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Pauta de Corrección del Control 3

1) Métodos para Sistemas de Ecuaciones No-Lineales.

(a) El polinomio de cuarto grado:

$$p(x) = x^4 + x^3 - x^2 - x - 1 = 0$$

tiene un cero cerca de 1.1. Determine este cero aplicando los métodos de:

i) La Bisección

iii) Newton-Raphson

(b) Resuelva el siguiente SENL mediante el Método de Newton-Kantorovich:

$$\begin{bmatrix} x \sin(y) \\ y \cos(x) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Usando una aritmética de 2 cifras significativas con redondeo y partiendo con $\vec{x}_0 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$, haga 2 iteraciones. Calcule el error absoluto de dicha solución.

Respuesta

1) Usaremos 4 cifras significativas con redondeo

(a) $p(x) = x^4 + x^3 - x^2 - x - 1$, $\bar{x} \in [1.1 - \varepsilon, 1.1 + \varepsilon]$

i. Sea $x_0 = 1$, $x_1 = 1.2 \Rightarrow p(1) = -1$, $p(1.2) = 1.2^4 + 1.2^3 - 1.2^2 - 1.2 - 1 = 0.1616$

$$x_2 = \frac{1+1.2}{2} = 1.1 \Rightarrow p(1.1) = 1.1^4 + 1.1^3 - 1.1^2 - 1.1 - 1 = -0.5149$$

$$x_3 = \frac{1.1+1.2}{2} = 1.15 \Rightarrow p(1.15) = 1.15^4 + 1.15^3 - 1.15^2 - 1.15 - 1 = -0.2026$$

$$x_4 = \frac{1.15+1.2}{2} = 1.175 \Rightarrow p(1.175) = 1.175^4 + 1.175^3 - 1.175^2 - 1.175 - 1 = -0.02727$$

$$x_5 = \frac{1.175+1.2}{2} = 1.188 \Rightarrow p(1.188) = 1.188^4 + 1.188^3 - 1.188^2 - 1.188 - 1 = 0.06922$$

$$x_6 = \frac{1.188+1.175}{2} = 1.182 \Rightarrow p(1.182) = 1.182^4 + 1.182^3 - 1.182^2 - 1.182 - 1 = 0.02423$$

$$x_7 = \frac{1.182+1.175}{2} = 1.179 \Rightarrow p(1.179) = 1.179^4 + 1.179^3 - 1.179^2 - 1.179 - 1 = 0.002031$$

ii. $p'(x) = 4x^3 + 3x^2 - 2x - 1 \Rightarrow x_{k+1} = x_k - \frac{x_k^4 + x_k^3 - x_k^2 - x_k - 1}{4x_k^3 + 3x_k^2 - 2x_k - 1}$

$$x_1 = 1.1 - \frac{1.1^4 + 1.1^3 - 1.1^2 - 1.1 - 1}{4 \times 1.1^3 + 3 \times 1.1^2 - 2 \times 1.1 - 1} = 1.1 - \frac{1.464 + 1.331 - 1.21 - 1.1 - 1}{5.324 + 3.63 - 2.2 - 1} = 1.1 - \frac{-0.515}{5.754} = 1.1 + 0.0895 = 1.19$$

$$x_2 = 1.19 - \frac{1.19^4 + 1.19^3 - 1.19^2 - 1.19 - 1}{4 \times 1.19^3 + 3 \times 1.19^2 - 2 \times 1.19 - 1} = 1.19 - \frac{2.005 + 1.685 - 1.416 - 1.19 - 1}{6.741 + 4.248 - 2.38 - 1} = 1.19 - \frac{0.084}{7.609} = 1.19 - 0.01104 = 1.179$$

$$x_3 = 1.179 - \frac{1.179^4 + 1.179^3 - 1.179^2 - 1.179 - 1}{4 \times 1.179^3 + 3 \times 1.179^2 - 2 \times 1.179 - 1} = 1.179 - \frac{1.932 + 1.639 - 1.390 - 1.179 - 1}{6.555 + 4.170 - 2.358 - 1} = 1.179 - \frac{0.002}{7.367} = 1.179 - 0.0002715 = 1.179$$

(b) Método de Newton-Kantorovich con $\vec{x}_0 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$:

$$\vec{G}(x, y) = \begin{bmatrix} x \sin(y) - x \\ y \cos(x) - y \end{bmatrix}, \nabla \vec{G}(x, y) = \begin{bmatrix} \sin(y) - 1 & x \cos(y) \\ -y \sin(x) & \cos(x) - 1 \end{bmatrix}$$

$$\vec{G}(\vec{x}_0) = \begin{bmatrix} 0.1 \sin(0.5) - 0.1 \\ 0.5 \cos(0.1) - 0.5 \end{bmatrix} = \begin{bmatrix} -0.052 \\ 0 \end{bmatrix}$$

$$\nabla \vec{G}(\vec{x}_0) = \begin{bmatrix} \sin(0.5) - 1 & 0.1 \cos(0.5) \\ -0.5 \sin(0.1) & \cos(0.1) - 1 \end{bmatrix} = \begin{bmatrix} -0.52 & 0.088 \\ -0.05 & 0 \end{bmatrix} \quad \det = 0.0044$$

$$\left[\nabla \vec{G}(\vec{x}_0) \right]^{-1} = \frac{1}{0.0044} \begin{bmatrix} 0 & -0.088 \\ 0.05 & -0.52 \end{bmatrix} = \begin{bmatrix} 0 & -20 \\ 11 & -120 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0 & -20 \\ 11 & -120 \end{bmatrix} \begin{bmatrix} -0.052 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}\vec{G}(\vec{x}_1) &= \begin{bmatrix} 0.1 \sin(1) - 0.1 \\ 1 \cos(0.1) - 1 \end{bmatrix} = \begin{bmatrix} -0.016 \\ 0 \end{bmatrix} \\ \nabla \vec{G}(\vec{x}_1) &= \begin{bmatrix} \sin(1) - 1 & 0.1 \cos(1) \\ -1 \sin(0.1) & \cos(0.1) - 1 \end{bmatrix} = \begin{bmatrix} -0.16 & 0.054 \\ -0.1 & 0 \end{bmatrix} \quad \det = 0.0054 \\ [\nabla \vec{G}(\vec{x}_1)]^{-1} &= \frac{1}{0.0054} \begin{bmatrix} 0 & -0.054 \\ 0.1 & -0.16 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 19 & -30 \end{bmatrix} \\ \vec{x}_2 &= \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & -10 \\ 19 & -30 \end{bmatrix} \begin{bmatrix} -0.016 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1.3 \end{bmatrix}\end{aligned}$$

$$E_{NK} = \|G(\vec{x}_2)\|_{\infty} = \left\| \begin{bmatrix} 0.1 \sin(1.3) - 0.1 \\ 1.3 \cos(0.1) - 1.3 \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} -3.6442 \times 10^{-3} \\ -6.4946 \times 10^{-3} \end{bmatrix} \right\|_{\infty} = 6.495 \times 10^{-3}$$

2) Métodos de Integración:

La Regla de Simpson Compuesta se obtiene al calcular la integral de una función $f(x)$ en un intervalo $[a, b]$ subdividiéndolo en los intervalos de igual largo definidos por los puntos equi-espaciados:

x_0, x_1, \dots, x_n : Si $n \geq 2$ par, $h = \frac{(b-a)}{n}$, $x_k = a + kh$, entonces:

$$\int_a^b f(t) dt \cong \frac{h}{3} \left[f(x_0) + 4f(x_1) + f(x_n) + \sum_{k=1}^{\frac{n}{2}-1} (2f(x_{2k}) + 4f(x_{2k+1})) \right]$$

Para calcular la integral impropia:

$$I = \int_0^1 \frac{g(x)}{x^p} dx$$

mediante los métodos vistos en clases se realiza el siguiente procedimiento:

Paso 1: Determinar el polinomio de Taylor de $g(x)$ en torno a $x_0 = 0$ de orden 4: $p_4(x) = \sum_{k=0}^4 \frac{g^{(k)}(0)}{k!} x^k$

Paso 2: $I = \int_0^1 \frac{g(x)-p_4(x)}{x^p} dx + \int_0^1 \frac{p_4(x)}{x^p} dx$

Paso 3: $\int_0^1 \frac{p_4(x)}{x^p} dx = \int_0^1 \frac{\sum_{k=0}^4 \frac{g^{(k)}(0)}{k!} x^k}{x^p} dx = \int_0^1 \sum_{k=0}^4 \frac{g^{(k)}(0)}{k!} x^{k-p} dx = \sum_{k=0}^4 \frac{g^{(k)}(0)}{k!(k-p+1)}$

Paso 4: $\int_0^1 \frac{g(x)-p_4(x)}{x^p} dx = \int_0^1 G(x) dx$ donde:

$$G(x) = \begin{cases} \frac{g(x)-p_4(x)}{x^p} & \text{si } 0 < x \leq 1 \\ 0 & \text{si } x = 0 \end{cases}$$

Se calcula $\int_0^1 G(x) dx$ mediante el M. de Simpson Compuesto con $n = 4$.

Calcule la siguiente integral impropia aplicando este método: $I = \int_0^1 \frac{e^x}{\sqrt{x}} dx$. Determine el error relativo al utilizar este método comparando con el valor exacto: $I = 2.9253035$

Respuesta

2) $g(x) = e^x$; $x^p = x^{\frac{1}{2}}$

$$p_4(x) = \sum_{k=0}^4 \frac{g^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\int_0^1 \frac{p_4(x)}{x^{\frac{1}{2}}} dx = \int_0^1 \frac{1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}}{x^{\frac{1}{2}}} dx = \int_0^1 \left(x^{-\frac{1}{2}} + x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{3}{2}} + \frac{1}{6}x^{\frac{5}{2}} + \frac{1}{24}x^{\frac{7}{2}} \right) dx = \left[2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{5}x^{\frac{5}{2}} + \frac{1}{21}x^{\frac{7}{2}} + \frac{1}{108}x^{\frac{9}{2}} \right] \Big|_0^1 = 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} = \frac{11051}{3780} = 2.923544973544973545$$

$$G(x) = \begin{cases} \frac{e^x - \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right)}{x^{\frac{1}{2}}} & \text{si } 0 < x \leq 1 \\ 0 & \text{si } x = 0 \end{cases}$$

$$n = 8; h = \frac{1}{4}; x_0 = 0; x_1 = \frac{1}{4}; x_2 = \frac{1}{2}; x_3 = \frac{3}{4}; x_4 = 1$$

$$\int_0^1 G(x) dx \cong \frac{h}{3} [G(x_0) + 4G(x_1) + 2G(x_2) + 4G(x_3) + G(x_4)]$$

$$\begin{aligned} &= \frac{1}{12} 0 + \frac{1}{12} 4 \frac{e^{\frac{1}{4}} - \left(1 + \frac{1}{4} + \frac{\left(\frac{1}{4}\right)^2}{2} + \frac{\left(\frac{1}{4}\right)^3}{6} + \frac{\left(\frac{1}{4}\right)^4}{24}\right)}{\left(\frac{1}{4}\right)^{\frac{1}{2}}} + \frac{1}{12} 2 \frac{e^{\frac{1}{2}} - \left(1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^4}{24}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} \\ &+ \frac{1}{12} 4 \frac{e^{\frac{3}{4}} - \left(1 + \frac{3}{4} + \frac{\left(\frac{3}{4}\right)^2}{2} + \frac{\left(\frac{3}{4}\right)^3}{6} + \frac{\left(\frac{3}{4}\right)^4}{24}\right)}{\left(\frac{3}{4}\right)^{\frac{1}{2}}} + \frac{1}{12} e^1 - \frac{1}{12} \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right) = \frac{1}{12} \times 2.1229460 \times 10^{-2} = \\ &1.769122 \times 10^{-3} \end{aligned}$$

$$I \simeq 2.923\,545 + 1.769\,122 \times 10^{-3} = 2.925\,314$$

$$E_{abs} = |2.9253035 - 2.925\,314| = 0.000\,010\,5$$

$$E_{rel} = \frac{|2.9253035 - 2.925\,314|}{|2.9253035|} = 3.589\,371\,154\,138\,365\,472\,2 \times 10^{-6}$$

3) Métodos para Ecuaciones Diferenciales Ordinarias:

La ecuación diferencial de Bernoulli de orden n (entero con $n \neq 1$) se define según:

$$\frac{dy(x)}{dx} + p(x)y(x) = q(x)y(x)^n \quad 0 \leq x \leq 1 \quad y(x=0) = y_0 \quad (B)$$

Esta edo es no lineal pero integrable. Multiplicando (B) por $(1-n)y^{-n}$ se tiene:

$$\frac{d}{dx}(y(x)^{1-n}) + (1-n)p(x)y(x)^{1-n} = q(x)(1-n)$$

Si se define $u(x) = y(x)^{1-n}$ se obtiene la edo de primer orden lineal en la variable $u(x)$:

$$\frac{du}{dx} + (1-n)p(x)u(x) = q(x)(1-n)$$

Para la sgte. ecuación tipo Bernoulli donde $n = -2$, $p(x) = x^2$, $q(x) = x^2$:

$$\frac{dy}{dx} + x^2y = \frac{x^2}{y^2} \quad 0 \leq x \leq 1 \quad y(x=0) = 3 \quad (*)$$

- Obtenga la solución analítica de (*).
- Obtenga la solución numérica de (*) mediante el método de Runge-Kutta de Orden 3 para $h = 0.2$:

$$\begin{aligned} q_k^1 &= hf(x_k, y_k) & q_k^2 &= hf(x_k + \frac{h}{2}, y_k + \frac{q_k^1}{2}) & q_k^3 &= hf(x_k + h, y_k - q_k^1 + 2q_k^2) \\ y_{k+1} &= y_k + \frac{1}{6}(q_k^1 + 4q_k^2 + q_k^3) & \forall k &= 0, 1, 2, \dots, n \end{aligned}$$

Determine la precisión de la solución numérica, comparando la solución analítica y numérica.

Respuesta

$$3) \frac{dy}{dx} + x^2y = \frac{x^2}{y^2} \quad 0 \leq x \leq 1 \quad y(x=0) = 3$$

$$(a) \frac{dy}{dx} + x^2y = \frac{x^2}{y^2} / * (1+2)y^2 \implies 3\frac{dy}{dx}y^2 + 3x^2y^3 = 3x^2$$

$$u(x) = y(x)^3 \implies \frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

$$3\frac{dy}{dx}y^2 + 3x^2y^3 = 3x^2 \implies \frac{du}{dx} + 3x^2u = 3x^2$$

$$\text{Homogenea: } \frac{du}{dx} + 3x^2u = 0 \implies \frac{du}{u} = -3x^2dx / \int \implies \ln u = -x^3 + K \implies u = ce^{-x^3}$$

$$\text{Particular: } \frac{du}{dx} + 3x^2u = 3x^2 / * e^{\int 3x^2dx} = e^{x^3} \implies \frac{du}{dx}e^{x^3} + 3x^2ue^{x^3} = 3x^2e^{x^3}$$

$$\implies \frac{d[ue^{x^3}]}{dx} = 3x^2e^{x^3} \implies u = e^{-x^3} \int 3x^2e^{x^3}dx = e^{-x^3}e^{x^3} = 1$$

$$u(x) = ce^{-x^3} + 1 \implies y(x) = [ce^{-x^3} + 1]^{\frac{1}{3}}$$

$$y(0) = 3 = [c + 1]^{\frac{1}{3}} \implies c = 3^3 - 1 = 26$$

$$y(x) = [26e^{-x^3} + 1]^{\frac{1}{3}}$$

$$(b) \text{ Runge-Kutta de Orden 3 para } h = 0.2; y_0 = 3; f(x, y) = \frac{x^2}{y^2} - x^2y$$

i. $k = 0; x_1 = 0.2$

$$q_0^1 = 0.2f(x_0, y_0) = 0.2 \left[\frac{0^2}{3^2} - 0^2 \cdot 3 \right] = 0$$

$$q_0^2 = 0.2f(x_0 + 0.1, y_0 + \frac{q_0^1}{2}) = 0.2 \left[\frac{0.1^2}{3^2} - 0.1^2 \times 3 \right] = -0.2 \times 2.889 \times 10^{-2} = -0.005\,778$$

$$y_0 - q_0^1 + 2q_0^2 = 3 - 0 - 2 \times 0.005\,778 = 2.988$$

$$q_0^3 = 0.2f(x_0 + 0.2, y_0 - q_0^1 + 2q_0^2) = 0.2 \left[\frac{0.2^2}{(2.988)^2} - 0.2^2 \times 2.988 \right] = -0.2 \times 0.115\,0 = -0.023$$

$$y_1 = y_0 + \frac{1}{6}(q_0^1 + 4q_0^2 + q_0^3) = 3 + \frac{1}{6}(0 + 4 \times (-0.005\,778) - 0.023) = 2.992$$

ii. $k = 1; x_2 = 0.4$

$$q_1^1 = 0.2f(x_1, y_1) = 0.2 \left[\frac{0.2^2}{2.992^2} - 0.2^2 \times 2.992 \right] = -0.2 \times 0.1152 = -0.02304$$

$$y_1 + \frac{q_1^1}{2} = 2.992 - \frac{0.02304}{2} = 2.98$$

$$q_1^2 = 0.2f(x_1 + 0.1, y_1 + \frac{q_1^1}{2}) = 0.2 \left[\frac{0.3^2}{2.98^2} - 0.3^2 \times 2.98 \right] = -0.2 \times 0.2581 = -0.05162$$

$$y_1 - q_1^1 + 2q_1^2 = 2.992 + 0.02304 - 2 \times 0.05162 = 2.912$$

$$q_1^3 = 0.2f(x_1 + 0.2, y_1 - q_1^1 + 2q_1^2) = 0.2 \left[\frac{0.4^2}{2.912^2} - 0.4^2 \times 2.912 \right] = -0.2 \times 0.4471 = -0.08942$$

$$y_2 = y_1 + \frac{1}{6}(q_1^1 + 4q_1^2 + q_1^3) = 2.992 + \frac{1}{6}(-0.02304 - 4 \times 0.05162 - 0.08942) = 2.939$$

iii. $k = 2; x_3 = 0.6$

$$q_2^1 = 0.2f(x_2, y_2) = 0.2 \left[\frac{0.4^2}{2.939^2} - 0.4^2 \times 2.939 \right] = -0.2 \times 0.4517 = -0.09034$$

$$y_2 + \frac{q_2^1}{2} = 2.939 - \frac{0.09034}{2} = 2.894$$

$$q_2^2 = 0.2f(x_2 + 0.1, y_2 + \frac{q_2^1}{2}) = 0.2 \left[\frac{0.5^2}{2.894^2} - 0.5^2 \times 2.894 \right] = -0.2 \times 0.6937 = -0.1387$$

$$y_2 - q_2^1 + 2q_2^2 = 2.939 + 0.09034 - 2 \times 0.1387 = 2.752$$

$$q_2^3 = 0.2f(x_2 + 0.2, y_2 - q_2^1 + 2q_2^2) = 0.2 \left[\frac{0.6^2}{2.752^2} - 0.6^2 \times 2.752 \right] = -0.2 \times 0.9432 = -0.1886$$

$$y_3 = y_2 + \frac{1}{6}(q_2^1 + 4q_2^2 + q_2^3) = 2.939 + \frac{1}{6}(-0.09034 - 4 \times 0.1387 - 0.1886) = 2.8$$

iv. $k = 3; x_4 = 0.8$

$$q_3^1 = 0.2f(x_3, y_3) = 0.2 \left[\frac{0.6^2}{2.8^2} - 0.6^2 \times 2.8 \right] = -0.2 \times 0.9621 = -0.1924$$

$$y_3 + \frac{q_3^1}{2} = 2.8 - \frac{0.1924}{2} = 2.704$$

$$q_3^2 = 0.2f(x_3 + 0.1, y_3 + \frac{q_3^1}{2}) = 0.2 \left[\frac{0.7^2}{2.704^2} - 0.7^2 \times 2.704 \right] = -0.2 \times 1.258 = -0.2516$$

$$y_3 - q_3^1 + 2q_3^2 = 2.8 + 0.1924 - 2 \times 0.2516 = 2.489$$

$$q_3^3 = 0.2f(x_3 + 0.2, y_3 - q_3^1 + 2q_3^2) = 0.2 \left[\frac{0.8^2}{2.489^2} - 0.8^2 \times 2.489 \right] = -0.2 \times 1.49 = -0.298$$

$$y_4 = y_3 + \frac{1}{6}(q_3^1 + 4q_3^2 + q_3^3) = 2.8 + \frac{1}{6}(-0.1924 - 4 \times 0.2516 - 0.298) = 2.551$$

v. $k = 4; x_5 = 1$

$$q_4^1 = 0.2f(x_4, y_4) = 0.2 \left[\frac{0.8^2}{2.551^2} - 0.8^2 \times 2.551 \right] = -0.2 \times 1.534 = -0.3068$$

$$y_4 + \frac{q_4^1}{2} = 2.551 - \frac{0.3068}{2} = 2.398$$

$$q_4^2 = 0.2f(x_4 + 0.1, y_4 + \frac{q_4^1}{2}) = 0.2 \left[\frac{0.9^2}{2.398^2} - 0.9^2 \times 2.398 \right] = -0.2 \times 1.802 = -0.3604$$

$$y_4 - q_4^1 + 2q_4^2 = 2.551 + 0.3068 - 2 \times 0.3604 = 2.137$$

$$q_4^3 = 0.2f(x_4 + 0.2, y_4 - q_4^1 + 2q_4^2) = 0.2 \left[\frac{1^2}{2.137^2} - 1^2 \times 2.137 \right] = -0.2 \times 1.918 = -0.3836$$

$$y_5 = y_4 + \frac{1}{6}(q_4^1 + 4q_4^2 + q_4^3) = 2.551 + \frac{1}{6}(-0.3068 - 4 \times 0.3604 - 0.3836) = 2.196$$

Precisión: $y(x) = \left[26e^{-x^3} + 1 \right]^{\frac{1}{3}}$

x	Valor Exacto	Valor R-K	Error Absoluto
0	$y(0) = 3.0$	$y_0 = 3.0$	$E_0 = 3 - 3 = 0$
0.2	$y(0.2) = 2.992$	$y_1 = 2.992$	$E_1 = 2.992 - 2.992 = 0$
0.4	$y(0.4) = 2.939$	$y_2 = 2.939$	$E_2 = 2.939 - 2.939 = 0$
0.6	$y(0.6) = 2.8$	$y_3 = 2.8$	$E_3 = 2.8 - 2.8 = 0$
0.8	$y(0.8) = 2.55$	$y_4 = 2.551$	$E_4 = 2.551 - 2.55 = 0.001$
1.0	$y(1) = 2.194$	$y_5 = 2.196$	$E_5 = 2.196 - 2.194 = 0.002$

$\max E_k = 0.002 \implies$ el método tiene 3 cifras significativas exactas de 4, y los errores son pequeños