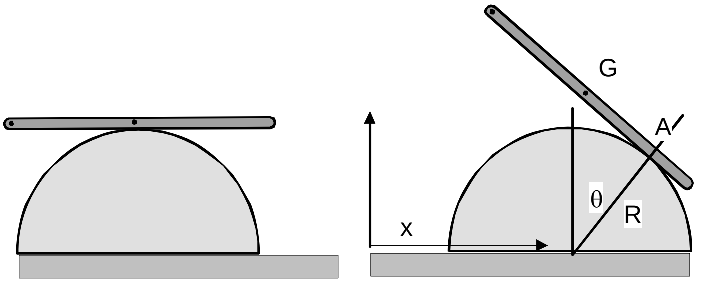
PAUTA CONTROL 3

Un punto base más los indicados. Los puntajes son referenciales para un camino de solución...

1.



Notando que $AG = R\theta$ las coordenadas de G son

$$
\begin{aligned}
x_G &= x + R\sin\theta - R\theta\cos\theta \\
y_G &= R\cos\theta + R\theta\sin\theta
\end{aligned}
$$

luego

$$
\begin{aligned}
\dot{x}_G &= \dot{x} + R\dot{\theta}\cos\theta - R\dot{\theta}\cos\theta + R\theta\dot{\theta}\sin\theta = \dot{x} + R\theta\dot{\theta}\sin\theta \quad (1\text{ p}) \\
\dot{y}_G &= -R\dot{\theta}\sin\theta + R\dot{\theta}\sin\theta + R\theta\dot{\theta}\cos\theta = R\theta\dot{\theta}\cos\theta \quad (1\text{ p})
\end{aligned}
$$

Entonces el Lagrangiano será

$$
\begin{aligned}
L &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(\dot{x}_G^2 + \dot{y}_G^2) + \frac{1}{2}\frac{1}{3}Ma^2\dot{\theta}^2 - Mg(R\cos\theta + R\theta\sin\theta) \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + 2\dot{x}R\theta\dot{\theta}\sin\theta + R^2\theta^2\dot{\theta}^2\sin\theta) + \\
&\quad + \frac{1}{2}\frac{1}{3}Ma^2\dot{\theta}^2 - Mg(R\cos\theta + R\theta\sin\theta) \\
&= M\dot{x}^2 + M\dot{x}R\theta\dot{\theta}\sin\theta + \frac{1}{2}MR^2\theta^2\dot{\theta}^2\sin\theta + \frac{1}{6}Ma^2\dot{\theta}^2 \quad (2\text{ p}) \\
&\quad - MgR(\cos\theta + \theta\sin\theta)
\end{aligned}
$$

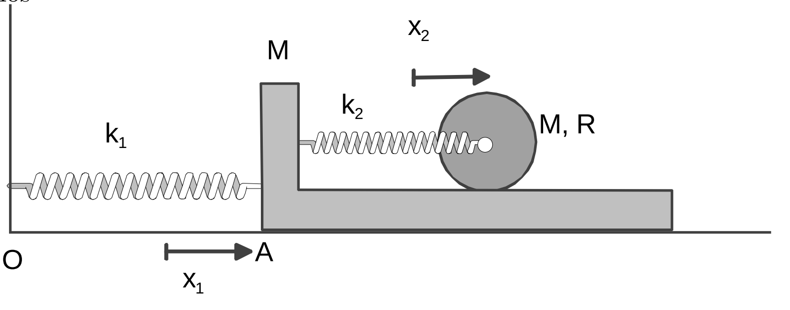
haciendo las derivadas correspondientes

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= M\dot{x} + MR\theta\dot{\theta}\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= M\dot{x}R\theta\sin\theta + MR^2\theta^2\dot{\theta}\sin\theta + \frac{1}{3}Ma^2\dot{\theta} \\
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \theta} &= M\dot{x}R\theta\dot{\theta}\cos\theta + MR^2\theta\dot{\theta}^2\sin\theta + \frac{1}{2}MR^2\theta^2\dot{\theta}^2\cos\theta - MgR\theta\cos\theta
\end{aligned}
$$

Luego las ecuaciones de Lagrange son (si he cometido algun error, lo reparan)

$$
\begin{aligned}
0 &= \frac{d}{dt}(M\dot{x} + MR\theta\dot{\theta}\sin\theta) && \text{(1 p)} \\
0 &= M\ddot{x}R\theta\sin\theta + MR\dot{x}\dot{\theta}\sin\theta + MR^2\theta^2\ddot{\theta}\sin\theta + MR^2\theta\dot{\theta}^2\sin\theta \\
&+\frac{1}{2}MR^2\theta^2\dot{\theta}^2\cos\theta + \frac{1}{3}Ma^2\ddot{\theta} + MgR\theta\cos\theta && \text{(1 p)}
\end{aligned}
$$

2. Tenemos



$$\omega = \frac{\dot{x}_2}{R},\ v_G = \dot{x}_1 + \dot{x}_2,\quad I_G = \frac{1}{2}MR^2,$$

entonces

$$
\begin{aligned}
K &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}\frac{1}{2}MR^2(\frac{\dot{x}_2}{R})^2 \\
&= M\dot{x}_1^2 + M\dot{x}_1\dot{x}_2 + \frac{3}{4}M\dot{x}_2^2 && \text{(1 p)} \\
V &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2
\end{aligned}
$$

No hay nada que aproximar. Las ecuaciones de Lagrange son

$$
\begin{aligned}
2M\ddot{x}_1 + M\ddot{x}_2 + k_1x_1 &= 0 \\
\frac{3}{2}M\ddot{x}_2 + M\ddot{x}_1 + k_2x_2 &= 0
\end{aligned}
$$

Pero $k_2 = \frac{26}{7}k_1$. llamando $\Omega_1^2 = \frac{k_1}{M},\ \Omega_2^2 = \frac{k_2}{M}$ellas son

$$
\begin{aligned}
2\ddot{x}_1 + \ddot{x}_2 + \frac{k_1}{M}x_1 &= 0 \\
\frac{3}{2}\ddot{x}_2 + \ddot{x}_1 + \frac{26}{7}\frac{k_1}{M}x_2 &= 0 && \text{(1 p)}
\end{aligned}
$$

con la sustitución $x_1 = a_1e^{-i\omega t},\ x_2 = a_2e^{-i\omega t}$

$$
\begin{aligned}
2\ddot{x}_1 + \ddot{x}_2 + \frac{k_1}{M}x_1 &= 0 \\
\frac{3}{2}\ddot{x}_2 + \ddot{x}_1 + \frac{26}{7}\frac{k_1}{M}x_2 &= 0
\end{aligned}
$$

$$\begin{aligned}(\frac{k_1}{M}-2\omega^2)a_1-\omega^2 a_2 &= 0\\ -\omega^2 a_1+(\frac{26}{7}\frac{k_1}{M}-\frac{3}{2}\omega^2)a_2 &= 0\end{aligned}$$

determinante

$$(\frac{k_1}{M}-2\omega^2)(\frac{26}{7}\frac{k_1}{M}-\frac{3}{2}\omega^2)-\omega^4=0$$

Frecuencias propias

$$\omega_1^2=4\frac{k_1}{M},\omega_2^2=\frac{13}{28}\frac{k_1}{M} \qquad \text{(1 p)}$$

razones

$$\frac{a_1}{a_2}=\frac{\omega^2}{\frac{k}{M}-2\omega^2}$$

$$\begin{aligned}\frac{a_{11}}{a_{21}} &= \frac{4\frac{k_1}{M}}{\frac{k_1}{M}-2\times 4\frac{k_1}{M}}=-\frac{4}{7}\\ \frac{a_{12}}{a_{22}} &= \frac{\frac{13}{28}\frac{k_1}{M}}{\frac{k_1}{M}-2\times\frac{13}{28}\frac{k_1}{M}}=\frac{13}{2}\end{aligned}$$

Luego

$$\begin{aligned}x_1 &= a_{11}C_1e^{-i\omega_1 t}+a_{12}C_2e^{-i\omega_2 t}\\ x_2 &= a_{21}C_1e^{-i\omega_1 t}+a_{22}C_2e^{-i\omega_2 t}\end{aligned}$$

tomemos arbitrariamente (da lo mismo)

$$a_{11}=1,\ a_{22}=1\Rightarrow a_{21}=-\frac{7}{4},\quad a_{12}=\frac{13}{2}$$

así

$$\begin{aligned}x_1 &= C_1e^{-i\omega_1 t}+\frac{13}{2}C_2e^{-i\omega_2 t}\\ x_2 &= -\frac{7}{4}C_1e^{-i\omega_1 t}+C_2e^{-i\omega_2 t}\end{aligned}$$

se despejan las coordenadas normales (o expresiones proporcionales)

$$\begin{aligned}\zeta_1 &= C_1e^{-i\omega_1 t}=-\frac{52}{99}x_2+\frac{8}{99}x_1\\ \zeta_2 &= C_2e^{-i\omega_2 t}=\frac{8}{99}x_2+\frac{14}{99}x_1\end{aligned} \qquad \text{(1 p)}$$

La solución en términos de las condiciones iniciales

$$\begin{aligned} x_1 &= \mathrm{Re}(C_1 e^{-i\omega_1 t} + \frac{13}{2} C_2 e^{-i\omega_2 t}) \\ x_2 &= \mathrm{Re}(-\frac{7}{4} C_1 e^{-i\omega_1 t} + C_2 e^{-i\omega_2 t}) \\ \dot{x}_1 &= \mathrm{Re}(-i\omega_1 C_1 e^{-i\omega_1 t} - i\omega_2 \frac{13}{2} C_2 e^{-i\omega_2 t}) \\ \dot{x}_2 &= \mathrm{Re}(i\omega_1 \frac{7}{4} C_1 e^{-i\omega_1 t} - i\omega_2 C_2 e^{-i\omega_2 t}) \end{aligned}$$

en $t = 0$

$$\begin{aligned} 0 &= \mathrm{Re}(C_1 + \frac{13}{2} C_2) \\ a &= \mathrm{Re}(-\frac{7}{4} C_1 + C_2) \\ 0 &= \mathrm{Re}(-i\omega_1 C_1 - i\omega_2 \frac{13}{2} C_2) \\ 0 &= \mathrm{Re}(i\omega_1 \frac{7}{4} C_1 - i\omega_2 C_2) \end{aligned}$$

de donde se deduce que las partes imaginarias de C_1 y C_2 son cero y las partes reales son

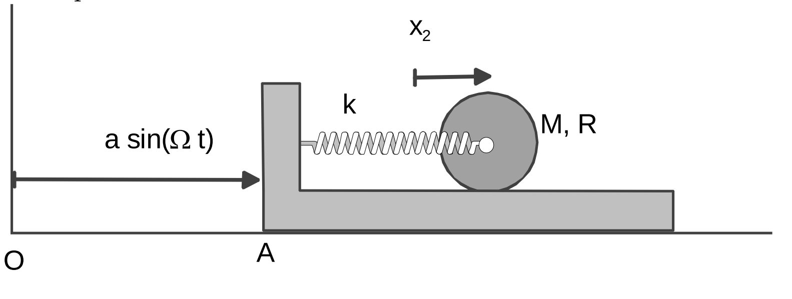
$$\mathrm{Re}(C_1) = -\frac{52}{99} a, \quad \mathrm{Re}(C_2) = \frac{8}{99} a$$

y finalmente

$$\begin{aligned} x_1 &= -\frac{52}{99} a \cos \omega_1 t + \frac{52}{99} a \cos \omega_2 t, \\ x_2 &= \frac{91}{99} a \cos \omega_1 t + \frac{8}{99} a \cos \omega_2 t. \end{aligned} \qquad (2 \text{ p})$$

Si alumno utiliza otros pasos, evalue usted más o menos de acuerdo a este método...

3. Es parecido pero....



$$x_A = a\sin\Omega t$$

$$\omega = \frac{\dot{x}_2}{R},\ v_G = \dot{x}_A + \dot{x}_2 = a\Omega\cos\Omega t + \dot{x}_2, \quad I_G = \frac{1}{2}MR^2,$$

entonces

$$\begin{aligned}
K &= \frac{1}{2}M(a\Omega\cos\Omega t + \dot{x}_2)^2 + \frac{1}{2}\frac{1}{2}MR^2(\frac{\dot{x}_2}{R})^2 \\
&= \frac{1}{2}Ma^2\Omega^2\cos^2\Omega t + Ma\Omega\,(\cos\Omega t)\,\dot{x}_2 + \frac{3}{4}M\dot{x}_2^2 \\
&\Rightarrow \frac{3}{4}M\dot{x}_2^2 + Ma\Omega\,(\cos\Omega t)\,\dot{x}_2 \\
V &= \frac{1}{2}kx_2^2
\end{aligned}$$

$$L = \frac{3}{4}M\dot{x}_2^2 + Ma\Omega\,(\cos\Omega t)\,\dot{x}_2 - \frac{1}{2}kx_2^2 \qquad (2\text{ p})$$

Luego

$$p_2 = \frac{3}{2}M\dot{x}_2 + Ma\Omega\,(\cos\Omega t)$$

La ecuación de Lagrange es

$$\frac{3}{2}M\ddot{x}_2 - Ma\Omega^2\,(\sin\Omega t) + kx_2 = 0 \qquad (2\text{ p})$$

El Hamiltoniano es (distinto a la energía)

$$\begin{aligned}
H &= p_2\dot{x}_2 - L \\
&= \frac{3}{2}M\dot{x}_2^2 + Ma\Omega\,(\cos\Omega t)\,\dot{x}_2 - (\frac{3}{4}M\dot{x}_2^2 + Ma\Omega\,(\cos\Omega t)\,\dot{x}_2 - \frac{1}{2}kx_2^2) \\
&= \frac{3}{4}M\dot{x}_2^2 + \frac{1}{2}kx_2^2 \\
&= \frac{3}{4}M(\frac{p_2 - Ma\Omega\,(\cos\Omega t)}{\frac{3}{2}M})^2 + \frac{1}{2}kx_2^2
\end{aligned}$$

Las ecuaciones de Hamilton serán

$$\begin{aligned}
\frac{\partial H}{\partial p_2} &= \frac{p_2 - Ma\Omega\,(\cos\Omega t)}{\frac{3}{2}M} = \dot{x}_2 \qquad (1\text{ p}) \\
\frac{\partial H}{\partial x_2} &= kx_2 = -\dot{p}_2 \qquad (1\text{ p})
\end{aligned}$$