

Pauta Control 2 Matemáticas Aplicadas.

Problema $\lambda \in \mathbb{C}$, $p^\lambda: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$
 $z \mapsto p^\lambda(z) = \exp(\lambda \log(z))$, $z \neq 0$.

i) Verificar que para $k \in \mathbb{Z}$, $p^k(z) = z^k$. Mostrar que $\overline{p^\lambda(z)} = p^{\bar{\lambda}}(\bar{z}) \quad \forall \lambda \in \mathbb{C}$.

Sol Primera forma: Inducción en $k \in \mathbb{N}$

$$\Rightarrow \underbrace{k=1}_{\text{Pasar a } k+1} \quad p^1(z) = \exp(\log(z)) = z = z^1 \quad //$$

$$\begin{aligned} \Rightarrow \underbrace{k=k+1}_{\text{Pasar a } k+1} \quad p^{k+1}(z) &= \exp((k+1)\log(z)) = \exp(k \log(z) + \log(z)) \\ &= \exp(k \log(z)) \exp(\log(z)) \\ &= p^k(z) \cdot z \stackrel{\text{H.I.}}{=} z^k \cdot z = z^{k+1} \end{aligned}$$

\Rightarrow Para k negativo, si $n = -k$, $n \geq 0$

$$p^{-n}(z) = z^n$$

$$(p^{-n}(z))^{-1} = z^{-n} = z^k$$

$$\begin{aligned} \text{pero } (p^{-n}(z))^{-1} &= (\exp(n \log(z)))^{-1} = \exp(-n \log(z)) = \exp(k \log(z)) \\ &= p^k(z) // \end{aligned}$$

Segunda forma De la definición de p^λ , si $\lambda = k$

$$p^k(z) = \exp(k \log(z)) = (\exp(\log(z)))^k = z^k \quad \text{sí tiene sentido si } k \in \mathbb{Z}.$$

Para los números complejos, basta notar que $\overline{\exp(z)} = \exp(\bar{z})$

$$\begin{aligned} \exp(z_1+z_2) &= \exp(z_1) \exp(z_2) \\ \Rightarrow \exp(kz) &= \exp(z_1+\dots+z_k) = (\exp(z))^k \end{aligned}$$

$$\begin{aligned} \exp(z) &= e^x (\cos(y) + i \sin(y)) \quad \text{si } x+iy = z, \quad x, y \in \mathbb{R} \\ &= e^x \cos(y) + i e^x \sin(y) \end{aligned}$$

$$\begin{aligned} \Rightarrow \overline{\exp(z)} &= e^x \cos(y) - i e^x \sin(y) \\ &= e^x \cos(y) + i e^x \sin(y) \\ &= e^x \cos(y) = e^{x-y} = \exp(\bar{z}) \end{aligned}$$

$$\text{Luego, } \log(\bar{z}) = \log(\overline{\exp(w)}) - \log(\exp(\bar{w})) = \bar{w} = \overline{\log(z)}$$

$$z = \exp(w)$$

$$\log(z) = w$$

$$\text{Entonces, } \overline{p^\lambda(z)} = \overline{\exp(\lambda \log(z))} = \exp(\bar{\lambda} \overline{\log(z)}) \\ = \exp(\bar{\lambda} \log(\bar{z})) = p^{\bar{\lambda}}(\bar{z}).$$

ii) $\lambda, \mu \in \mathbb{C}$, verificar que $p^{\lambda+\mu}(z) = p^\lambda(z) p^\mu(z)$. Determinar dominio donde p^λ es holomorfa y probar que $(p^\lambda)' = \lambda p^{\lambda-1}$.

$$\begin{aligned} \text{Sol} \quad p^{\lambda+\mu}(z) &= \exp((\lambda+\mu) \log(z)) = \exp(\lambda \log(z) + \mu \log(z)) \\ &= \exp(\lambda \log(z)) \exp(\mu \log(z)) \\ &= p^\lambda(z) \cdot p^\mu(z). \end{aligned}$$

El dominio de analiticidad de \exp es \mathbb{C} , y el dominio donde \log es holomorfa es D , $D = \mathbb{C} \setminus (\{\text{eje real negativo}\} \cup \{0\})$.

Así, Dom.Holomorfa(p^λ) = $\mathbb{C} \cap D = D$. (por la composición de funciones).

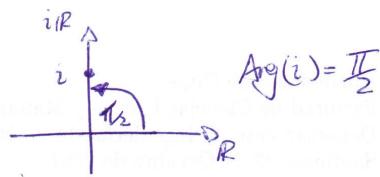
Además, si $z \in D$,

$$\begin{aligned} (p^\lambda)'(z) &= \frac{d}{dz} \left(\exp(\lambda \log(z)) \right) = \exp'(\lambda \log(z)) \cdot \frac{d}{dz}(\lambda \log(z)) \\ &= \exp(\lambda \log(z)) \cdot \lambda \cdot \frac{1}{z} \quad (\exp' = \exp \text{ y } (\log'(z) = z^{-1})) \\ &= \lambda p^\lambda(z) \cdot z^{-1} = \lambda p^\lambda(z) \cdot p^{-1}(z) \quad (\text{de la parte (i)}) \\ &= \lambda p^{\lambda-1}(z) \quad (\text{por lo anteriormente hecho}) \end{aligned}$$

iii) Si $\alpha, \beta \in \mathbb{R}$, y $t > 0$, entonces $t^{\alpha+i\beta} = t^\alpha (\cos(\beta \log(t)) + i \sin(\beta \log(t)))$.
Además, $i^i = e^{-\pi/2}$.

$$\begin{aligned} \text{Sol} \quad t^{\alpha+i\beta} &= p^{\alpha+i\beta}(t) = \exp((\alpha+i\beta) \log(t)) = \exp(\underbrace{\alpha \log(t)}_{\in \mathbb{R}} + i \underbrace{\beta \log(t)}_{\in \mathbb{R}}) \\ &= \exp(\alpha \log(t)) \exp(i \beta \log(t)) \\ &= e^{\alpha \log(t)} (\cos(\beta \log(t)) + i \sin(\beta \log(t))) \\ &= t^\alpha (\cos(\beta \log(t)) + i \sin(\beta \log(t))) \quad t^\alpha \text{ en el sentido real.} \end{aligned}$$

$$\begin{aligned}
 i^i &= \exp(i \log(i)) \\
 &= \exp(i(\ln(|i|) + i \arg(i))) \\
 &= \exp(i \ln\sqrt{2} + i^2 \operatorname{Arg}(i)) \\
 &= \exp(-\frac{\pi}{2}) = e^{-\frac{\pi}{2}} //
 \end{aligned}$$



• Derivative of $\ln(z)$ is $\frac{1}{z}$ (for $z \neq 0$)

• Impact of changing initial point: if we change the starting point of the path, the final value will change by a constant multiple of $2\pi i$. This is because the derivative of the function is constant, so the difference in the final value is proportional to the length of the path.

• If we go around a closed curve, we get back to where we started.

• $\ln(z)$ is not defined for $z = 0$

• $\ln(z)$ is not differentiable at $z = 0$

• $\ln(z)$ is not continuous at $z = 0$ (it has a jump discontinuity)

$\ln(z)$ is not analytic at $z = 0$ (it has a singularity)

• $\ln(z)$ is not bounded near $z = 0$ (it goes to negative infinity)

• $\ln(z)$ is not bounded near $z = \infty$ (it goes to positive infinity)

• $\ln(z)$ is not differentiable at $z = \infty$

• $\ln(z)$ is not continuous at $z = \infty$

$\ln(z)$ is not analytic at $z = \infty$

• $\ln(z)$ is not bounded near $z = \infty$

• $\ln(z)$ is not bounded near $z = -\infty$ (it goes to negative infinity)

• $\ln(z)$ is not bounded near $z = \infty$ (it goes to positive infinity)

• $\ln(z)$ is not differentiable at $z = -\infty$

• $\ln(z)$ is not continuous at $z = -\infty$

$\ln(z)$ is not analytic at $z = -\infty$

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$\ln(z)$ is not analytic at $z = \infty$