2.2 Laplace transform

Let us first state a few important points about the application of Laplace transform in solving differential equations (Fig. 2.1). After we have formulated a model in terms of a *linear or linearized* differential equation, dy/dt = f(y), we can solve for y(t). Alternatively, we can transform the equation into an algebraic problem as represented by the function G(s) in the Laplace domain and solve for Y(s). The time domain solution y(t) can be obtained with an inverse transform, but we rarely do so in control analysis.

What we argue (of course it is true) is that the Laplace-domain function Y(s) must contain the same information as y(t). Likewise, the function G(s) contains the same dynamic information as the original differential equation. We will see that the function G(s) can be "clean" looking if the differential equation has zero initial conditions. That is one of the reasons why we always pitch a control problem in terms of deviation variables.[1] We can now introduce the definition.

The Laplace transform of a function *f*(*t*) is defined as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt \tag{2-4}$$

where *s* is the transform variable.[2] To complete our definition, we have the inverse transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)\, e^{st}\, ds \tag{2-5}$$

where γ is chosen such that the infinite integral can converge.[3] Do not be intimidated by (2-5). In a control class, we never use the inverse transform definition. Our approach is quite simple. We construct a table of the Laplace transform of some common functions, and we use it to do the inverse transform using a look-up table.

An important property of the Laplace transform is that it is a *linear* operator, and contribution of individual terms can simply be added together (superimposed):

$$\mathcal{L}[a\, f_1(t) + b\, f_2(t)] = a\,\mathcal{L}[f_1(t)] + b\,\mathcal{L}[f_2(t)] = aF_1(s) + bF_2(s) \tag{2-6}$$

> **Note:**
> The linear property is one very important reason why we can do partial fractions and inverse transform using a look-up table. This is also how we analyze more complex, but linearized, systems. Even though a text may not state this property explicitly, we rely heavily on it in classical control.

We now review the Laplace transform of some common functions—mainly the ones that we come across frequently in control problems. We do not need to know all possibilities. We can consult a handbook or a mathematics textbook if the need arises. (A summary of the important ones is in Table 2.1.) Generally, it helps a great deal if you can do the following common ones

---

1 But! What we measure in an experiment is the "real" variable. We have to be careful when we solve a problem which provides real data.

2 There are many acceptable notations of Laplace transform. We choose to use a capitalized letter, and where confusion may arise, we further add (*s*) explicitly to the notation.

3 If you insist on knowing the details, they can be found on our *Web Support*.

without having to look up a table. The same applies to simple algebra such as partial fractions and calculus such as linearizing a function.

1. **A constant**

$$f(t) = a, \qquad F(s) = \frac{a}{s} \tag{2-7}$$

The derivation is:

$$\mathcal{L}[a] = a\int_0^\infty e^{-st}\,dt = -\frac{a}{s}e^{-st}\Big|_0^\infty = a\left[0+\frac{1}{s}\right] = \frac{a}{s}$$

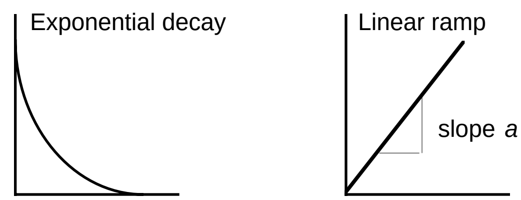

Figure 2.2. Illustration of exponential and ramp functions.

2. **An exponential function** (Fig. 2.2)

$$f(t) = e^{-at} \text{ with } a > 0, \qquad F(s) = \frac{1}{(s+a)} \tag{2-9}$$

$$\mathcal{L}[e^{-at}] = a\int_0^\infty e^{-at}\,e^{-st}\,dt = \frac{-1}{(s+a)}\,e^{-(a+s)t}\Big|_0^\infty = \frac{1}{(s+a)}$$

3. **A ramp function** (Fig. 2.2)

$$f(t) = at \ \text{ for } t \ge 0 \text{ and } a = \text{constant}, \qquad F(s) = \frac{a}{s^2} \tag{2-8}$$

$$\mathcal{L}[at] = a\int_0^\infty t\,e^{-st}\,dt = a\left[-t\,\frac{1}{s}\,e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}\,e^{-st}\,dt\right] = \frac{a}{s}\int_0^\infty e^{-st}\,dt = \frac{a}{s^2}$$

4. **Sinusoidal functions**

$$f(t) = \sin\omega t, \qquad F(s) = \frac{\omega}{(s^2+\omega^2)} \tag{2-10}$$

$$f(t) = \cos\omega t, \qquad F(s) = \frac{s}{(s^2+\omega^2)} \tag{2-11}$$

We make use of the fact that $\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$ and the result with an exponential function to derive

$$\mathcal{L}[\sin\omega t] = \frac{1}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})\,e^{-st}\,dt = \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}\,dt - \int_0^\infty e^{-(s+j\omega)t}\,dt\right]$$
$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

The Laplace transform of cosωt is left as an exercise in the Review Problems. If you need a review on complex variables, our *Web Support* has a brief summary.

---

5. **Sinusoidal function with exponential decay**

$$f(t) = e^{-at}\sin\omega t, \qquad F(s) = \frac{\omega}{(s+a)^2 + \omega^2} \tag{2-12}$$

Making use of previous results with the exponential and sine functions, we can pretty much do this one by inspection. First, we put the two exponential terms together inside the integral:

$$\int_0^\infty \sin\omega t\, e^{-(s+a)t}\,dt = \frac{1}{2j}\left[\int_0^\infty e^{-(s+a-j\omega)t}\,dt - \int_0^\infty e^{-(s+a+j\omega)t}\,dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right]$$

The similarity to the result of $\sin\omega t$ should be apparent now, if it was not the case with the LHS.

---

6. **First order derivative**, df/dt,

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \tag{2-13}$$

and the **second order derivative**,

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0) \tag{2-14}$$

We have to use integration by parts here,

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}\,e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt = -f(0) + sF(s)$$

and

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \int_0^\infty \frac{d}{dt}\left(\frac{df}{dt}\right)e^{-st}\,dt = \frac{df}{dt}e^{-st}\Big|_0^\infty + s\int_0^\infty \frac{df}{dt}\,e^{-st}\,dt = -\frac{df}{dt}\Big|_0 + s\left[sF(s) - f(0)\right]$$

We can extend these results to find the Laplace transform of higher order derivatives. The key is that if we use deviation variables in the problem formulation, all the initial value terms will drop out in Eqs. (2-13) and (2-14). This is how we can get these “clean-looking” transfer functions later.

---

7. **An integral**,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2-15}$$

We also need integration by parts here

$$\int_0^\infty\left[\int_0^t f(t)\,dt\right]e^{-st}\,dt = -\frac{1}{s}e^{-st}\int_0^t f(t)\,dt\,\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt = \frac{F(s)}{s}$$

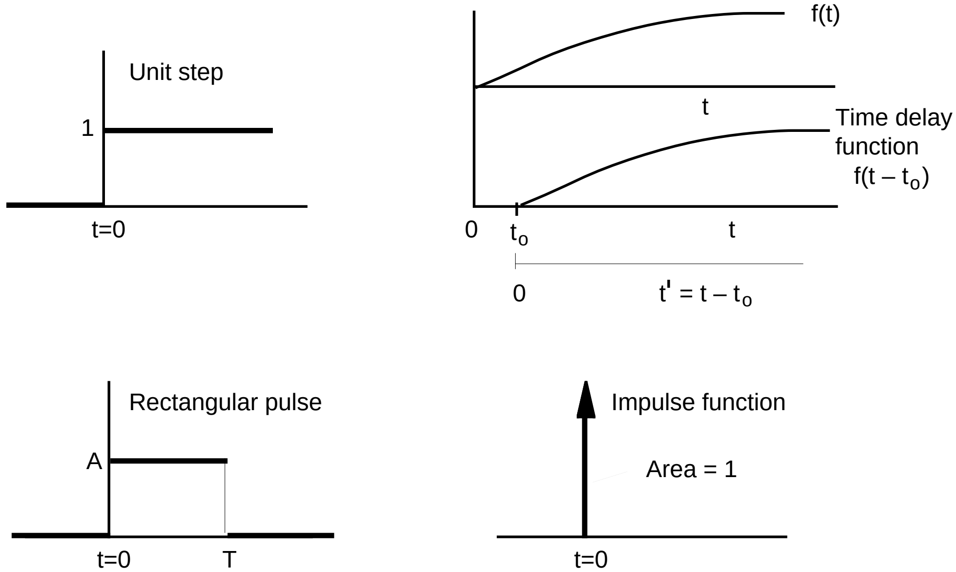

Figure 2.3. Depiction of unit step, time delay, rectangular, and impulse functions.

2.3 Laplace transforms common to control problems

We now derive the Laplace transform of functions common in control analysis.

1. **Step function**

$$f(t) = Au(t), \qquad F(s) = \frac{A}{s} \tag{2-16}$$

We first define the **unit step function** (also called the Heaviside function in mathematics) and its Laplace transform:[1]

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}; \qquad \mathcal{L}[u(t)] = U(s) = \frac{1}{s} \tag{2-17}$$

The Laplace transform of the unit step function (Fig. 2.3) is derived as follows:

$$\mathcal{L}[u(t)] = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} u(t)\, e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt = \frac{-1}{s}\, e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

With the result for the unit step, we can see the results of the Laplace transform of any step function f(t) = Au(t).

$$f(t) = A\,u(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases}; \qquad \mathcal{L}[Au(t)] = \frac{A}{s}$$

The Laplace transform of a step function is essentially the same as that of a constant in (2-7). When you do the inverse transform of A/s, which function you choose depends on the context of the problem. Generally, a constant is appropriate under most circumstances.

---

1 Strictly speaking, the step function is discontinuous at t = 0, but many engineering texts ignore it and simply write u(t) = 1 for t ≥ 0.

2. **Dead time function** (Fig. 2.3)

$f(t - t_o)$, $\qquad \mathcal{L}[f(t - t_o)] = e^{-st_o}F(s)$ (2-18)

The dead time function is also called the **time delay**, **transport lag**, translated, or time shift function (Fig. 2.3). It is defined such that an original function f(t) is "shifted" in time t_o, and no matter what f(t) is, its value is set to zero for $t < t_o$. This time delay function can be written as:

$$f(t - t_o) = \begin{cases} 0 & , t - t_o < 0 \\ f(t - t_o) & , t - t_o > 0 \end{cases} = f(t - t_o)\, u(t - t_o)$$

The second form on the far right is a more concise way to say that the time delay function $f(t - t_o)$ is defined such that it is zero for $t < t_o$. We can now derive the Laplace transform.

$$\mathcal{L}[f(t - t_o)] = \int_0^\infty f(t - t_o)\, u(t - t_o)\, e^{-st}\, dt = \int_{t_o}^\infty f(t - t_o)\, e^{-st}\, dt$$

and finally,

$$\int_{t_o}^\infty f(t - t_o)\, e^{-st}\, dt = e^{-st_o}\int_{t_o}^\infty f(t - t_o)\, e^{-s(t - t_o)}\, d(t - t_o) = e^{-st_o}\int_0^\infty f(t')\, e^{-st'}\, dt' = e^{-st_o}F(s)$$

where the final integration step uses the time shifted axis $t' = t - t_o$.

3. **Rectangular pulse function** (Fig. 2.3)

$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 < t < T \\ 0 & t > T \end{cases} = A[u(t) - u(t - T)] \;, \qquad \mathcal{L}[f(t)] = \frac{A}{s}\left[1 - e^{-sT}\right] \qquad \text{(2-19)}$$

The rectangular pulse can be generated by subtracting a step function with dead time T from a step function. We can derive the Laplace transform using the formal definition

$$\mathcal{L}[f(t] = \int_0^\infty f(t)\, e^{-st}\, dt = A\int_{0^+}^T e^{-st}\, dt = A\,\frac{-1}{s}\, e^{-st}\Big|_0^T = \frac{A}{s}\left[1 - e^{-sT}\right]$$

or better yet, by making use of the results of a step function and a dead time function

$$\mathcal{L}[f(t] = \mathcal{L}[A\, u(t) - A\, u(t - T)] = \frac{A}{s} - e^{-sT}\frac{A}{s}$$

4. **Unit rectangular pulse function**

$$f(t) = \begin{cases} 0 & t < 0 \\ 1/T & 0 < t < T \\ 0 & t > T \end{cases} = \frac{1}{T}[u(t) - u(t - T)] \;, \qquad \mathcal{L}[f(t)] = \frac{1}{sT}\left[1 - e^{-sT}\right] \qquad \text{(2-20)}$$

This is a prelude to the important impulse function. We can define a rectangular pulse such that the area is unity. The Laplace transform follows that of a rectangular pulse function

$$\mathcal{L}[f(t] = \mathcal{L}\left[\frac{1}{T}\, u(t) - \frac{1}{T}\, u(t - T)\right] = \frac{1}{T\,s}\left[1 - e^{-sT}\right]$$

5. **Impulse function** (Fig. 2.3)

$$\mathcal{L}[\delta(t)] = 1, \text{ and } \mathcal{L}[A\delta(t)] = A \tag{2-21}$$

The (unit) impulse function is called the **Dirac** (or simply delta) function in mathematics.[1] If we suddenly dump a bucket of water into a bigger tank, the impulse function is how we describe the action mathematically. We can consider the impulse function as the unit rectangular function in Eq. (2-20) as T shrinks to zero while the height 1/T goes to infinity:

$$\delta(t) = \lim_{T \to 0} \frac{1}{T}[u(t) - u(t - T)]$$

The area of this "squeezed rectangle" nevertheless remains at unity:

$$\lim_{T \to 0} (T \frac{1}{T}) = 1 \text{, or in other words } \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

The impulse function is rarely defined in the conventional sense, but rather via its important property in an integral:

$$\int_{-\infty}^{\infty} f(t)\, \delta(t)\, dt = f(0) \text{, and } \int_{-\infty}^{\infty} f(t)\, \delta(t - t_o)\, dt = f(t_o) \tag{2-22}$$

The Laplace transform of the impulse function is obtained easily by taking the limit of the unit rectangular function transform (2-20) with the use of L'Hospital's rule:

$$\mathcal{L}[\delta(t] = \lim_{T \to 0} \frac{1 - e^{-sT}}{T\,s} = \lim_{T \to 0} \frac{s\,e^{-sT}}{s} = 1$$

From this result, it is obvious that $\mathcal{L}[A\delta(t)] = A$.

2.4 Initial and final value theorems

We now present two theorems which can be used to find the values of the time-domain function at two extremes, $t = 0$ and $t = \infty$, without having to do the inverse transform. In control, we use the final value theorem quite often. The initial value theorem is less useful. As we have seen from our very first example in Section 2.1, the problems that we solve are defined to have exclusively zero initial conditions.

Initial Value Theorem: $$\lim_{s \to \infty} [sF(s)] = \lim_{t \to 0} f(t) \tag{2-23}$$

Final Value Theorem: $$\lim_{s \to 0} [sF(s)] = \lim_{t \to \infty} f(t) \tag{2-24}$$

The final value theorem is valid provided that a final value exists. The proofs of these theorems are straightforward. We will do the one for the final value theorem. The proof of the initial value theorem is in the Review Problems.

Consider the definition of the Laplace transform of a derivative. If we take the limit as s approaches zero, we find

---

[1] In mathematics, the unit rectangular function is defined with a height of 1/2T and a width of 2T from –T to T. We simply begin at t = 0 in control problems. Furthermore, the impulse function is the time derivative of the unit step function.

$$\lim_{s \to 0} \int_0^{\infty} \frac{d f(t)}{dt} e^{-st} \, dt = \lim_{s \to 0} \left[s\, F(s) - f(0) \right]$$

If the infinite integral exists,[1] we can interchange the limit and the integration on the left to give

$$\int_0^{\infty} \lim_{s \to 0} \frac{d f(t)}{dt} e^{-st} \, dt = \int_0^{\infty} d f(t) = f(\infty) - f(0)$$

Now if we equate the right hand sides of the previous two steps, we have

$$f(\infty) - f(0) = \lim_{s \to 0} \left[s\, F(s) - f(0) \right]$$

We arrive at the final value theorem after we cancel the f(0) terms on both sides.

✎ **Example 2.1**: Consider the Laplace transform $F(s) = \dfrac{6\,(s-2)\,(s+2)}{s\,(s+1)\,(s+3)\,(s+4)}$. What is $f(t=\infty)$?

$$\lim_{s \to 0} s \frac{6\,(s-2)\,(s+2)}{s\,(s+1)\,(s+3)\,(s+4)} = \frac{6\,(-2)\,(2)}{(3)\,(4)} = -2$$

✎ **Example 2.2**: Consider the Laplace transform $F(s) = \dfrac{1}{(s-2)}$. What is $f(t=\infty)$?

Here, $f(t) = e^{2t}$. There is no upper bound for this function, which is in violation of the existence of a final value. The final value theorem does not apply. If we insist on applying the theorem, we will get a value of zero, which is meaningless.

✎ **Example 2.3**: Consider the Laplace transform $F(s) = \dfrac{6\,(s^2-4)}{(s^3+s^2-4s-4)}$. What is $f(t=\infty)$?

Yes, another trick question. If we apply the final value theorem without thinking, we would get a value of 0, but this is meaningless. With MATLAB, we can use

```
roots([1 1 -4 -4])
```

to find that the polynomial in the denominator has roots –1, –2, and +2. This implies that f(t) contains the term e^{2t}, which increases without bound.

As we move on, we will learn to associate the time exponential terms to the roots of the polynomial in the denominator. From these examples, we can gather that to have a meaningful, *i.e.*, finite bounded value, the roots of the polynomial in the denominator must have *negative real parts*. This is the basis of stability, which will formerly be defined in Chapter 7.

---

[1] This is a key assumption and explains why Examples 2.2 and 2.3 do not work. When a function has no bound—what we call unstable later—the assumption is invalid.

2.5 Partial fraction expansion

Since we rely on a look-up table to do reverse Laplace transform, we need the skill to reduce a complex function down to simpler parts that match our table. In theory, we should be able to "break up" a ratio of two polynomials in *s* into simpler partial fractions. If the polynomial in the denominator, p(s), is of an order higher than the numerator, q(s), we can derive [1]

$$F(s) = \frac{q(s)}{p(s)} = \frac{\alpha_1}{(s + a_1)} + \frac{\alpha_2}{(s + a_2)} + \ldots \frac{\alpha_i}{(s + a_i)} + \ldots \frac{\alpha_n}{(s + a_n)} \tag{2-25}$$

where the order of p(s) is n, and the a_i are the negative values of the roots of the equation p(s) = 0. We then perform the inverse transform term by term:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{\alpha_1}{(s + a_1)}\right] + \mathcal{L}^{-1}\left[\frac{\alpha_2}{(s + a_2)}\right] + \ldots \mathcal{L}^{-1}\left[\frac{\alpha_i}{(s + a_i)}\right] + \ldots \mathcal{L}^{-1}\left[\frac{\alpha_n}{(s + a_n)}\right] \tag{2-26}$$

This approach works because of the linear property of Laplace transform.

The next question is how to find the partial fractions in Eq. (2-25). One of the techniques is the so-called **Heaviside expansion**, a fairly straightforward algebraic method. We will illustrate three important cases with respect to the roots of the polynomial in the denominator: (1) distinct real roots, (2) complex conjugate roots, and (3) multiple (or repeated) roots. In a given problem, we can have a combination of any of the above. Yes, we need to know how to do them all.

2.5.1 Case 1: p(s) has distinct, real roots

✎ **Example 2.4**: Find f(t) of the Laplace transform $F(s) = \frac{6s^2 - 12}{(s^3 + s^2 - 4s - 4)}$.

From Example 2.3, the polynomial in the denominator has roots –1, –2, and +2, values that will be referred to as poles later. We should be able to write F(s) as

$$\frac{6s^2 - 12}{(s + 1)\,(s + 2)\,(s - 2)} = \frac{\alpha_1}{(s + 1)} + \frac{\alpha_2}{(s + 2)} + \frac{\alpha_3}{(s - 2)}$$

The Heaviside expansion takes the following idea. Say if we multiply both sides by (s + 1), we obtain

$$\frac{6s^2 - 12}{(s + 2)\,(s - 2)} = \alpha_1 + \frac{\alpha_2}{(s + 2)}\,(s + 1) + \frac{\alpha_3}{(s - 2)}\,(s + 1)$$

which should be satisfied by any value of s. Now if we choose s = –1, we should obtain

$$\alpha_1 = \left.\frac{6s^2 - 12}{(s + 2)\,(s - 2)}\right|_{s = -1} = 2$$

Similarly, we can multiply the original fraction by (s + 2) and (s – 2), respectively, to find

$$\alpha_2 = \left.\frac{6s^2 - 12}{(s + 1)\,(s - 2)}\right|_{s = -2} = 3$$

and

---

[1] If the order of q(s) is higher, we need first carry out "long division" until we are left with a partial fraction "residue." Thus the coefficients α_i are also called residues. We then expand this partial fraction. We would encounter such a situation only in a mathematical problem. The models of real physical processes lead to problems with a higher order denominator.

$$\alpha_3 = \left.\frac{6s^2 - 12}{(s+1)(s+2)}\right|_{s=2} = 1$$

Hence, $F(s) = \frac{2}{(s+1)} + \frac{3}{(s+2)} + \frac{1}{(s-2)}$, and using a look-up table would give us

$$f(t) = 2e^{-t} + 3e^{-2t} + e^{2t}$$

When you use MATLAB to solve this problem, be careful when you interpret the results. The computer is useless unless we know what we are doing. We provide only the necessary statements.[1] For this example, all we need is:

```
[a,b,k]=residue([6 0 -12],[1 1 -4 -4])
```

✎ **Example 2.5**: Find f(t) of the Laplace transform $F(s) = \frac{6s}{(s^3 + s^2 - 4s - 4)}$.

Again, the expansion should take the form

$$\frac{6s}{(s+1)(s+2)(s-2)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+2)} + \frac{\alpha_3}{(s-2)}$$

One more time, for each term, we multiply the denominators on the right hand side and set the resulting equation to its root to obtain

$$\alpha_1 = \left.\frac{6s}{(s+2)(s-2)}\right|_{s=-1} = 2\,,\ \alpha_2 = \left.\frac{6s}{(s+1)(s-2)}\right|_{s=-2} = -3\ \text{, and } \alpha_3 = \left.\frac{6s}{(s+1)(s+2)}\right|_{s=2} = 1$$

The time domain function is

$$f(t) = 2e^{-t} - 3e^{-2t} + e^{2t}$$

Note that f(t) has the identical functional dependence in time as in the first example. Only the coefficients (residues) are different.

The MATLAB statement for this example is:

```
[a,b,k]=residue([6 0],[1 1 -4 -4])
```

✎ **Example 2.6**: Find f(t) of the Laplace transform $F(s) = \frac{6}{(s+1)(s+2)(s+3)}$.

This time, we should find

$$\alpha_1 = \left.\frac{6}{(s+2)(s+3)}\right|_{s=-1} = 3\ ,\ \alpha_2 = \left.\frac{6}{(s+1)(s+3)}\right|_{s=-2} = -6\ ,\ \alpha_3 = \left.\frac{6}{(s+1)(s+2)}\right|_{s=-3} = 3$$

The time domain function is

---

[1] Starting from here on, it is important that you go over the MATLAB sessions. Explanation of `residue()` is in Session 2. While we do not print the computer results, they can be found on our *Web Support*.

$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}$$

The e^{-2t} and e^{-3t} terms will decay faster than the e^{-t} term. We consider the e^{-t} term, or the pole at s = –1, as more dominant.

We can confirm the result with the following MATLAB statements:

```
p=poly([-1 -2 -3]);
[a,b,k]=residue(6,p)
```

Note:

(1) The time dependence of the time domain solution is derived entirely from the roots of the polynomial in the denominator (what we will refer to later as the **poles**). The polynomial in the numerator affects only the coefficients α_i. This is one reason why we make qualitative assessment of the dynamic response characteristics entirely based on the poles of the characteristic polynomial.

(2) Poles that are closer to the origin of the complex plane will have corresponding exponential functions that decay more slowly in time. We consider these poles more dominant.

(3) We can generalize the Heaviside expansion into the fancy form for the coefficients

$$\alpha_i = (s + a_i)\left.\frac{q(s)}{p(s)}\right|_{s = -a_i}$$

but we should always remember the simple algebra that we have gone through in the examples above.

2.5.2 Case 2: p(s) has complex roots [1]

Example 2.7: Find f(t) of the Laplace transform $F(s) = \dfrac{s+5}{s^2 + 4s + 13}$.

We first take the painful route just so we better understand the results from MATLAB. If we have to do the chore by hand, we much prefer the completing the perfect square method in Example 2.8. Even without MATLAB, we can easily find that the roots of the polynomial $s^2 + 4s + 13$ are $-2 \pm 3j$, and F(s) can be written as the sum of

$$\frac{s+5}{s^2 + 4s + 13} = \frac{s+5}{[s - (-2+3j)]\,[s - (-2-3j)]} = \frac{\alpha}{s - (-2+3j)} + \frac{\alpha^*}{s - (-2-3j)}$$

We can apply the same idea formally as before to find

$$\alpha = \left.\frac{s+5}{[s - (-2-3j)]}\right|_{s = -2+3j} = \frac{(-2+3j)+5}{(-2+3j)+2+3j} = \frac{(j+1)}{2j} = \frac{1}{2}(1-j)$$

and its complex conjugate is

$$\alpha^* = \frac{1}{2}(1+j)$$

The inverse transform is hence

---

1 If you need a review of complex variable definitions, see our *Web Support*. Many steps in Example 2.7 require these definitions.

$$f(t) = \frac{1}{2}(1-j)\,e^{(-2+3j)t} + \frac{1}{2}(1+j)\,e^{(-2-3j)t}$$
$$= \frac{1}{2}e^{-2t}\left[(1-j)\,e^{j\,3t} + (1+j)\,e^{-j\,3t}\right]$$

We can apply Euler's identity to the result:

$$f(t) = \frac{1}{2}e^{-2t}\left[(1-j)(\cos 3t + j\sin 3t) + (1+j)(\cos 3t - j\sin 3t)\right]$$
$$= \frac{1}{2}e^{-2t}\left[2(\cos 3t + \sin 3t)\right]$$

which we further rewrite as

$$f(t) = \sqrt{2}\,e^{-2t}\sin(3t+\phi) \quad \text{where} \quad \phi = \tan^{-1}(1) = \pi/4 \text{ or } 45°$$

The MATLAB statement for this example is simply:

```
[a,b,k]=residue([1 5],[1 4 13])
```

Note:

(1) Again, the time dependence of f(t) is affected only by the roots of p(s). For the general complex conjugate roots $-a \pm bj$, the time domain function involves e^{-at} and (cos bt + sin bt). The polynomial in the numerator affects only the constant coefficients.

(2) We seldom use the form (cos bt + sin bt). Instead, we use the phase lag form as in the final step of Example 2.7.

✎ **Example 2.8**: Repeat Example 2.7 using a look-up table.

In practice, we seldom do the partial fraction expansion of a pair of complex roots. Instead, we rearrange the polynomial p(s) by noting that we can complete the squares:

$$s^2 + 4s + 13 = (s+2)^2 + 9 = (s+2)^2 + 3^2$$

We then write F(s) as

$$F(s) = \frac{s+5}{s^2+4s+13} = \frac{(s+2)}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

With a Laplace transform table, we find

$$f(t) = e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

which is the answer with very little work. Compared with how messy the partial fraction was in Example 2.7, this example also suggests that we want to leave terms with conjugate complex roots as one second order term.

2.5.3 Case 3: p(s) has repeated roots

✎ **Example 2.9**: Find f(t) of the Laplace transform $F(s) = \frac{2}{(s+1)^3(s+2)}$.

The polynomial p(s) has the roots –1 repeated three times, and –2. To keep the numerator of each partial fraction a simple constant, we will have to expand to

$$\frac{2}{(s+1)^3(s+2)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3} + \frac{\alpha_4}{(s+2)}$$

To find α_3 and α_4 is routine:

$$\alpha_3 = \left.\frac{2}{(s+2)}\right|_{s=-1} = 2\,, \text{ and } \quad \alpha_4 = \left.\frac{2}{(s+1)^3}\right|_{s=-2} = -2$$

The problem is with finding α_1 and α_2. We see that, say, if we multiply the equation with (s+1) to find α_1, we cannot select s = –1. What we can try is to multiply the expansion with $(s+1)^3$

$$\frac{2}{(s+2)} = \alpha_1(s+1)^2 + \alpha_2(s+1) + \alpha_3 + \frac{\alpha_4(s+1)^3}{(s+2)}$$

and then differentiate this equation with respect to s:

$$\frac{-2}{(s+2)^2} = 2\,\alpha_1(s+1) + \alpha_2 + 0 + [\alpha_4 \text{ terms with } (s+1)]$$

Now we can substitute s = –1 which provides $\alpha_2 = -2$.

We can be lazy with the last α_4 term because we know its derivative will contain (s + 1) terms and they will drop out as soon as we set s = –1. To find α_1, we differentiate the equation one more time to obtain

$$\frac{4}{(s+2)^3} = 2\,\alpha_1 + 0 + 0 + [\alpha_4 \text{ terms with } (s+1)]$$

which of course will yield $\alpha_1 = 2$ if we select s = –1. Hence, we have

$$\frac{2}{(s+1)^3(s+2)} = \frac{2}{(s+1)} + \frac{-2}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{-2}{(s+2)}$$

and the inverse transform via table-lookup is

$$f(t) = 2\left[\left(1 - t + \frac{t^2}{2}\right)e^{-t} - e^{-2t}\right]$$

We can also arrive at the same result by expanding the entire algebraic expression, but that actually takes more work(!) and we will leave this exercise in the Review Problems.

The MATLAB command for this example is:

```
p=poly([-1 -1 -1 -2]);
[a,b,k]=residue(2,p)
```

Note:

In general, the inverse transform of repeated roots takes the form

$$\mathcal{L}^{-1}\left[\frac{\alpha_1}{(s+a)} + \frac{\alpha_2}{(s+a)^2} + \ldots \quad \frac{\alpha_n}{(s+a)^n}\right] = \left[\alpha_1 + \alpha_2 t + \frac{\alpha_3}{2!}t^2 + \ldots \quad \frac{\alpha_n}{(n-1)!}t^{n-1}\right]e^{-at}$$

The exponential function is still based on the root s = –a, but the actual time dependence will decay slower because of the ($\alpha_2 t$ + …) terms.