Pauta control 3 Matemáticas Aplicadas MA26B-02 Prim'07

Problema 2 Encuentre la solución $u = u(x,t)$ de la ecuación

$$u_{tt} - u_{xx} = 0 \qquad t>0 \;,\; 0 \le x \le 1$$

bajo las condiciones

$$\begin{cases} u(t,0) = u(t,1) = 0 & t>0 \quad (c1) \\ u(x,0) = \begin{cases} x & 0 \le x \le 1/2 \\ 1-x & 1/2 \le x \le 1 \end{cases} & (c2) \\ u_t(0,x) = 3\,\mathrm{sen}(2\pi x) & 0 \le x \le 1 \quad (c3) \end{cases}$$

Sol Aplicamos separación de variables: $u(x,t) = X(x)\,T(t)$. La ecuación queda

$$X(x)\,T''(t) - T(t)\,X''(x) = 0$$

$t>0$, $0 \le x \le 1$, la ( )' denota la derivada c/r a la variable de la función.

Donde $X(x), T(t) \neq 0$, dividimos por $X(x)T(t)$

$$\Rightarrow \frac{T''(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0 \iff \underbrace{\frac{T''(t)}{T(t)}}_{\text{Depende de }(t)} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{Depende de }(x)}$$

$$\Rightarrow \frac{T''(t)}{T(t)} = -k = \frac{X''(x)}{X(x)} \qquad k \in \mathbb{R}$$

La ecuación queda

$$\begin{cases} T''(t) = -k\,T(t) & (1) \\ X''(x) = -k\,X(x) & (2) \end{cases}$$

$\Rightarrow$ Si $k=0$, $\quad T(t) = at+b \;;\; X(x) = cx+d$

De la (c1) $\quad u(t,0) = (at+b)\,d = 0 \quad \forall t \Rightarrow d=0$

$\quad u(t,1) = (at+b)\underbrace{(c+d)}_{0} = 0 \quad \forall t \Rightarrow c=0$

$\Rightarrow$ la solución es la nula.

$\Rightarrow$ si $k \neq 0$,

$$\begin{cases} T(t) = A\cos(\sqrt{k}\,t) + B\,\mathrm{sen}(\sqrt{k}\,t) \\ X(x) = C\cos(\sqrt{k}\,x) + D\,\mathrm{sen}(\sqrt{k}\,x) \end{cases}$$

$$\Rightarrow u(x,t) = \left(A\cos(\sqrt{k}\,t) + B\,\mathrm{sen}(\sqrt{k}\,t)\right)\left(C\cos(\sqrt{k}\,x) + D\,\mathrm{sen}(\sqrt{k}\,x)\right)$$

Imponiendo (c1), $u(t,0) = T(t)\cdot C = 0 \quad \forall t$

$$\Rightarrow C = 0$$

$$u(t,1) = T(t)\cdot D\,\mathrm{sen}(\sqrt{k}) = 0 \quad \forall t$$

$$\Rightarrow \cancel{D=0} \quad \vee \quad \mathrm{sen}(\sqrt{k}) = 0$$

$$\Rightarrow \sqrt{k} = n\pi \qquad n \in \mathbb{Z}.$$

Entonces, $k = (n\pi)^2 \in \mathbb{R}_+$ (o sea k es positivo, pues si $k<0$, se tiene la solución nula)

Basta tomar $n \in \mathbb{N}$.

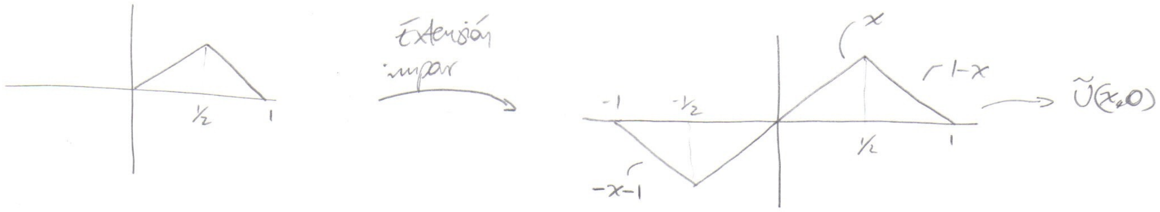
$$\Rightarrow \quad U_n(t,x) = T_n(t)\, X_n(x) = \underbrace{\left(A_n \cos(n\pi t) + B_n \operatorname{sen}(n\pi t)\right)}_{T_n(t)} \cdot \underbrace{D_n \operatorname{sen}(n\pi x)}_{X_n(x)}$$

La solución general es

$$U(t,x) = \sum_{n\geq 1} U_n(t,x) = \sum_{n\geq 1} \tilde{A}_n \cos(n\pi t) \operatorname{sen}(n\pi x) + \tilde{B}_n \operatorname{sen}(n\pi t) \operatorname{sen}(n\pi x)$$

$\tilde{A}_n = A_n D_n$, $\tilde{B}_n = B_n \cdot D_n$.

De la (2), se tiene $U(x,0) = \begin{cases} x & 0 \leq x \leq 1/2 \\ 1-x & 1/2 \leq x \leq 1 \end{cases}$, que se ve de la forma



Se pasa a la extensión impar para poder calcular la serie de Fourier de $U(x,0)$

$$S_{\tilde{U}(0,\cdot)}(x) = a_0 + \sum_{k\geq 1} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \operatorname{sen}\left(\frac{k\pi x}{L}\right), \quad \text{con } L=1$$

como $\tilde{U}(0,\cdot)$ es impar, $a_k = 0 \ \forall k$

y $$b_k = \frac{1}{1}\int_{-1}^{1} \tilde{U}(y,0) \operatorname{sen}\left(\frac{k\pi}{1} y\right) dy$$

$$= \underbrace{\int_{-1}^{-1/2} -(y+1) \operatorname{sen}(k\pi y)\, dy}_{I_{1k}} + \underbrace{\int_{-1/2}^{1/2} y \operatorname{sen}(k\pi y)\, dy}_{I_{2k}} + \underbrace{\int_{1/2}^{1} (1-y) \operatorname{sen}(k\pi y)\, dy}_{I_{3k}}$$

• $I_{1k} = \int_{y=-1}^{-1/2} (-y-1) \operatorname{sen}(k\pi y)\, dy \overset{(CV)}{=} \int_{w=1}^{1/2} (w-1) \operatorname{sen}(k\pi(-w))(-dw)$ y seno es impar

CV: $y=-w \Rightarrow dy=-dw$
$-y = w$

$$= +\int_{w=1}^{1/2} (w-1) \operatorname{sen}(k\pi w)\, dw = \int_{w=1/2}^{1} (1-w) \operatorname{sen}(k\pi w)\, dw$$

$$= \int_{1/2}^{1} \operatorname{sen}(k\pi w)\, dw - \int_{1/2}^{1} w \operatorname{sen}(k\pi w)\, dw$$

$u = w \quad du = dw$
$dv = \operatorname{sen}(k\pi w)\, dw \Rightarrow v = \frac{-\cos(k\pi w)}{k\pi}$

$$= \int_{1/2}^{1} \operatorname{sen}(k\pi w)\, dw - \left(-\frac{w \cos(k\pi w)}{k\pi}\Big|_{1/2}^{1} + \int_{1/2}^{1} \frac{\cos(k\pi w)}{k\pi}\, dw \right)$$

$$= -\frac{\cos(k\pi w)}{k\pi}\Big|_{1/2}^{1} + \left(\frac{\cos(k\pi)}{k\pi} - \frac{1}{2}\cos\left(\frac{k\pi}{2}\right) \cdot \frac{1}{k\pi} \right) - \frac{1}{(k\pi)^2} \operatorname{sen}(k\pi w)\Big|_{1/2}^{1}$$

$$= -\left(\frac{\cancel{\cos(k\pi\cdot 1)}}{\cancel{k\pi}} - \frac{\cos(k\pi/2)}{k\pi}\right) + \left(\frac{\cancel{\cos(k\pi)}}{\cancel{k\pi}} - \frac{\cos(k\pi/2)}{2k\pi}\right) - \frac{1}{(k\pi)^2}\Big(\underbrace{\sin(k\pi)}_{0} - \sin\left(\frac{k\pi}{2}\right)\Big)$$

$$= \cos\left(\frac{k\pi}{2}\right)\left(\frac{1}{k\pi} - \frac{1}{2k\pi}\right) - \frac{1}{(k\pi)^2}\,\sin\left(\frac{k\pi}{2}\right)$$

$$= \frac{1}{2k\pi}\cos\left(\frac{k\pi}{2}\right) - \frac{1}{(k\pi)^2}\,\sin\left(\frac{k\pi}{2}\right)$$

Si k par, $\quad I_{1k} = \frac{1}{2k\pi}\cos\left(\frac{k\pi}{2}\right) \overset{k=2n}{=} \frac{1}{4n\pi}\cos(n\pi) = \frac{1}{4n\pi}(-1)^n$

Si k impar, $\quad I_{1k} = -\frac{1}{(k\pi)^2}\sin\left(\frac{k\pi}{2}\right) \overset{k=2n-1}{=} -\frac{1}{(2n-1)^2\pi^2}\sin\left(n\pi - \frac{\pi}{2}\right) = -\frac{1}{(2n-1)^2\pi^2}(-1)^{n-1}$

$$= \frac{1}{(2n-1)^2\pi^2}(-1)^n$$

- $I_{1k} = I_{3k}$
- $I_{2k} = \int_{-1/2}^{1/2} y\,\sin(k\pi y)\,dy = \underbrace{-y\frac{\cos(k\pi y)}{k\pi}\Big|_{-1/2}^{1/2}}_{0} + \int_{-1/2}^{1/2}\frac{\cos(k\pi y)}{k\pi}\,dy$

$$= \frac{1}{(k\pi)^2}\,\sin(k\pi y)\Big|_{-1/2}^{1/2} = \frac{1}{(k\pi)^2}\left(\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right)$$

$$= \frac{2}{(k\pi)^2}\,\sin\left(\frac{k\pi}{2}\right)$$

Si k impar, $\quad I_{2k} = \frac{2}{(2n-1)^2\pi^2}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{2}{(2n-1)^2\pi^2}(-1)^{n-1}$

Si k par, $\quad I_{2k} = 0$

$$\Rightarrow \quad b_k = \begin{cases} \frac{1}{2n\pi}(-1)^n = \frac{1}{k\pi}(-1)^{k/2} & k = 2n \\[2ex] \frac{2}{(2n-1)^2\pi^2}(-1)^n + \frac{2}{(2n-1)^2\pi^2}(-1)^{n-1} = 0 & k = 2n-1 \end{cases}$$

$$\Rightarrow \quad S_{\tilde{U}(\cdot,0)}(x) = \sum_{n\geq 1}\frac{1}{2n\pi}(-1)^n\,\sin(2n\pi x) \qquad (\tilde{U}(\cdot,0) \text{ es diferenciable en } x=0).$$

Luego, $\quad \tilde{U}(x,0) = \sum_{n\geq 1}\tilde{A}_n\,\underbrace{\cos(n\pi 0)}_{1}\,\sin(n\pi x) + \tilde{B}_n\,\underbrace{\sin(n\pi 0)}_{0}\,\sin(n\pi x) = S_{\tilde{U}(0,\cdot)}(x)$

$$= \sum_{\substack{n\geq 1\\ n \text{ pares}}}\tilde{A}_n\sin(n\pi x) + \sum_{\substack{n\geq 1\\ n \text{ impares}}}\tilde{A}_n\sin(n\pi x) = \sum_{\substack{k\geq 1\\ k \text{ pares}}}\frac{1}{k\pi}(-1)^{k/2}\sin(k\pi x)$$

$$\Rightarrow \begin{cases} \tilde{A}_n = 0 & \text{si } n \text{ impar} \\ \tilde{A}_n = \frac{1}{n\pi}(-1)^{n/2} & \text{si } n \text{ par} \end{cases}$$

$$\Rightarrow U(t,x) = \sum_{\substack{n\geq 1 \\ n \text{ par}}} \frac{1}{n\pi}(-1)^{n/2} \cos(n\pi t)\,\mathrm{sen}(n\pi x) + \sum_{n\geq 1} \tilde{B}_n \,\mathrm{sen}(n\pi t)\,\mathrm{sen}(n\pi x)$$

Entonces $U_t(0,x) = \sum_{\substack{n\geq 1 \\ n \text{ par}}} \frac{-1}{n\pi}(-1)^{n/2} n\pi \,\underbrace{\mathrm{sen}(n\pi 0)}_{0}\,\mathrm{sen}(n\pi x) + \sum_{n\geq 1} \tilde{B}_n \underbrace{\cos(n\pi 0)}_{1} \cdot n\pi \cdot \mathrm{sen}(n\pi x)$

$$\Rightarrow 3\,\mathrm{sen}(2\pi x) = \sum_{n\geq 1} \tilde{B}_n \cdot n\pi\, \mathrm{sen}(n\pi x)$$

$\Rightarrow \cdot)\ \tilde{B}_n = 0 \quad \forall n \neq 2$

$\cdot)\ 3 = \tilde{B}_2 \cdot 2\pi \quad \Rightarrow \boxed{\tilde{B}_2 = \frac{3}{2\pi}}$

$$\Rightarrow U(t,x) = \sum_{\substack{n\geq 1 \\ n \text{ par}}} \frac{1}{n\pi}(-1)^{n/2} \cos(n\pi t)\,\mathrm{sen}(n\pi x) + \frac{3}{2\pi}\,\mathrm{sen}(2\pi t)\,\mathrm{sen}(2\pi x).$$

$$* \left\{ U(t,x) = \frac{3}{2\pi}\,\mathrm{sen}(2\pi t)\,\mathrm{sen}(2\pi x) + \sum_{n\geq 1} \frac{1}{2n\pi}(-1)^n \cos(2n\pi t)\,\mathrm{sen}(2n\pi x) \right\} *$$

Obs Para calcular los coeficientes de Fourier, se puede hacer

$\tilde{U}(0,x) = \sum_{n\geq 1} \tilde{A}_n \underbrace{\cos(n\pi 0)}_{1}\,\mathrm{sen}(n\pi x) + \tilde{B}_n \underbrace{\mathrm{sen}(n\pi 0)}_{0}\,\mathrm{sen}(n\pi x) = \tilde{f}(x)$, $\tilde{f}$ la extensión impar de (C2).

$\sum_{n\geq 1} \tilde{A}_n \,\mathrm{sen}(n\pi x) = \tilde{f}(x) \quad /\cdot\ \mathrm{sen}(k\pi x) \quad / \int_{-1}^{1}(\ ) \quad$ / cambiar $\int$ con Σ

$$\underbrace{\sum_{n\geq 1} \tilde{A}_n \int_{-1}^{1} \mathrm{sen}(n\pi x)\,\mathrm{sen}(k\pi x)\,dx}_{\text{solo queda } n=k} = \int_{-1}^{1} \tilde{f}(x)\,\mathrm{sen}(k\pi x)\,dx$$

$$\tilde{A}_k \cdot 1 = \int_{-1}^{1} \tilde{f}(x)\,\mathrm{sen}(k\pi x)\,dx$$

$\Rightarrow \boxed{\tilde{A}_k = \int_{-1}^{1} \tilde{f}(x)\,\mathrm{sen}(k\pi x)\,dx}$ y calcular la integral

$= 2\int_0^1 f(x)\,\mathrm{sen}(k\pi x)\,dx$ por ser $\tilde{f}$ impar, sen impar.

$\Rightarrow \tilde{f} \cdot \mathrm{sen}(k\pi \cdot)$ par.