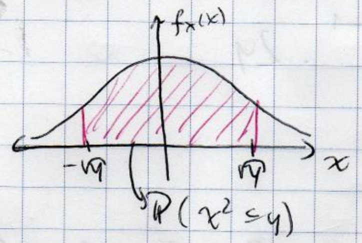
Pauta Auxiliar 7 MA34A-03

Fecha / Date: 11/10/07

a) $X \sim N(0,1)$ $\qquad Y = X^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$



$$= 1 - 2P(X \le -\sqrt{y})$$

$$= 1 - 2F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -2\frac{d}{dy}F_X(-\sqrt{y}) = -2 f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}}$$

$$= f_X(-\sqrt{y})\, y^{-1/2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \cdot y^{-1/2} = \sqrt{\frac{1}{2}} \cdot y^{-1/2} \cdot \frac{1}{\sqrt{\pi}} e^{-\frac{y}{2}} \;\Big|\begin{matrix}\cdot 1/2 \\ \cdot 1/2\end{matrix}$$

$$f_Y(y) = \frac{1}{2}\left(\frac{1}{2}y\right)^{-1/2} \frac{1}{\sqrt{\pi}} e^{-y/2} \qquad \sqrt{\pi} \rightarrow \Gamma(1/2)$$

$$= \alpha e^{-\alpha y} \frac{(\alpha y)^{r-1}}{\Gamma(r)} \qquad \text{con } r = 1/2,\ \alpha = 1/2$$

$\Rightarrow Y \sim G(1/2, 1/2)$

b) $Y = \sqrt{X_1^2 + X_2^2 + X_n^2}$

$$P(Y \le y) = P(0 \le X_1^2 + X_2^2 + \dots + X_n^2 \le y^2)$$

Pero $X_i^2 \sim G(1/2, 1/2) \quad \forall i$

$Z = \sum X_i^2 \sim G\left(\frac{n}{2}, 1/2\right)$ (visto en clases)

$$F_Y(y) = P(0 \le Z \le y^2) = F_Z(y^2)$$

$\Rightarrow \quad F_Y(y) = F_Z(y^2)$

$$f_Y(y) = \frac{d}{dy} F_Y = f_Z(y^2) \cdot 2y$$

$$= \frac{\frac{1}{2} e^{-\frac{1}{2}y^2} \left(\frac{1}{2}y^2\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} \cdot 2y$$

Caso $n=3$ $\quad Y = \sqrt{X_1^2+X_2^2+X_3^2}$ $\quad$ Distribución de Maxwell

$$f_Y(y) = \frac{\cancel{\frac{1}{2}} e^{-\frac{1}{2}y^2} \left(\frac{1}{2}y^2\right)^{\frac{1}{2}} \cdot 2y}{\cancel{\frac{1}{2}}\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \cdot 2y^2 \quad *$$

* Si $\sigma \neq 1 \Rightarrow$ aplicaciones en FIESTA ($\sigma = \sigma(T)$)

D2) Persona cualquiera $\rightarrow P(\text{Glaucoma}) = 1/10$

$P(\text{No glaucoma}) = 9/10$

a) $$P(\text{Glaucoma} \mid P=22.5) = \frac{P(P=22.5 \mid \text{Glaucoma}) \cdot P(\text{Glaucoma})}{P(P=22.5)}$$

$$P(p=22.5 \mid \text{glaucoma}) = \int_{22.5}^{22.5+\Delta x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-25)^2}{2}} dx$$

$\downarrow$ T.V.M

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x'-25)^2}{2}} \cdot \Delta x \qquad x' \in [22,5, 22,5+\Delta x]$$

$$P(P=22.5 \mid \text{normal}) = \int_{22.5}^{22.5+\Delta x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-20)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x'-20)^2}{2}} \Delta x \qquad x' \in [22,5; 22,5+\Delta x]$$

$$P(P=22.5) = P(P=22.5 \mid \text{Glaucoma}) \cdot \frac{1}{10} + P(P=22.5 \mid \text{normal}) \cdot \frac{9}{10}$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(x'-25)^2}{2}} \cdot \frac{1}{10} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(x'-20)^2}{2}} \frac{9}{10}\right) \Delta x$$

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$$\Rightarrow \mathbb{P}(\text{glaucoma} \mid p=22.5) = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{(x'-25)^2}{2}}\, \cancel{\Delta x} \cdot 1/10}{\cancel{\frac{1}{\sqrt{2\pi}}} \cdot \cancel{\Delta x} \left(e^{-\frac{(x'-25)^2}{2}} \cdot \frac{1}{10} + e^{-\frac{(x'-22,5)^2}{2}} \cdot \frac{9}{10} \right)}$$

$$\lim_{\Delta x \to 0} \quad \Rightarrow \quad x' \longrightarrow 22,5$$

$$\Rightarrow \quad e^{-\frac{(x'-25)^2}{2}} \longrightarrow e^{-\frac{(-2,5)^2}{2}}$$

$$e^{-\frac{(x'-20)^2}{2}} \longrightarrow e^{-\frac{(2,5)^2}{2}} \quad \Bigg] \text{ iguales}$$

$$\Rightarrow \mathbb{P}(\text{Glaucoma} \mid p=22.5) = 1/10$$

b) Queremos n tal que $\left|\frac{1}{n}\sum_{i=1}^{n} x_i - 25\right| \leq 0.5$ con probabilidad 0.95

$$\Rightarrow \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} x_i - 25\right| \leq 0.5\right) = 0.95$$

$$\Rightarrow \mathbb{P}\left(-0.5 \leq \frac{1}{n}\sum_{i=1}^{n} x_i - 25 \leq 0.5\right) = 0.95$$

Ahora bien $Z = \dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i - 25}{1/\sqrt{n}}$, $Z \sim N(0,1)$

$$\Rightarrow \mathbb{P}\left(\frac{-0.5}{1/\sqrt{n}} \leq Z \leq \frac{0,5}{1/\sqrt{n}}\right) = 1 - 2\,\mathbb{P}(Z \geq 0.5\sqrt{n}) = 0.95$$

$$\mathbb{P}(Z \geq 0.5\sqrt{n}) = 0.025 \Rightarrow 0.5\sqrt{n} = 1.96$$

$$\sqrt{n} = 3,92$$

$n = 15,3664 \Rightarrow n = 16$ cumple lo requerido

a) Queremos $Z = \frac{\frac{1}{n}\sum x_i - \mu}{\sigma^2/\sqrt{n}} \sim N(0,1)$

$$P\left(\left|\frac{1}{n}\sum x_i - \mu\right| < 2\right) = 0.95$$

$$P\left(|Z| < \frac{2}{1/\sqrt{n}}\right) = 0.95$$

$$1 - 2P\left(Z > \frac{2}{5/\sqrt{n}}\right) = 0.95$$

$$P\left(Z > \frac{2}{5}\cdot\sqrt{n}\right) = 0.025$$

$$\Rightarrow \frac{2}{5}\cdot\sqrt{n} = 1.96 \Rightarrow \sqrt{n} = 4.9$$

$$\Rightarrow n = 24.01$$

$n = 25$ cumple lo pedido

b) Buscamos

$$P(x_P - x_B > 5)$$

$x_P \sim N(50, 25)$
$x_B \sim N(40, 16)$ $\Rightarrow$ $x_P - x_B \sim N(10, 25+16)$

$\sim N(10, 41)$

$\rightarrow \sigma = \sqrt{41}$

$$P(x_P - x_B > 5) = P\left(\frac{\overbrace{x_P - x_B}^{x_2} - 10}{\sqrt{41}} > \frac{-5}{\sqrt{41}}\right)$$

$$Z = \frac{x_P - x_B - 10}{\sqrt{41}} \sim N(0,1)$$

$$\Rightarrow P\left(Z > \frac{-5}{\sqrt{41}}\right) = P(Z > -0.7808) =$$

$$= 1 - P(Z < -0.7808) =$$

$$1 - 0.22 = 0.78$$

4. Sean las variables N_i: Nota control i-ésimo
x_i: Puntos " " "

$x_i \rightarrow B(n, p)$ $n = 6$ $p = 2/5$

$N_i = 1.0 + x_i$

Se deduce del enunciado

¿Cómo se distribuye $\frac{1}{20}\sum_{i=1}^{20} N_i = \underset{\text{nota final}}{\mathbb{N}}$?

<u>Usando T.C.L</u>

$$\mathbb{N} = \frac{1}{20}\sum_{i=1}^{20} N_i \rightsquigarrow N(\bar{N}, \bar{\sigma}^2)$$

$\bar{N}$ = Promedio de cualquier control (esperanza), pues son idénticamente distribuidos

$\bar{\sigma}^2$: Varianza de la nota de cualquier control, dividido por $n \Rightarrow \bar{\sigma}^2 = \sigma^2/n = \sigma^2/20$

<u>O bien la otra forma del T.C.L</u>

$$Z = \frac{\mathbb{N} - \bar{N}}{\sigma/\sqrt{n}} \rightsquigarrow N(0,1)$$

σ: $\sqrt{\text{varianza}}$ = desviación estándar

¿$\bar{N}$? $\bar{N} = \mathbb{E}(1.0 + x_i) = 1.0 + \mathbb{E}(x_i)$

$\mathbb{E}(x_i) = np = 6 \cdot \frac{2}{5} = \frac{12}{5} \Rightarrow \mathbb{E}(N_i) = \frac{12}{5} + 1 = \frac{17}{5}$

¿σ^2? $\sigma^2 = Var(1.0 + x_i) = Var(x_i) = np(1-p) =$
$= 6 \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{36}{25} \Rightarrow \sigma = 6/5$

Buscamos $\mathbb{P}(\mathbb{N} \geq 4.0) = \mathbb{P}\left(\frac{\mathbb{N} - \bar{N}}{\sigma/\sqrt{20}} \geq \overbrace{\frac{4.0 - \bar{N}}{\sigma/\sqrt{20}}}^{\alpha}\right)$

$Z \rightsquigarrow N(0,1)$ $= \mathbb{P}(Z \geq \alpha)$

Ahora simplemente necesitamos calcular α y buscar el valor de $\mathbb{P}(Z > \alpha)$ en la tabla

$$\alpha = \frac{4.0 - \bar{N}}{\sigma/\sqrt{20}} = \frac{\sqrt{20}\cdot(4-\frac{17}{5})}{6/5} = \frac{\sqrt{20}\cdot 3/5}{6/5} = \frac{\sqrt{20}}{2} = 2.236$$

$$\Rightarrow \mathbb{P}(\text{aprobar}) = \mathbb{P}(Z \geq 2.236) \approx 0{,}0127$$

b) Ahora buscaremos repetir el mismo procedimiento con la nueva distribución de N_i

$$N_i = \underset{\substack{\uparrow\\ \text{buenas}}}{1.0 + x_i} - \frac{1}{4}(\underset{\substack{\uparrow\\ \text{malas}}}{6 - x_i}) = \frac{5}{4}x_i - 0.5$$

$$\mathbb{E}(N_i) = \frac{5}{4}\mathbb{E}(x_i) - \frac{1}{2}$$

$$\mathbb{E}(x_i) = 6\,\frac{2}{3} = \frac{12}{3} = 4 \Rightarrow \mathbb{E}(N_i) = \boxed{4.5}$$

$$Var(N_i) = \frac{25}{16}Var(x_i) = \frac{25}{\cancel{16}_{9}}\cdot\cancel{6}\,\frac{2}{\cancel{3}}\cdot\frac{1}{3} = \frac{25}{12} = \sigma^2$$

$$\Rightarrow \sigma = \frac{5}{2\sqrt{3}}$$

$$\therefore\ Z = \frac{\bar{N} - 4.5}{5/2\sqrt{3}\sqrt{20}} \sim N(0,1)$$

$$\mathbb{P}(\text{aprobar}) = \mathbb{P}(\bar{N} \geq 4.0) = \mathbb{P}\left(Z \geq -\frac{0.5\cdot 2\cdot\sqrt{60}}{5}\right)$$

$$= \mathbb{P}(Z \geq -1.549) = 1 - \mathbb{P}(Z < -1.549)$$
$$= 1 - \mathbb{P}(Z > 1.549)$$
$$= 1 - 0.0606$$
$$= 0.9394$$

normal es simétrica c/r 0 (0,1)

5. Recordamos que $M_x(t) = \mathbb{E}(e^{-tx})$ y que cumple que:

$$\mathbb{E}(x) = M_x'(t)\Big|_{t=0}$$

(Notar que se cumple si $M_x(t)$ existe.)

$$\mathbb{E}(x^2) = M_x''(t)\Big|_{t=0}$$

y además

$$\vdots$$

$$\mathbb{E}(x^n) = M^n_x(t)\Big|_{t=0}$$

$M_x(t)\big|_{t=0}$ siempre existe y vale 1

Buscaremos $\left[\dfrac{d^2}{dt^2} \ln(M_x(t))\right]\Big|_{t=0}$

Notar que, por supuesto, debemos derivar 1^{ero} y luego evaluar.

$$\frac{d}{dt}\left[\ln(M_x(t))\right] = \frac{1}{M_x(t)} \cdot M_x'(t)$$

→ Regla de la cadena

$$\frac{d^2}{dt^2}\left[\ln(M_x(t))\right] = -\frac{1}{M_x(t)^2} \cdot M_x'(t)^{*} \cdot M_x'(t) + \frac{1}{M_x(t)} \cdot M_x''(t)$$

regla del producto (y no olvidar *)

$$\Rightarrow \frac{d^2}{dt^2}\left[\ln[M_x(t)]\right] = \frac{M_x''(t)}{M_x(t)} - \frac{1}{M_x(t)^2} \cdot M_x'(t)^2$$

Ahora evaluamos para $t=0$

$$\Rightarrow \frac{d^2}{dt^2}\left[\ln[M_x(t)]\right]\Big|_{t=0} = \mathbb{E}(x^2) - (\mathbb{E}(x))^2 = Var(x)$$