightarrow \lambda_1 = \frac{1}{\sqrt{2}}$

$\lambda_2 = \frac{-1}{\sqrt{2}}$

$$\boxed{\begin{aligned} P_2 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) \\ P_3 &= \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right) \end{aligned}}$$

Luego, los puntos obtenidos son:

$P_1 = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)$

$P_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

$P_3 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1\right)$

Evaluando

$f(P_1) = \frac{-1}{2}$

$f(P_2) = 1+\sqrt{2}$

$f(P_3) = 1-\sqrt{2}$

Con lo cual, f restringido a A alcanza su

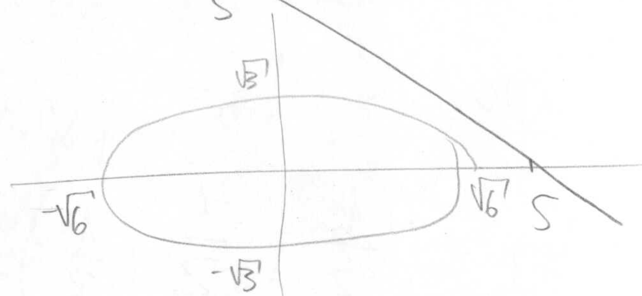
- máximo en $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ y vale $1+\sqrt{2}$

- mínimo en $\left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)$ y vale $\frac{-1}{2}$.

b) $\min\ (x_1-x_2)^2+(y_1-y_2)^2$

s.a $\quad x_1^2+2y_1^2=6$

$\quad\quad x_2+y_2=5$



$$\mathcal{L}=(x_1-x_2)^2+(y_1-y_2)^2-\lambda(x_1^2+2y_1^2-6)-\mu(x_2+y_2-5)$$

$\frac{\partial \mathcal{L}}{\partial x_1}=0 \Leftrightarrow 2(x_1-x_2)-2\lambda x_1=0$ (1)

$\frac{\partial \mathcal{L}}{\partial y_1}=0 \Leftrightarrow 2(y_1-y_2)-4\lambda y_1=0$ (2)

$\frac{\partial \mathcal{L}}{\partial x_2}=0 \Leftrightarrow -2(x_1-x_2)-\mu=0$ (3)

$\frac{\partial \mathcal{L}}{\partial y_2}=0 \Leftrightarrow -2(y_1-y_2)-\mu=0$ (4)

$x_1^2+2y_1^2=6$ (5)

$x_2+y_2=5$ (6)

De (2) y (4) $\Rightarrow -4\lambda y_1=\mu$

(1) y (3) $\Rightarrow -2\lambda x_1=\mu$

$\Rightarrow \boxed{2y_1=x_1} \xrightarrow{(5)} 4y_1^2+2y_1^2=6$

$\boxed{y_1^2=1}$

Claramente, la distancia mínima se tiene en $y>0$

$\Rightarrow \boxed{y_1^*=1}\ \boxed{x_1^*=2} \xrightarrow{(1)\wedge(2)} x_1-x_2=y_1-y_2$

$\Rightarrow x_2=y_2+1$

$\xrightarrow{(6)} \boxed{y_2^*=2} \Rightarrow \boxed{x_2^*=3} \Rightarrow \boxed{d=\sqrt{2}}$