Numerical methods for BSDE

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1 Introduction and basic facts

Nonlinear backward stochastic differential equations were first introduced in 1990 by Pardoux and Peng [35]. Under the Lipschitz assumption of the generator f, the authors stated the first existence and uniqueness result. Later on (1991-1994) they developed the BSDE theory relaxing hypothesis that ensured existence and uniqueness on this type of equations and giving applications to optimal control problems, a reference [36]. In [33] the authors proved a result of existence and uniqueness of BSDE under weaker conditions than the Lipschitz property.

The interest in BSDEs comes form their connections with PDEs, (see [14], [41]); stochastic control and mathematical finance (see [17], [18], among others). In particular, as shown in [16], BSDEs are a useful tool in the pricing of a European option, which consists of a contract which pays the amount ξ at time T. In a complete market, the price process Y of ξ is a solution of a BSDE.

On the other hand, a result of existence and uniqueness of BSDE equations under the assumption that the generator is locally Lipschitz conditions can be found in [20]. A similar result was obtained in the case when the coefficient is continuous with Linear growth [24]. The same authors [25] generalized these results under the assumption

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that the coefficients has a super-linear quadratic growth. Other extensions of existence and uniqueness of BSDE can be found in [21], [26] and [33]. Studies of stability of solutions for BSDE have been studied, for example in [1] the authors study stability under disturbances in the filtration.

In [6] it is considered standard BSDEs when the noise is driven by a Brownian motion and an independent Poisson random measure. They have shown the existence and uniqueness of the solution, in addition, the link with integral-partial differential equations is studied.

The BSDEs with jumps, beginning with an existence theorem given in [26] and [39]. The authors stated such a theorem for Lipschitzian generators, which can be proved by a fixed point techniques. Other interesting paper on BSDEs with jumps is given in [38] and [40].

Since in a very few cases BSDE solutions are explicit, it is logical to ask for numerical methods approximating the unique solution of this type of equations and to know the associated type of convergence. On this matter some methods of approximation have been developed.

An algorithm of four steps proposed in [28], to solve equations of the Forward-Backward type, relating the type of approximation to the partial differential equations theory. On the other hand in [3] it is proposed a method of random discretization in the time for a BSDE, where the convergence of the method for the solution (Y, Z) only needs regularity assumptions, but for the simulation studies it is necessary multiple approximations. See also [10] [29] and [13] for a FBSDE solutions; [19] for a regression-based Monte Carlo method and [42] for approximating solutions of BSDEs, and [37] for Monte Carlo valuation of American Options.

On the other hand in [11], [9], [2] and [27] the authors replace the Brownian motion by simple random walks in order to define a numerical approximations for BSDE. This technique allows to simplify the computation of the conditional expectations involved at each time step.

A quantization technique was suggested in [4] and [5] for the resolution of reflected backward SDEs when the generator f does not depend on the control variable z. This method is based on the approximation of the continuous time processes on a finite grid, and requires a further estimation of the transition probabilities on the grid.

For a class of RBSDE, in [8] the authors propose a discrete-time approximation for a forwardbackward stochastic differential equations. The L^p norm of the error is shown to be of the order of the time step. On the other hand a numerical approximation for a class of RBSDE based on the numerical approximation for BSDE and the approximation given in [30] can be found in [31] and [34].

One of the most recently work in the numerical scheme for a class of with Jumps is given in [23] and is based in the approximation for the Brownian motion and a Poisson process by two simple random walk. Finally for decoupled forward-backward SDEs with jumps a numercially scheme was proposed in [7].

Let $\Omega = \mathcal{C}\left([0,1], \mathbb{R}^d\right)$ and consider the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, in which $B_t(\omega) = \omega(t)$ is a standard d-dimensional Brownian motion. We consider he following Backward Stochastic Differential Equation (BSDE in short).

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{1}$$

where ξ is a $\mathcal{F}_T$-measurable square integrable random variable and f is Lipschitz continuous in the space variable with Lipschitz constant L. The solution of (1) is a pair of adapted processes (Y, Z) which satisfies the equation.

2 Numerical methods for BSDE

One idea to get a numerical scheme for solving BSDE is based upon a discretization of the equation (1) by replacing B with a simple random walk. To be more precise, let us consider the symmetric random walk W^n:

$$W_t^n := \frac{1}{\sqrt{n}} \sum_{k=0}^{c_n(t)} \zeta_k^n, \quad 0 \leq t \leq T,$$

where $\{\zeta_k^n\}_{1\leq k\leq n}$ is an i.i.d. Bernoulli symmetric sequence. We define $\mathcal{G}_k^n := \sigma(\zeta_1^n, \ldots, \zeta_k^n)$. Throughout this section $c_n(t) = [nt]/n$, and ξ^n denotes a square integrable random variable, measurable w.r.t. $\mathcal{G}_n^n$ that should converge to ξ. We assume that W^n and B are defined in the same probability space.

In [27], the authors consider the case when the generator depends only on the variable Y, which makes simpler the analysis. In this situation the BSDE (1) is given by

$$Y_t = \xi + \int_t^T f(Y_s)ds - \int_t^T Z_s dB_s, \tag{2}$$

whose solution is given by:

$$Y_t = \mathbb{E}\left(\xi + \int_t^T f(Y_s)ds \,/\mathcal{F}_t\right), \tag{3}$$

which can be discretized in time with step-size $h = T/n$ by solving a discrete backward stochastic differential equation given by:

$$Y_{t_i}^n = \xi^n + \frac{1}{n}\sum_{j=i}^{n} f(Y_{t_j}^n) - \sum_{j=i}^{n-1} Z_{t_j}^n \triangle W_{t_{j+1}}^n, \tag{4}$$

This equation has a unique solution (Y_t^n, Z_t^n) since the martingale W^n has the predictable representation property. It can be checked that solving this equation is equivalent to finding a solution to the following implicit iteration problem:

$$Y_{t_i}^n = \mathbb{E}\left\{Y_{t_{i+1}}^n + \frac{1}{n}f(Y_{t_i}^n)/\mathcal{G}_i^n\right\},$$

which due to the adaptedness condition, is equivalent to

$$Y_{t_i}^n - \frac{1}{n}f(Y_{t_i}^n) = \mathbb{E}\left\{Y_{t_{i+1}}^n/\mathcal{G}_i^n\right\}. \tag{5}$$

Furthermore, once $Y_{t_{i+1}}^n$ is determined, $Y_{t_i}^n$ is solved via (5) by a fixed point technique:

$$\begin{cases} X^0 = & \mathbb{E}\left\{Y_{t_{i+1}}|\mathcal{G}_i^n\right\} \\ X^1 = & X^0 + \frac{1}{n}f(X^k). \end{cases}$$

It is standard to show that if f is uniformly Lipschitz in the spatial variable x with Lipschitz constant L (we also assume that f is bounded by R) then the iterations of this procedure will converge to the true solution of (5) at a geometric rate L/n. Therefore, in the case when n is large enough, one iteration would already give us the error estimate: $|Y_{t_i}^n - X^1| \leq \frac{LR}{n^2}$, producing a good approximate solution of (5). Consequently, the explicit numerical scheme is given by:

$$\begin{cases} \hat{Y}_T^n = & \xi^n; \hat{Z}_T^n = 0 \\ X_{t_i}^n = & \mathbb{E}\left\{\hat{Y}_{t_{i+1}}|\mathcal{G}_i^n\right\} \\ \hat{Y}_{t_i}^n = & X_{t_i}^n + \frac{1}{n}f(X_{t_i}^n) \\ \hat{Z}_{t_i}^n = & \mathbb{E}\left\{\left[\hat{Y}_{t_{i+1}} + \frac{1}{n}f(\hat{Y}_{t_i}^n) - \hat{Y}_{t_i}^n\right](\triangle W_{t_{i+1}}^n)^{-1}|\mathcal{G}_i^n\right\} \end{cases}$$

The convergence of $\hat{Y}^n$ to Y is proved in the sense of the Skorohod topology in [27] and [9]. In [11] it is proved the convergence of the sequence Y^n using the tool of convergence of Filtrations. See also [3] for the case where f depends on both variables y and z.

3 Application to European options

3.1 Black and Scholes Model

Let us assume that the price S of an asset evolves according to the following linear SDE

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{6}$$

which is the continuous version of

$$\frac{S_{t+\triangle t} - S_t}{S_t} \approx \mu \triangle t + \sigma \triangle B_t, \tag{7}$$

where the relative return has a linear growth plus a random perturbation. σ is called the volatility and it is a measure of uncertainty. In this particular case, S has an explicit solution given by the Doleans-Dade exponential

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2 t\right) + \sigma B_t}. \tag{8}$$

The other element of this simple model is the existence of a riskless asset whose evolution is given by $\beta_t = \beta_0 e^{rt}$, where r : is the interest rate, that we assume is constant over time. Then β satisfies the ODE:

$$\beta_t = \beta_0 + r \int_0^t \beta_s ds \tag{9}$$

A portfolio is a couple of adapted processes (a_t, b_t) that represents the amount of investment in both assets at time t (both can be positive and negative). The wealth process is then given by

$$Y_t = a_t S_t + b_t \beta_t \tag{10}$$

A main assumption is that Y is self-financing which in mathematical terms means that

$$dY_t = a_t dS_t + b_t d\beta_t. \tag{11}$$

A call option is a financial contract between two parties. The option gives you the right to buy an agreed quantity of a particular commodity S at a certain time (the expiration date, T) for a certain price (the strike price K). Of course you have to pay a fee (called a premium q) for this right. If the option can be exercised only at T, the option is called *European*. If it can be exercised at any time before T is called *American*. The main question is what is the right price for an option in both cases? Mathematically q is determined by the existence of a replication with the initial value q and the final value $(S_T - K)^+$, that is to find (a_t, b_t) such that

$$Y_t = a_t S_t + b_t \beta_t \quad Y_T = (S_T - K)^+ \quad Y_0 = q. \tag{12}$$

We look for a solution to this problem of the form $Y_t = w(t, S_t)$ with $w(T, x) = (x - K)^+$. Using Itô's formula we get

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \tfrac{\partial w}{\partial x} dS_s + \int_0^t \tfrac{\partial^2 w}{\partial x^2} d[S,S]_s + \int_0^t \tfrac{\partial w}{\partial t} ds \\ &= Y_0 + \int_0^t \tfrac{\partial w}{\partial x} \{\mu S_s ds + \sigma S_s dB_s\} + \int_0^t \tfrac{1}{2} \tfrac{\partial^2 w}{\partial x^2} \sigma^2 S_s^2 ds + \int_0^t \tfrac{\partial w}{\partial t} ds \\ &= Y_0 + \int_0^t \tfrac{\partial w}{\partial x} \sigma S_s dB_s + \int_0^t \left(\tfrac{1}{2} \tfrac{\partial^2 w}{\partial x^2} \sigma^2 S_s^2 + \mu S_s \tfrac{\partial w}{\partial x} + \tfrac{\partial w}{\partial t} \right) ds. \end{aligned}$$

Using the self-financing hypothesis we obtain

$$\begin{aligned} Y_t &= Y_0 + \int_0^t a_s dS_s + \int_0^t b_s d\beta_s = Y_0 + \int_0^t a_s \{\mu S_s ds + \sigma S_s dB_s\} + \int_0^t b_s d\beta_s \\ &= Y_0 + \int_0^t a_s \sigma S_s dB_s + \int_0^t (r b_s \beta_s + a_s \mu S_s) ds. \end{aligned}$$

Using the uniqueness in the predictable representation property for the Brownian Motion we obtain that

$$\begin{aligned} a_s \sigma S_s &= \sigma S_s \tfrac{\partial w}{\partial x} \\ r b_s \beta_s + a_s \mu S_s &= \tfrac{1}{2} \sigma^2 S_s^2 \tfrac{\partial^2 w}{\partial x^2} + \mu S_s \tfrac{\partial w}{\partial x} + \tfrac{\partial w}{\partial t} \\ a_s &= \tfrac{\partial w}{\partial x}(s, S_s) \\ b_s &= \tfrac{Y_s - a_s S_s}{\beta_s}. \end{aligned}$$

Since $r \frac{(Y_s - a_s S_s)}{\beta_s} \beta_s + a_s \mu S_s = \frac{1}{2} \sigma^2 S_s^2 \frac{\partial^2 w}{\partial x^2} + \mu S_s \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}$, the equation for w is:

$$\begin{aligned} r\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2} &= -rx\frac{\partial w}{\partial x} + rw \\ w(T, x) &= (x - K)^+. \end{aligned} \tag{13}$$

The solution of this PDE is related to a BSDE which we deduce now. Let us start again from the self-financing assumption

$$\begin{array}{rcl}(S_T-K)^+=Y_T & = & Y_t+\int_t^T \frac{\partial w}{\partial x}dS_s+\int_t^T r(Y_s-S_s\frac{\partial w}{\partial x})ds \\ & = & Y_t+\int_t^T \sigma S_s\frac{\partial w}{\partial x}dB_s+\int_t^T(rY_s+(\mu-r)S_s\frac{\partial w}{\partial x})ds,\end{array}$$

from where we deduce

$$Y_t=\xi+\int_t^T(\alpha Z_s-rY_s)ds-\int_t^T Z_s dB_s,$$

with $\alpha=\frac{r-\mu}{\sigma}$, $\xi=(S_0e^{(\mu-\frac{1}{2}\sigma^2T)+\sigma B_T}-K)^+$ and $Z_s=\sigma S_s\frac{\partial w}{\partial x}$. In this case we have an explicit solution for w given by

$$\begin{array}{rcl}Y_0 & = & S_0\Phi(g(T,S_0))-Ke^{-rT}\Phi(h(T,S_0)) \\ w(t,x) & = & x\Phi(g(T-t,x))-Ke^{-r(T-t)}\Phi(h(T-t,x))\end{array},$$

where $g(t,x)=\frac{ln(x/K)+(r+1/2\sigma^2)t}{\sigma\sqrt{t}}$, $h(t,x)=g(t,x)-\sigma\sqrt{t}$ and $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{\frac{-y^2}{2}}dy$ is the standard normal distribution. In general, for example when σ may depend on time an (S_t), we obtain a BSDE for (Y_t) coupled with a Forward equation for (S_t), that can be solved numerically.

4 Numerical methods for RBSDE

In this section we are interested in the numerical approximation of backward stochastic differential equation with reflection (in short RBSDE). We present here the case of one lower barrier, which we assume is an Itô process (a sum of a Brownian martingale and a continuous finite variation process).

$$\begin{array}{rcl}Y_t & = & \xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_s dB_s+K_T-K_t, \quad 0\le t\le T, \\ Y_t & \ge & L_t, \quad 0\le t\le T, \quad \text{and} \quad \int_0^T (Y_t-L_t)\,dK_t=0.\end{array} \tag{14}$$

where as before f is the generator, ξ is the terminal condition, $L=(L_t)$ is the reflecting barrier. Under the Lipschitz assumption of f (see [14] and for generalizations see [33], [12], [22]) there is a unique solution (Y,Z,K) of adapted processes, with the condition that (K) is increasing and minimal in the sense that is supported on the times (Y) touches the boundary.

The numerical scheme for RBSDE that present here is based on a penalization of the equation (14) (see [14]) and then use the standard Euler scheme. The penalization equation is given by

$$Y_t^\varepsilon = \xi + \int_t^1 f(s, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^1 Z_s^\varepsilon dB_s + \frac{1}{\varepsilon}\int_t^1 (L_s - Y_s^\varepsilon)^+ ds,$$

where ξ and f satisfy the above assumptions **(A1), (A2)**. In this framework, we define

$$K_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t (L_s - Y_s^\varepsilon)^+ ds, \quad 0 \le t \le 1,$$

where ε is the penalization parameter. In order to have an explicit iteration we include an extra Picard's iteration and the numerical procedure is then

$$Y_{t_i}^{\varepsilon,p+1,n} = Y_{t_{i+1}}^{\varepsilon,p+1,n} + \frac{1}{n} f(t_i, Y_{t_i}^{\varepsilon,p,n}, Z_{t_i}^{\varepsilon,p,n}) + \frac{1}{n\varepsilon}(L_{t_i} - Y_{t_i}^{\varepsilon,p,n})^+ - \frac{1}{\sqrt{n}} Z_{t_i}^{\varepsilon,p+1,n}\zeta_{i+1} \quad (15)$$

$$K_{t_{i+1}}^{\varepsilon,p+1,n} - K_{t_i}^{\varepsilon,p+1,n} := \frac{1}{n\varepsilon}\left(S - \ddot{Y}_{t_i}^{\varepsilon,p+1,n}\right)^+ \quad \text{for} \quad i \in \{n-1, \dots, 0\}. \quad (16)$$

Theorem 1 *Under the assumptions*

> **(A1)** f *is Lipschitz continuous and bounded.*
>
> **(A2)** L *is assumed to be an Itô process.*
>
> **(A3)** $\lim_{n\to+\infty} \mathbb{E}\left[\sup_{s\in[0,T]} \left|\mathbb{E}[\xi|\mathcal{F}_s] - \mathbb{E}[\xi^n|\mathcal{G}^n_{c_n(s)}]\right|\right] = 0.$

the triplet $(\xi^n, Y^{\varepsilon,p,n}, Z^{\varepsilon,p,n}, K^{\varepsilon,p,n})$ *converges in the Skorohod topology towards the solution* (ξ, Y, Z, K) *of the RBSDE (14) (the order is first* $p \to \infty$*, then* $n \to \infty$ *and finally* $\varepsilon \to 0$*).*

4.1 A procedure based on Ma and Zhang's method

In this subsection we introduce a numerical scheme based on an idea given in [30]. The new ingredient is to use a standard BSDE with no reflection and then impose in the final condition on every step of the discretization that the solution must be above the barrier. Schematically we have

- $Y_1^n := \xi^n$.

- for $i = n, n-1, \ldots 1$ let $\left(\tilde{Y}^n, Z^n\right)$ be the solution of the BSDE:

$$\tilde{Y}^n_{t_{i+1}} = Y^n_{t_i} + \frac{1}{n} f(s, \tilde{Y}^n_s, Z^n_s) - Z^n_s (W^n_{t_{i+1}} - W^n_{t_i}). \tag{17}$$

- define $Y^n_{t_{i+1}} = \tilde{Y}^n_{t_{i+1}} \vee L_{t_{i+1}}$
- let $K^n_0 = 0$ and define $K^n_{t_i} := \sum_{j=1}^{i} (Y^n_{t_{j-1}} - \tilde{Y}^n_{t_{j-1}})$.

Clearly, K^n is predictable and we have

$$Y^n_{t_{i-1}} = Y^n_{t_i} + \int_{t_{i-1}}^{t_i} f\left(s, \tilde{Y}^n_s, Z^n_s\right) ds - \int_{t_{i-1}}^{t_i} Z^n_s dW^n_s + K^n_{t_i} - K^n_{t_{i-1}}.$$

Theorem 2 *Under the assumptions* **A1**, **A2** *of Theorem 1 and*

$$\lim_{n \to +\infty} \mathbb{E}\left[\sup_{s \in [0,T]} \left|\mathbb{E}[\xi | \mathcal{F}_s] - \mathbb{E}[\xi^n | \mathcal{G}^n_{c_n(s)}]\right|\right]^2 = 0,$$

we have

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{0 \le i \le n} \left|Y_{t_i} - Y^n_{t_i}\right|^2 + \int_0^1 \left|Z_t - Z^n_t\right|^2 dt\right] = 0.$$

5 Application to American Options

An American option is a one that can be exercised at any time between the purchase date and the expiration date T, which we assume is non-random and for the sake of simplicity we take $T = 1$. This situation is far more general than the European-style option, which can only be exercised on the date of expiration. Since an American option provides an investor with a greater degree of flexibility the premium for this option is higher than the premium for a European-style option.

We consider an economy with a set of dynamically financial markets, described by a Filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{0 \le t \le T}, \mathbb{P})$. As before we consider the following adapted processes: The price of the risk asset $S = (S_t)_{0 \le t \le T}$ and the wealth process $Y = (Y_t)_{0 \le t \le T}$. We assume that the rate interest r is constant. The aim is to obtain Y_0, the value of the American Option.

We assume that there exists a unique risk-neutral measure allowing one to compute prices of all contingent claims as the expected value of their discounted cash flows.

This requires some regularity assumptions (see for example [38] or [15]). The equation that describes the evolution of Y is given by a linear reflected BSDE coupled with the forward equation for S.

$$\begin{aligned} Y_t &= (K - S_1)^+ - \int_y^1 (rY_s + (\mu - r)Z_s)\, ds + K_1 - K_t - \int_t^1 Z_s dB_s, \\ S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dB_s. \end{aligned}$$

The increasing process K keeps the process Y above the barrier $L_t = (S_t - K)^+$ (for a call option) in a minimal way, that is $Y_t \geq L_t$, $dK_t \geq 0$ and

$$\int_0^1 (Y_t - L_t) dK_t = 0.$$

The exercise random time is given by the following stopping time $\tau = \inf\{t; Y_t - L_t < 0\}$ that represents the exit time from market, for the investor. As usual we take $\tau = 1$ if Y never touch the boundary L. At τ he/she will buy the stock if $\tau < 1$, otherwise the investor do not exercise the option. In this problem, we are interested in finding Y_t, Z_t, and τ. In the following table and picture we summarize the results of a simulation for the American option.

n	$S_0 = 80$	$S_0 = 100$	$S_0 = 120$
1	20	11.2773	4.1187
2	22.1952	10.0171	3.8841
3	21.8707	10.7979	3.1489
4	22.8245	10.1496	3.9042
⋮	⋮	⋮	⋮
15	22.6775	10.8116	3.7119
16	22.6068	10.6171	3.6070
17	22.7144	10.7798	3.6811
18	22.6271	10.6125	3.6364
Real Values	**21.6059**	**9.9458**	**4.0611**

Table 1: Numerical Scheme for American Option with 18 steps, $K = 100$ $r = 0.06$, $\sigma = 0.4$ and $T = 0.5$ and different values of S_0.

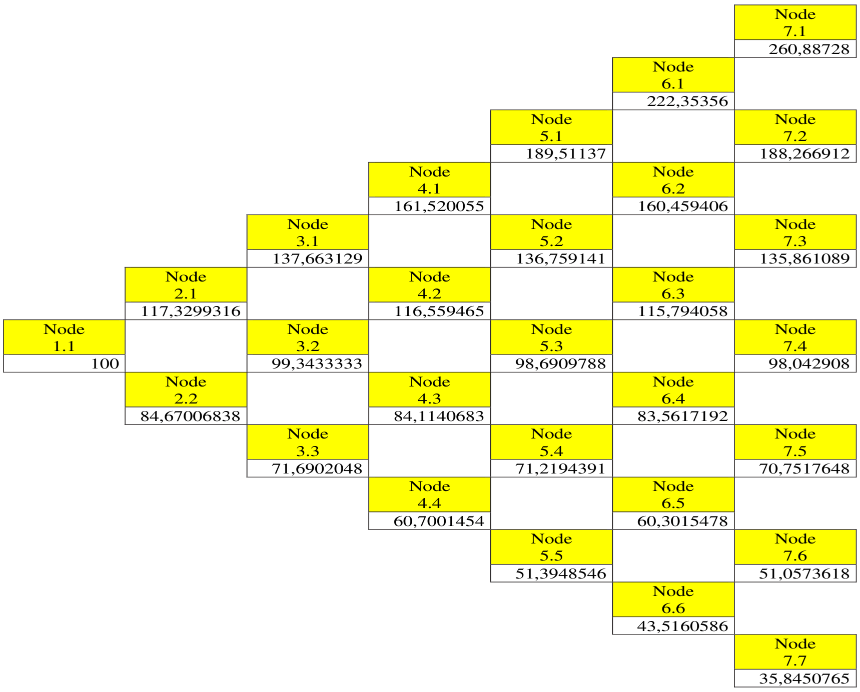

Figure 1: Binomial tree for 6 time steps, $r = 0.06$, $\sigma = 0.4$ and $T = 0.5$.

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