Pauta Control 2, Matemáticas Aplicadas.

Pregunta 1 $\lambda \in \mathbb{C}$, $p^\lambda : \mathbb{C}\setminus\{0\} \to \mathbb{C}$

$$z \mapsto p^\lambda(z) = \exp(\lambda \log(z)) \quad , z \neq 0.$$

i) Verificar que para $k \in \mathbb{Z}$, $p^k(z) = z^k$. Mostrar que $\overline{p^\lambda(\bar{z})} = p^{\bar{\lambda}}(z)$ $\forall \lambda \in \mathbb{C}$.

Sol Primera forma: Inducción en $k \in \mathbb{N}$

- $k=1$) $p^1(z) = \exp(\log(z)) = z = z^1$ //
- $k \Rightarrow k+1$) $p^{k+1}(z) = \exp((k+1)\log(z)) = \exp(k\log(z) + \log(z))$

$$= \exp(k\log(z))\exp(\log(z))$$

$$= p^k(z)\cdot z \overset{H.I.}{=} z^k \cdot z = z^{k+1}$$

- Para k negativo, si $n = -k$, $n \geq 0$

$$p^n(z) = z^n$$

$$(p^n(z))^{-1} = z^{-n} = z^k$$

pero $(p^n(z))^{-1} = (\exp(n\log(z)))^{-1} = \exp(-n\log(z)) = \exp(k\log(z))$

$$= p^k(z) \quad //$$

Segunda forma: De la definición de p^λ, si $\lambda = k$

$$p^k(z) = \exp(k\log(z)) \overset{(*)}{=} (\exp(\log(z)))^k = z^k.$$ si tiene sentido si $k \in \mathbb{Z}$.

$\exp(z_1+z_2) = \exp(z_1)\exp(z_2)$

$\Rightarrow \exp(kz) = \exp(\underbrace{z+\dots+z}_{k\text{ veces}}) = (\exp(z))^k$

Para lo de los conjugados, basta notar que $\overline{\exp(z)} = \exp(\bar{z})$

$$\exp(z) = e^x(\cos(y) + i\,\mathrm{sen}(y)) \qquad \text{si } x+iy = z, \; x,y \in \mathbb{R}$$

$$= e^x\cos(y) + i e^x \mathrm{sen}(y)$$

$$\Rightarrow \overline{\exp(z)} = e^x\cos(y) - i e^x\mathrm{sen}(y)$$

$$= e^x\cos(y) + i e^x \mathrm{sen}(-y)$$

$$= e^x \operatorname{cis}(-y) = e^{x-iy} = \exp(\bar{z})$$

Luego, $\log(\bar{z}) = \log(\overline{\exp(w)}) = \log(\exp(\bar{w})) = \bar{w} = \overline{\log(z)}$

$z = \exp(w)$

$\log(z) = w$

Entonces, $\overline{p^{\lambda}(\overline{z})} = \overline{\exp(\lambda \log(\overline{z}))} = \exp(\overline{\lambda}\ \overline{\log(\overline{z})})$

$= \exp(\overline{\lambda} \log(z)) = p^{\overline{\lambda}}(z).$

ii) $\lambda, \mu \in \mathbb{C}$, verificar que $p^{\lambda+\mu}(z) = p^{\lambda}(z)\, p^{\mu}(z)$. Determinar dominio donde p^{λ} es holomorfa y pruebe que $(p^{\lambda})' = \lambda p^{\lambda-1}$.

Sol

$$p^{\lambda+\mu}(z) = \exp((\lambda+\mu)\log(z)) = \exp(\lambda\log(z) + \mu\log(z))$$

$$= \exp(\lambda \log(z))\, \exp(\mu \log(z))$$

$$= p^{\lambda}(z) \cdot p^{\mu}(z).$$

El dominio de analiticidad de exp es $\mathbb{C}$, y el dominio donde log es holomorfa es D,

$D = \mathbb{C} \setminus (\{\text{eje real negativo}\} \cup \{0\})$.

Así, Dom Holomorfa $(p^{\lambda}) = \mathbb{C} \cap D = D$. (por la composición de funciones).

Además, si $z \in D$,

$$(p^{\lambda})'(z) = \frac{d}{dz}(\exp(\lambda \log(z))) = \exp'(\lambda \log(z)) \cdot \frac{d}{dz}(\lambda \log(z))$$

$$= \exp(\lambda \log(z)) \cdot \lambda \cdot \frac{1}{z} \qquad (\exp' = \exp) \text{ y } (\log'(z) = z^{-1}).$$

$$= \lambda p^{\lambda}(z) \cdot z^{-1} = \lambda p^{\lambda}(z) \cdot p^{-1}(z) \qquad (\text{de la parte (i)})$$

$$= \lambda p^{\lambda-1}(z) \qquad (\text{por lo anteriormente hecho})$$

iii) Si $\alpha, \beta \in \mathbb{R}$, y $t>0$, entonces $t^{\alpha+i\beta} = t^{\alpha}(\cos(\beta\log(t)) + i \operatorname{sen}(\beta \log(t)))$.
Además, $i^{i} = e^{-\pi/2}$.

Sol

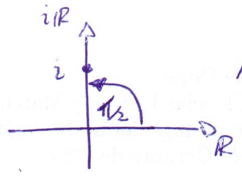
$$t^{\alpha+i\beta} = p^{\alpha+i\beta}(t) = \exp((\alpha+i\beta)\log(t)) = \exp(\underbrace{\alpha\log(t)}_{\in\mathbb{R}} + i\underbrace{\beta\log(t)}_{\in\mathbb{R}})$$

$$= \exp(\alpha \log(t))\, \exp(i\beta\log(t))$$

$$= e^{\alpha\log(t)}(\cos(\beta\log(t)) + i\operatorname{sen}(\beta\log(t)))$$

$$= t^{\alpha}(\cos(\beta\log(t)) + i \operatorname{sen}(\beta\log(t))) \qquad t^{\alpha} \text{ en el sentido real.}$$

$$
\begin{aligned}
i^i &= \exp(i \log(i)) \\
&= \exp\left(i\left(\ln(|i|) + i \operatorname{Arg}(i)\right)\right) \\
&= \exp\left(i \underbrace{\cancel{\ln(1)}}_{0} + i^2 \operatorname{Arg}(i)\right) \\
&= \exp(-\pi/2) = e^{-\pi/2} \quad //
\end{aligned}
$$



$\operatorname{Arg}(i) = \frac{\pi}{2}$