A THEORY OF THE ECONOMICS OF TIME[1]

THIS study represents a theory of consumer behaviour, specifically designed to handle economic problems wherein a time dimension is relevant. Over the years there have been a number of attempts to modify neoclassical consumer theory to deal with problems of this nature, but none of these works achieved the level of sophistication of the traditional approach, whereby testable properties of demand functions are deduced. In part, this has been intentional. These studies were more concerned with specific problems such as the decision to work more or fewer hours [19], the effect of foregone earnings upon consumer choice [1] and the valuation of travel time [10, 17], rather than with the general properties of demand functions. More importantly, however, the restrictions on demand functions derived from neoclassical theory cannot be derived from existing theories of the time dimension in consumer choice. This is due not to any property intrinsic to the time dimension but to the fact that these theories are improperly specified.

The essential features of the model presented in this paper are: (1) utility is a function not only of commodities but also of the time allocated to them; (2) the individual's decision is subject to two resource constraints, a money constraint and a time constraint; and (3) the decision to consume a specified amount of any commodity requires that some minimum amount of time be allocated to it, but the individual may spend more time in that activity if he so desires. Under these specifications, all the implications of neoclassical theory are preserved and many additional results, applicable to situations involving a time dimension, are generated.

I. THE MODEL

Neoclassical consumer theory analyses individual preferences among alternative commodity bundles, $X = (X_1, \ldots, X_n)$, within an attainable set defined by the individual's income and a set of parametric money prices. Once a time dimension is introduced, the field of choice expands considerably: commodities might be consumed one at a time, or concurrently, or pure time might be consumed independently of consumer goods, etc. For simplicity, we shall consider only the case in which goods are consumed one at a time and all the time available to the individual is spent in the consumption of some commodity.

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Consider a set of commodity bundles, $X = (X_1, \ldots, X_n, T_1, \ldots, T_n)$, in the positive orthant of a Euclidean space of dimension $2n$. The variable, X_i, denotes some quantity of the ith consumption good, while T_i denotes the amount of time allocated to the ith good. Assume the individual possesses a complete, consistent preference ordering among alternative commodity bundles and exhibits rational behaviour. That is, from an attainable set of commodity bundles, $Z(Z \in X)$, the individual will select a bundle, X^* $(X^* \in Z)$, such that $X^* RX^i$ for all $X^i \in Z$, where R is a binary relation which reads " is preferred or indifferent to." The attainable set is defined by the constraints, to be specified below. The individual's preferences are assumed to be representable by a continuous, twice-differentiable real valued utility function, $U(X)$.

Following neoclassical consumer theory, we assume that the consumer receives a fixed positive amount of money income (Y), and chooses to spend all of it on consumption goods during the time period under consideration. Thus, total expenditures on consumption goods must be equal to the money endowment:

$$Y = \sum_{i=1}^{n} P_i X_i, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.1)$$

where $P_i \geqslant 0$ is the money price of the ith consumption good.

Similarly, the individual receives a fixed time endowment (T°) equal to the length of the decision period.[1] Since money income is a flow concept, the time endowment must be consistent with that of income. The time constraint requires that the amounts of time allocated to each specific use add up to the time available:

$$T^\circ = \sum_{i=1}^{n} T_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.2)$$

This relationship follows directly from our assumption that goods are consumed one at a time and that all available time is allocated to consumption. It is important to note that (1.1) and (1.2), as specified, are independent of each other. Each represents a resource constraint, but unlike earlier attempts to describe the consumer's time allocation problem, parametric time prices of the goods are absent from the time constraint. This is in keeping

[1] The assumption of fixed endowments of money and time might appear to preclude the ability of the individual to exchange time for money through the work decision. This interpretation would reflect an institutionally determined work week, wherein T° is redefined as the time endowment net of working hours $(T^\circ = \bar{T} - \bar{W})$ and Y redefined to include wage and non-wage income $(Y = I + \bar{w}W$, where $\bar{w}$ is the individual's wage rate).

However, this need not be the case. The model could be modified to include the work decision, either by defining one of the commodities (X_w) as work $(P_w < 0)$, where $P_w X_w$ augments income and $T_w = a_w X_w$ diminishes the time endowment, or by including a pure time commodity, W, in the utility function and in the two resource constraints. In neither case are the qualitative properties of the model altered. The modifications affect the solution vector only indirectly through the structure of the H matrix, defined below. In short, any attention to the work decision tends to obscure the model's more important properties without adding anything substantial. See [6], pp. 130–4, 139–40.

with our specification of the T_i's as decision variables distinct from the X_i's. The use of time prices would reduce the number of decision variables from $2n$ to n, for the choice of any X_i would determine, by means of the time price, the corresponding T_i. Time prices are excessively rigid and unnecessary.

In reality, the amount of time allocated to the consumption of any commodity is partly a matter of choice and partly a matter of necessity. The fact that the consumption of goods generally requires the expenditure of time as well as money does not place an upper bound upon the amount of time an individual may spend consuming the chosen amount of the good. Thus we shall assume that the choice of a positive amount of any X_i places only a lower bound upon the amount T_i consumed. For simplicity, we shall also assume these relationships to be linear. Mathematically, they take the form of the inequalities

$$T_i \geqslant a_i X_i,\ i = 1, \ldots, n, \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

where a_i may be interpreted as a technologically or institutionally determined minimum amount of time required to consume one unit of X_i. For simplicity, we shall assume that the vector of a_i's, like the vector of prices, is known with certainty.

Henceforth, we shall refer to the n inequalities of (1.3) as the *time consumption constraints*, as distinguished from the *time resource constraint*, equation (1.2). They are specified as inequalities because the individual is free to allocate more than the required amount of time to any activity. Whether the equality is binding or not is a matter of individual preference, although common experience suggests that the constraint will be binding for nearly all individuals in certain activities, due to the nature of these activities. Examples of a binding technological or physical constraint are a round of golf, movies, meals, road congestion, reading a book, etc. Examples of the institutional type of constraint are speed limits, rigid work weeks, banquets, etc. These constraints, be they physical or institutional, must be made explicit in the formal maximisation model; along with (1.1) and (1.2), they define the attainable set of commodity bundles.

A. *First-order Conditions*

The individual's problem of efficiently allocating his time and money resources may be expressed as the maximisation of the Lagrange function,

$$L = U(X_1, \ldots, X_n, T_1, \ldots, T_n) + \lambda(Y - \sum_{i=1}^{n} P_i X_i) + \mu(T^\circ - \sum_{i=1}^{n} T_i) + \sum_{i=1}^{n} K_i(T_i - a_i X_i)$$

where $K_i \geqslant 0$, $i = 1, \ldots, n$, and $\mu, \lambda, > 0$. For simplicity, we shall assume that all commodities and usages of time are consumed in positive amounts, although the model could easily be modified to allow for corner solutions.

Under these specifications, the following conditions, plus (1.1) and (1.2), are necessary for maximisation:

$$U_i = \lambda P_i + K_i a_i, \quad i = 1, \ldots, n, \qquad (1.4)$$

$$U_{n+i} = \mu - K_i, \quad i = 1, \ldots, n, \qquad (1.5)$$

$$K_i(T_i - a_i X_i) = 0, \quad i = 1, \ldots, n; \qquad (1.6)$$

that is, either $T_i = a_i X_i$, or $K_i = 0, \quad i = 1, \ldots, n.$

The subscripts i and $n + i$ denote the partial derivatives of the utility function with respect to X_i and T_i, respectively. The Lagrangian multipliers, λ and μ, are shadow variables representing the marginal utility of money and the marginal utility of time, respectively. The ratio, μ/λ, the marginal rate of substitution between time and money, may be interpreted as the value of time. The generation of this value as an equilibrium condition is an important feature of the model,[1] but at this juncture the effect of the time consumption constraints upon the equilibrium conditions warrants our full attention. If the time consumption constraint is binding, the first-order conditions of traditional consumer theory cease to apply. In goods space, the marginal rate of substitution between two goods is not equal to their price ratios. Consider diagrammatically a two-good case. For expository purposes, we shall assume the individual's preferences are such that the time consumption constraint is binding between X_2 and T_2, but ineffective between X_1 and T_1. Unfortunately, even the two-good case requires a four-dimensional picture. As an alternative, the interaction among the constraints is illustrated in Figs. 1(*a*), (*b*), (*c*) and (*d*) by means of four cross-sectional diagrams. In Fig. 1(*a*), the budget constraint is truncated at point A, the intersection of the budget line and the line, $X_1 = (T°/a_1) - (a_2X_2/a_1)$.[1] That portion of the budget constraint lying above A represents combinations of X_1 and X_2 which are incompatible with the time constraint. The requirement that X_2 and T_2 be consumed in fixed proportions, illustrated in Fig. 1(*b*), causes non-tangency solutions in both Fig. 1(*a*) (goods space) and Fig. 1(*d*) (time space). In equilibrium, the rate of substitution between X_1 and X_2 is less than the price ratio, while the rate of substitution between T_1 and T_2 exceeds unity (the absolute value of the slope of the time constraint). Although a partial equilibrium analysis of Figs. 1(*a*) and 1(*d*) indicates that the consumer could improve his position by substituting X_2 for X_1 and substituting T_1 for T_2, the two substitutions cannot be made concurrently, because of the time consumption constraint. The total equilibrium generates the shadow variables, λ, μ, K_1 and K_2.

The economic interpretation of this equilibrium position reveals some rather interesting aspects of the problem of time valuation. Dividing through (1.5) by λ, we get

$$\frac{U_{n+i}}{\lambda} = \frac{\mu}{\lambda} - \frac{K_i}{\lambda} \qquad (1.7)$$

[1] Johnson [10] and Oort [17] drew the same conclusion from their respective models.

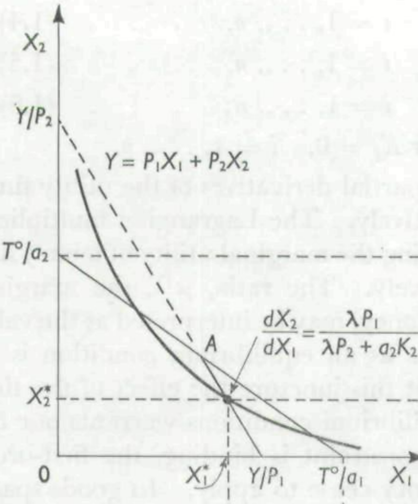

Fig. 1 (*a*)

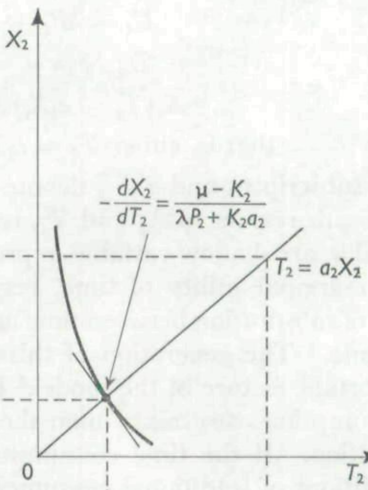

Fig. 1 (*b*)

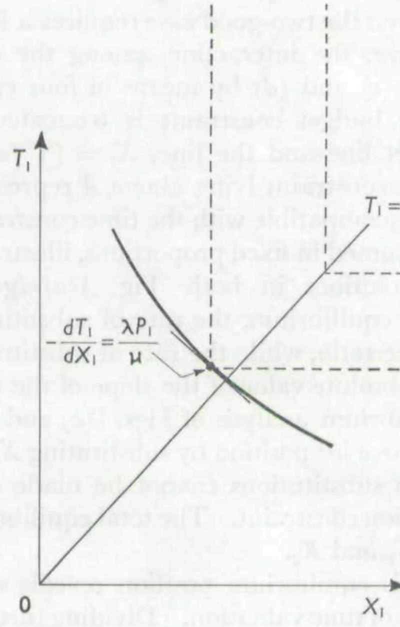

Fig. 1 (*c*)

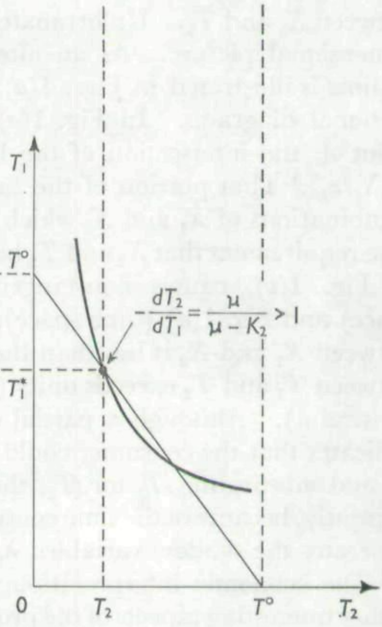

Fig. 1 (*d*)

The term, U_{n+i}/λ, the marginal rate of substitution of T_i for money, represents the value of time allocated to the consumption of X_i. It is the value of time as a *commodity*, not as a resource. The distinction between the two is crucial. They are equal if, and only if, $K_i = 0$. This condition will prevail if the individual elects to spend more than the required amount of time consuming X_i.[1]

B. *The Value of Time and the Value of Saving Time*

Since each of the time consumption constraints represents the requirement to spend time consuming some commodity, relaxing the *i*th time consumption constraint is equivalent to saving time in the *i*th consumption activity. Therefore, K_i may be interpreted as the marginal utility of saving time and the ratio, K_i/λ, the value of saving time. This interpretation acquires greater intuitive appeal in light of the distinction made between the value of time as a resource and the value of time as a commodity. Attributing a positive value to saving time from any activity presupposes that the time saved can be transferred to some alternative usage of greater value. The algebraic difference between the value of time in alternative uses and the value of time in any particular use (U_{n+i}/λ) determines the value of saving time from that activity:

$$\text{Value of Saving Time Consuming } X_i = \frac{\mu}{\lambda} - \frac{U_{n+i}}{\lambda} = \frac{K_i}{\lambda} \qquad (1.8)$$

Only this latter value, the value of saving time, has any empirical content. The value of time as a resource derives from its scarcity; it is an opportunity cost which is positive because the time resource is available in a limited amount. However, it makes little sense to consider the value associated with acquiring more of it, for such an increment is not possible, either under the specifications of this model or in reality, in any meaningful sense. Time saving, on the other hand, is both possible and observable in many facets of human activity. Thus the formal distinction between these two values is a most important feature of our model.

C. *Leisure and the Price of Travel Time*

An earlier study by M. Bruce Johnson [10] suggested that the marginal rate of substitution of time for money, μ/λ, is an appropriate theoretical representation of both the value of leisure and the value of travel time. Although this conclusion apparently contradicts ours, it can be demonstrated that Johnson's conclusion follows from a special case of our model. Let us

[1] The equilibrium conditions do not preclude the possibility that $K_i = 0$ and $T_i = a_i X_i$. Strictly speaking, then, the individual decision to spend more than the required amount of time consuming X_i is sufficient but not necessary for U_{n+i}/λ to equal μ/λ. That is, $T_i > a_i X_i$ implies that $K_i = 0$, but $K_i = 0$ does not imply that $T_i > a_i X_i$.

first consider the concept of leisure within the context of our model. Economists have traditionally defined leisure as the residual of work. This practice has proved useful in the theoretical analysis of work, but its usefulness in the analysis of other theoretical questions pertaining to time has more recently been challenged. Addressing the problem of travel time valuation specifically, David Tipping asks:

> How should one distinguish between, and measure the time which is spent in enjoying the consumption of goods and services . . . and the time which is spent in making such final consumption possible? Perhaps we need a category of " intermediate consumption " to cover such activities as travelling to the theatre, doing the shopping . . . [22, p. 848].

The distinction made by Tipping between " leisure " and " intermediate " goods can be formalised in terms of our general model. Leisure is conventionally defined as *free time* which may be used for rest, recreation, etc. This definition suggests a freedom from responsibility, a specific case of which is freedom from work (hence, the traditional economic definition of leisure as non-work). In terms of our model, the primary responsibility of the time resource is with respect to the consumption of goods. Given the solution vector of the n goods, $(X_1^*, \ldots, X_n^*)$, a minimum amount of time, specified by the parameters, $a_1, \ldots, a_n$, must be allocated to their consumption. Freedom from this responsibility, which inheres in the choice to allocate more time to any particular good than is required, thus constitutes leisure. Thus, those goods for which the time consumption constraint is ineffective, *i.e.*, $K_i = 0$, may be classified as leisure goods, while all other goods may be classified as intermediate goods: $L(i) = \{i \mid K_i = 0\}$, denotes leisure goods; $I(X) = \{i \mid K_i > 0\}$, denotes intermediate goods. All time allocated to leisure goods might thus be defined as leisure time:

$$L = \sum_{i \in L(X)} T_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.9)$$

Thus the concept of leisure is developed from the model. The definition is meaningful in an economic sense and is also consistent with a more philosophical concept of leisure.

Returning to Johnson's results, we find that the equilibrium conditions of our model support the conclusion that the value of leisure time is equal to μ/λ. This follows trivially from (1.7) and (1.9). However, they do not in general support the conclusion that the value of commuting time is equal to μ/λ. This would be the case only if the journey to work were a leisure good, but, of course, it is not. However, Johnson was considering not the value of commuting time, but the value of *saving* commuting time, which we shall henceforth call the *price* of time. Under the specifications of his model, the marginal utility of commuting time must equal zero, for commuting time does not appear in the utility function (see [10], p. 138). Given that the marginal utility of commuting time is equal to zero, (1.7) indicates that the price of commuting time would indeed equal μ/λ, as Johnson suggested. However, assigning a specific value to any partial derivative of the utility function can add nothing to the analysis. Neither the assumption nor its implication is empirically verifiable, for utility is not measurable in any meaningful sense and μ/λ cannot be related to any set of empirical data. On the other hand, a relationship can be derived between the measure, K_i/λ, and empirically observable data. This relationship is but one of many meaningful qualitative results that follow from our specification of the consumer's time allocation problem. It is to these results that we now turn.

II. Comparative Statics and the Demand Function

From our equilibrium conditions, (1.1), (1.2), (1.4), (1.5) and (1.6), the quantity demanded of each commodity may be derived as a function of the parameters of the system:

$$X_i = X_i(P_1, \ldots, P_n, a_1, \ldots, a_n, Y, T^\circ), i = 1, \ldots, n$$

In order to derive refutable qualitative properties of these demand functions, we shall assume, following neoclassical theory, that the consumer is in the neighbourhood of a constrained optimum and consider the effects upon quantity demanded of small changes in the parameters. Because we are considering only the properties of an optimal position, ineffective constraints may be ignored, whereby the total differential of the first-order conditions reduces to:

$$\sum_{j=1}^{n} U_{i\,j} dX_j + \sum_{j=1}^{n} U_{i\,n+j}\, dT_j - P_i d\lambda - a_i dK_i = \lambda dP_i + K_i da_i, \quad i = 1, \ldots, n, \quad . \quad (2.1)$$

$$\sum_{j=1}^{n} U_{n+i\,j} dX_j + \sum_{j=1}^{n} U_{n+i\,n+j} dT_j - d\mu + dK_i = 0,\ i = 1, \ldots, n, \quad . \quad (2.2)$$

$$dY = \sum_{i=1}^{n} P_i dX_i + \sum_{i=1}^{n} X_i dP_i, \quad . \quad . \quad . \quad (2.3)$$

$$\sum_{i=1}^{n} dT_i = dT^\circ = 0, \quad . \quad . \quad . \quad . \quad (2.4)$$

$$-a_i dX_i + dT_i = X_i da_i,\ i \in I(X), \quad . \quad (2.5)$$

where $K_i = dK_i = 0$, for all $i \in L(X)$.[1] In the limiting case in which all

[1] The set of commodities, $\{X_i \mid K_i = 0 \text{ and } T_i = a_i X_i\}$ is assumed to be empty and, further, it is assumed that small changes in the parameters do not alter the composition of the sets $I(X)$ and $L(X)$. This assumption does not appear to be excessively restrictive since, for most applications of the theory, the commodities under analysis could be clearly labelled as leisure or intermediate goods. Strictly speaking, the total differential of (1.6) is

$$-a_i K_i dX_i + K_i dT_i = K_i X_i da_i + (a_i X_i - T_i) dK_i,\ i = 1, \ldots, n. \quad . \quad (2.5')$$

In the case of leisure goods, every term in the equation is equal to zero; in the case of intermediate goods, $T_i - a_i X_i = 0$, whereupon the K_i's cancel out, leaving (2.5).

goods are intermediate goods, this system may be rewritten as $Hy = c$, where

$$
H = \begin{bmatrix}
U_{11} & \cdots & U_{1n} & U_{1n+1} & \cdots & U_{1\,2n} & -P_1 & 0 & -a_1 & & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \\
U_{n1} & \cdots & U_{nn} & U_{nn+1} & \cdots & U_{n\,2n} & -P_n & 0 & 0 & & -a_n \\
U_{n+1\,1} & \cdots & U_{2n\,n} & U_{n+1\,n+1} & \cdots & U_{n+1\,2n} & 0 & -1 & 1 & & 0 \\
\vdots & & \vdots & \vdots & & \vdots & & & & \ddots & \\
U_{2n\,1} & \cdots & U_{n+1\,n} & U_{2n\,n+1} & \cdots & U_{2n\,2n} & 0 & -1 & 0 & & 1 \\
-P_1 & \cdots & -P_n & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & -1 & \cdots & -1 & \vdots & & & & \vdots \\
-a_1 & & 0 & 1 & & 0 & \vdots & & & & \vdots \\
 & \ddots & & & \ddots & & \vdots & & & & \vdots \\
0 & & -a_n & 0 & & 1 & 0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}
$$

$$
y = \begin{bmatrix} dX_1 \\ \vdots \\ dX_n \\ dT_1 \\ \vdots \\ dT_n \\ d\lambda \\ d\mu \\ dK_1 \\ \vdots \\ dK_n \end{bmatrix}_{3n+2\times 1}, \text{ and } c = \begin{bmatrix} \lambda dP_1 + K_1 da_1 \\ \vdots \\ \lambda dP_n + K_n da_n \\ 0 \\ \vdots \\ 0 \\ -dY + \sum_{i=1}^{n} X_i dP_i \\ 0 \\ X_1\, da_1 \\ \vdots \\ X_n\, da_n \end{bmatrix}_{3n+2\times 1}.
$$

In this case, the rank of H is $3n + 2$, although there will normally be leisure goods as well as intermediate goods. In this general case, the $2n + 2 + i$th row and column of H as well as the $2n + 2 + i$th element y and c are deleted, for all $i \in L(X)$, and the rank of H reduces to $2n + 2 + h$, h representing the number of intermediate goods in the neighbourhood of solution.

Using Cramer's rule, this system may be solved for the differentials, dX_i, $i = 1, \ldots, n$, $d\lambda$, and dK_i, $i \in I(X)$.[1] They are as follows:

$$\begin{aligned} dX_i = \lambda \sum_{j=1}^{n} \frac{D_{ji}}{D} dP_j &+ \sum_{j=1}^{n} K_j \frac{D_{ji}}{D} da_j \\ &+ (-dY + \sum_{j=1}^{n} X_j dP_j) \frac{D_{2n+1i}}{D} \\ &+ \sum_{j \in I(X)} X_j \frac{D_{2n+2+ji}}{D} da_j, \; i = 1, \ldots, n \quad . \quad (2.6) \end{aligned}$$

$$\begin{aligned} d\lambda = \sum_{j=1}^{n} (\lambda dP_j + K_j da_j) \frac{D_{j2n+1}}{D} \\ + (-dY + \sum_{j=1}^{n} X_j dP_j) \frac{D_{2n+1\;2n+1}}{D} \\ + \sum_{j \in I(X)} X_j \frac{D_{2n+2+j\;2n+1}}{D} da_j \quad . \quad . \quad . \quad . \quad (2.7) \end{aligned}$$

$$\begin{aligned} dK_i = \sum_{j=1}^{n} (\lambda dP_j + K_j da_j) \frac{D_{j\;2n+2+i}}{D} \\ + (-dY + \sum_{j=1}^{n} X_j dP_j) \frac{D_{2n+1\;2n+2+i}}{D} \\ + \sum_{j \in I(X)} X_j \frac{D_{2n+2+j\;2n+2+j}}{D} da_j, \; i \in I(X) \quad . \quad (2.8) \end{aligned}$$

where D is the determinant of H and D_{rk}, $r, k = 1, \ldots, 2n + 2 + h$ is the minor determinant of H, formed by deleting the rth row and the kth column. As a condition of optimisation, H must be negative semi-definite (see [5], or [13], p. 53 ff.). Thus

$$\frac{D_{rr}}{D} \leqslant 0, \; r = 1, \ldots, 2n \quad . \quad . \quad . \quad (2.9)$$

Moreover, since H is symmetrical,

$$\frac{D_{rk}}{D} = \frac{D_{kr}}{D}, \; r, k = 1, \ldots, 2n + 2 + h \quad . \quad . \quad (2.10)$$

A. *Price and Income Effects, and the Substitution Theorem*

All of the meaningful theorems derived from the neoclassical theory of demand retain their validity despite the fact that a time dimension has been added to the consumer's decision.

It follows directly from (2.6) that

$$\frac{\partial X_i}{\partial P_j} = \lambda \frac{D_{ji}}{D} + X_j \frac{D_{2n+1i}}{D} \quad . \quad . \quad . \quad (2.11)$$

and

$$\frac{D_{2n+1i}}{D} = -\frac{\partial X_i}{\partial Y} \quad . \quad . \quad . \quad . \quad (2.12)$$

[1] The system may also be solved for the differentials, dT_i, $i = 1, \ldots, n$, and $d\mu$, but these variables are not used in the derivation of any theorems. They are, therefore, not included.

Substituting (2.12) into (2.11), and rearranging, we get

$$\frac{\partial X_i}{\partial P_j} + X_j \frac{\partial X_i}{\partial Y} = \lambda \frac{D_{ji}}{D} \quad . \quad . \quad . \quad . \quad (2.13)$$

In the special case where $i = j$, it follows from (2.9) that

$$\frac{\partial X_i}{\partial P_i} + X_i \frac{\partial X_i}{\partial Y} = \lambda \frac{D_{ii}}{D} \leqslant 0 \quad . \quad . \quad . \quad (2.14)$$

This inequality is the well-known Slutsky equation [21], and can be tested empirically. However, the left side of the inequality does not represent the substitution effect, except in the trivial case in which all goods are leisure goods.[1] In light of the fact that the consumer's equilibrium position is no longer characterised by a tangency between an indifference surface and a budget hyperplane, this result should not be surprising. Moreover, strictly from the standpoint of hypothetical restrictions on demand functions, it would not appear to be particularly important, since (2.14) is valid in any case. Nevertheless, it demonstrates the invalidity of applying the substitution theorem to situations in which the time dimension is relevant (see Becker [1], p. 304 ff.).

B. *Homogeneity and Symmetry*

Homogeneity of degree zero in prices and income, as well as the Slutsky symmetry conditions, follow from (2.11). From the theorem of expansion by alien co-factors, it follows that

$$\sum_{i=1}^{n} P_i \frac{D_{ij}}{D} = 0, j = 1, \ldots, n \quad . \quad . \quad . \quad (2.15)$$

By direct substitution of (2.13), this becomes

$$\sum_{i=1}^{n} P_i \left[\frac{\partial X_j}{\partial P_i} + X_i \frac{\partial X_j}{\partial Y}\right] = \sum_{i=1}^{n} P_i \frac{\partial X_j}{\partial P_i} + Y \frac{\partial X_i}{\partial Y} = 0 \quad (2.16)$$

[1] Suppose the price change were accompanied by an income change which kept the individual at the same level of utility. Then

(i) $dU = \sum_{i=1}^{n} U_i dX_i + \sum_{i=1}^{n} U_{n+i} dT_i = 0.$

Substituting (1.4) and (1.5), we get

(ii) $dU = \sum_{i=1}^{n} (\lambda P_i + a_i K_i) dX_i + \sum_{i=1}^{n} (\mu - K_i) dT_i = 0$

$= \lambda \sum_{i=1}^{n} P_i dX_i + \sum_{i=1}^{n} a_i K_i dX_i + \mu \sum_{i=1}^{n} dT_i - \sum_{i=1}^{n} K_i dT_i.$

Substituting (2.3), (2.4) and (2.5′), and dividing through by λ, we get

(iii) $-dY + \sum_{j=1}^{n} X_i dP_i = -\frac{1}{\lambda} \sum_{i=1}^{n} K_i X_i da_i.$

It follows from (2.6) and (2.14) that the term, $\lambda D_{ii}/D$, represents the substitution effect if and only if $-dY + \sum_{i=1}^{n} X_i dP_i$ goes to zero when utility is held constant. But it follows from (iii) that this condition holds only in the trivial case in which all goods are leisure goods.

which, by Euler's theorem on homogeneous functions, is equivalent to $X_i(P_1, \ldots, P_n, Y) = X_i(\beta P_1, \ldots, \beta P_n, \beta Y), i = 1, \ldots, n, \beta > 0$. Equation (2.16) also implies that the sum of the elasticities of any good with respect to each price is equal in absolute value, but opposite in sign to the income elasticity of demand for that good. Dividing through (2.16) by X_j yields this result, which adds another n restrictions on the demand functions.

The symmetry conditions follow from (2.13) and (2.10):

$$\frac{\partial X_i}{\partial P_j} + X_j \frac{\partial X_i}{\partial Y} = \frac{\partial X_j}{\partial P_i} + X_i \frac{\partial X_j}{\partial Y}, i, j = 1, \ldots, n \qquad (2.17)$$

C. *Demand as a Function of Time Costs*

Economic literature is full of examples of demand curves depicting quantity demanded as a function of per unit time costs. This type of demand function may be found in models of highway congestion in which an equilibrium level of road usage critically depends on anticipated time costs [9, 23]. In addition, queuing models have acknowledged that the distribution of demand over the course of a day responds not only to explicit price changes but also to the price changes implicit in the variation of waiting time [15]. This notion has been discussed in the context of the demand for telephone communication [15] and the effect of queuing costs upon the demand for air transport [7].

The type of demand function commonly specified is in terms of own price and own " time price." In our notation,

$$\begin{aligned} X_i &= X_i(\bar{P}_1, \ldots, P_i, \ldots, \bar{P}_n, \bar{a}_1, \ldots, a_i, \ldots, \bar{a}_n, \bar{Y}, T^\circ) \\ &= X_i(P_i, a_i) \end{aligned}$$

Although our model admits of this type of demand function, X_i need not respond to changes in its own time consumption parameter (a_i). In the case of leisure goods, the own time consumption constraint is not binding and hence exerts no influence on the equilibrium solution. Consequently,

$$\frac{\partial X_i}{\partial a_i} = 0, i \in L(X) \quad . \quad . \quad . \quad . \qquad (2.18)$$

More rigorously, it is demonstrated in the appendix that

$$\frac{\partial X_i}{\partial a_i} = K_i \frac{D_{ii}}{D} - X_i \frac{K_i}{\lambda} \frac{\partial X_i}{\partial Y} + X_i \left[\frac{\partial}{\partial P_i} \left(\frac{K_i}{\lambda} \right) + X_i \frac{\partial}{\partial Y} \left(\frac{K_i}{\lambda} \right) \right] \qquad (2.19)$$

Since K_i/λ represents the price of time in the ith activity, $\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right)$ and $\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right)$ are the rates of change of that price with respect to P_i and Y, respectively. Since K_i is constant and equal to zero for all leisure goods, (2.18) immediately follows.

Despite the simplicity of (2.18), it is a rather important feature of our model. For a large majority of consumption activities, individuals do not consider time as part of the price of the commodity being purchased. The fact that our model generates a specific set of commodities whose demand functions are not " time elastic " assumes greater significance in light of this body of casual empirical evidence.[1] More importantly, what does our model enable us to say about the sign of $\partial X_i/\partial a_i$ in the case of intermediate goods, in which a_i may be interpreted as a parametric time price? Unfortunately, very little. The time effect, like the price effect, includes the strictly non-positive term, D_{ii}/D, which is enforced by an income effect so long as the good is not inferior. However, a positive income effect does not ensure a negative time effect, as it does a negative price effect. The term, $\frac{\partial X_i}{\partial a_i} + X_i \frac{K_i}{\lambda} \frac{\partial X_i}{\partial Y}$, might exceed zero due to the variability of the price of time. The model therefore provides little theoretical justification for the practice of drawing time-elastic demand curves with a downward slope, for the empirical significance of $\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right)$ and $\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right)$ cannot be ignored. One study which explicitly considered variations in the price of travel time [14] measured relationships between the price of time and both income and time costs. These relationships were statistically significant at the 95% level.

III. Measurement of the Price of Travel Time

A large majority of empirical works dealing with time valuation use what might be termed a revealed preference approach or a " trade-off " approach. The essence of this method is to identify a situation in which the traveller reveals, usually via questionnaire, a preference between alternatives involving a trade-off between higher (lower) money costs and lower (higher) time costs. Situations which have been used for this purpose involve choices among alternative routes [4], alternative modes of travel [2, 18], and alternative speeds [16]. A preference for a slower, less expensive alternative is interpreted to mean that the respondent values his time at a rate no greater than the measured trade-off, and vice versa. However, the assumptions necessary to justify these conclusions are very restrictive. Non-economic factors which play such an important role in the foundations of demand theory, namely subjective preferences, play no role in this type of analysis, for the only choice criterion is the relative costs of the various alternatives. The importance of subjective preference in this type of decision is suggested in the results of Beesley's study wherein a significant number of " inconsistent " choices were found. Inconsistent choices are defined by Beesley

[1] Throughout this paper, we shall use the term " time-elastic," to indicate that demand is responsive to changes in time costs per unit (changes in the parameter, a_i). It does not refer to any numerical value of the elasticity of demand with respect to a_i.

as those in which the preferred journey required both more time and more money or more of one, the other cost being the same for each journey [11, p. 178, Table 1].

As an alternative, our model suggests that the price of travel time is measurable directly from the demand function for trips. Dividing (2.19) by (2.14) and rearranging terms yields the relation,

$$\frac{K_i}{\lambda} = \frac{\dfrac{\partial X_i}{\partial a_i} - X_i\left[\dfrac{\partial}{\partial P_i}\left(\dfrac{K_i}{\lambda}\right) + X_i \dfrac{\partial}{\partial Y}\left(\dfrac{K_i}{\lambda}\right)\right]}{\dfrac{\partial X_i}{\partial P_i}} \quad . \quad . \quad (3.1)$$

where X_i might represent the demand for trips on a particular route over a specified period of time. If X_i is specified as a function of time costs, own price and income, the terms, $\partial X_i/\partial a_i$ and $\partial X_i/\partial P_i$, are reflected in the regression coefficients of the demand estimating equation. Obtaining estimates of the rates of change of the price of time with respect to income and the money price of the trip is not quite so easy, however.

One way of handling these two terms would be to ignore them on grounds that they would tend to offset each other. Indeed, it would appear that they do act in opposite directions. One would expect, *ceteris paribus*, that an individual would be willing to pay more money for the purpose of saving time, the lower the price he is already paying for the journey and the higher his income. Nevertheless, however reasonable it appears to hypothesise opposite algebraic signs for these two terms, there is no reason to lead us to believe they would offset each other. Such an assumption would be no less arbitrary than assuming them both to be equal to zero.

As an alternative, one could attempt to infer the values of these terms from the demand estimating model. If, for example, the estimating equation were of the general log linear form,

$$\log X_i = \log \alpha_i + \beta_i \log P_i + \gamma_i \log a_i + \delta_i \log Y + u_i \quad . \quad (3.2)$$

the price of time, derived from (3.2), would be

$$\frac{K_i}{\lambda} = \frac{\dfrac{\gamma_i}{a_i} - \left[\dfrac{\partial}{\partial P_i}\left(\dfrac{K_i}{\lambda}\right) + X_i \dfrac{\partial}{\partial Y}\left(\dfrac{K_i}{\lambda}\right)\right]}{\dfrac{\beta_i}{P_i}} \quad . \quad . \quad (3.3)$$

Ignoring second-order effects, this equation may be solved directly for the relevant rates of change:

$$\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right) = \frac{\partial}{\partial P_i}\left[\frac{\gamma_i}{\beta_i}\frac{P_i}{a_i}\right] \quad . \quad . \quad . \quad (3.4)$$

$$\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right) = \frac{\partial}{\partial Y}\left[\frac{\gamma_i}{\beta_i}\frac{P_i}{a_i}\right] \quad . \quad . \quad . \quad (3.5)$$

Aside from the significant estimating difficulties involved, this procedure suffers from the theoretical problem that it requires that $\frac{\partial^2}{\partial P_i^2}\left(\frac{K_i}{\lambda}\right)$ and $\frac{\partial^2}{\partial Y^2}\left(\frac{K_i}{\lambda}\right)$ be assumed equal to zero or, at least, negligible. Such an assumption is without empirical or theoretical foundation, although it might be avoided if an iterative means of solving for the relevant rates of change were adopted. Despite these difficulties, the procedure has considerable merit. Its prime advantages are threefold: (1) the estimated demand equation reflects, at least theoretically, the preferences of individuals as a whole. The important " non-economic " factors, such as comfort and convenience are therefore implicitly considered; (2) the validity of the aggregation technique implicit in the estimation of demand functions does not depend on any arbitrary assumptions about the individuals comprising the group; (3) most importantly, the measure itself is compatible with the hypothesis of utility maximisation. No other measure can make that claim.

This method might be contrasted with a recent attempt by Reuben Gronau to estimate the price of time from the demand function for air passenger travel [8]. The estimating equation, derived from a model similar to the one developed by Gary Becker [1], suggests that demand is a function of " full price " $(P_j + kW_iT_j)$ and income (Y_i):

$$\log X_{ij} = \beta_{0j} + \beta_{1j} \log (P_j + kW_iT_j) + \beta_{2j} \log Y_i + u_{ij} \qquad (3.6)$$

where the subscripts, i and j, denote the ith income group and align the jth destination, and u_{ij} denotes the stochastic disturbance term. In Gronau's model, time value estimation reduces to selecting the proper value of k (presumed to be constant), the ratio of the individual's price of time to his hourly wage (W_i). Gronau's T_i is equivalent to a_i in our model.

A number of problems arise in connection with this procedure. First, k cannot be estimated directly. As a means of determining k, Gronau selects the value that yields the highest adjusted R^2. Secondly, there is no reason to suspect that the price of time is a constant. Both our model and Gronau's suggest that this is not the case, since the price of time is the ratio of two shadow prices. Because of the variability of the price of time, Gronau's measure contains a theoretical bias. From (3.6), it may be demonstrated that

$$kW_i = \frac{\partial X_i}{\partial a_i} \bigg/ \frac{\partial X_i}{\partial P_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.7)$$

which is not the same as (3.1), the measure implicit in the individual's utility maximising decision. Thus, even granting the validity of the R^2 test, his conclusion that individuals behave as if they value time at their respective wage rates is suspect (see [8], pp. 52–3).

Moreover, it is suggested that any alleged relationship of this type between the price of time and the individual's wage is artificial. Though there is likely to be a high positive correlation between the two, it is unlikely that there exists any common proportionality constant for any significant proportion of the population. The fact that Gronau's estimating equations revealed virtually no difference among R^2 values would seem to bear this out (see [8], p. 48, Table 7).

IV. Summary and Conclusions

The model developed in this study is neither a theory of working time, nor a theory of leisure time, nor a theory of travel time, but a theory of time. Particular usages of time are not intrinsic to the role of time in affecting consumer decisions. The single feature which distinguishes this model from others dealing with this problem is the time consumption constraints, which allow for the fact that the amount of time spent in any activity is partly a matter of choice and partly a matter of necessity. When it becomes a matter of necessity, an additional constraint becomes binding upon the consumer's preferences and this constraint must be made explicit. When it is solely a matter of choice, the constraint is not effective and " time prices " have no effect upon the consumer's decision. The non-linear programming model is the only way to capture both features.

All of the implications of our model that relate to the time dimension follow directly from these constraints, although most of these results are not new. The distinction between the value of time and the value of time saving as well as the artificiality of defining leisure as non-work have been brought out by Tipping and others. The responsiveness of demand for certain activities to changes in time costs is the basic premise underlying the search for a socially optimal level of use of highways and airport facilities during periods of congestion. That demand is not responsive to changes in time costs in all situations has clearly been recognised. The fact that the timeless neoclassical theory has endured for over half a century is ample evidence of this. At the conclusion of his study, Johnson wrote: " As important as they are in the context of transportation, leisure, and work decisions, the time dimensions of activities may be irrelevant for many problems in neoclassical economic theory " [10, p. 143].

Yet, while these concepts are not new in themselves, they have never been explicitly derived from a formal model of consumer choice. The fact that all of these implications are already widely accepted is a tribute to the accuracy of the model's predictions. Moreover, the wide applicability of the timeless neoclassical theory lends greater import to another significant feature of our model. If all the time consumption constraints are assumed to be ineffective in the neighbourhood of solution, our model is qualitatively indistinguishable from the timeless one. Thus, our model applies in all those situations where neoclassical theory is applicable, as well as in a wide variety of other situations.

Finally, the most important new result generated by our model is the relationship established between the price of time and the demand function. With this type of measure, the important " non-economic " factors, which are largely ignored in the revealed preference approach to time value estimation, are implicit in the regression coefficients of the demand equation. The prospects of estimating the price of time in this manner at an acceptable level of statistical confidence may not be viewed with total optimism, however, for we have been unable to suggest any clearly acceptable method of estimating the *rates of change* of the price of time, which directly affect the price of time itself. Moreover, there is considerable doubt that the price of time, even if accurately measurable, is sufficiently stable (either with respect to time or with respect to parameter changes) to be used as a basis for solving any practical economic problem. Nevertheless, these difficulties are attributable not to any shortcoming of the theoretical analysis, but to the nature of the beast about which we have been theorising. The model has enabled us to define clearly many of the difficult methodological and conceptual problems that must be dealt with if ever we are to arrive at a useful measure of the price of time. It therefore establishes a firm analytical foundation by which future theoretical and empirical studies of this subject will most assuredly profit.

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Appendix: The Time Effect

From (2.6) in the text, it follows that

$$\frac{\partial X_i}{\partial a_i} = K_i \frac{D_{ii}}{D} + X_i \frac{D_{2n+2+i\,i}}{D} \quad . \quad . \quad . \quad \text{(A.1)}$$

But, from (2.8) and (2.10), it follows that

$$\frac{D_{2n+2+i\,i}}{D} = \frac{D_{i\,2n+2+i}}{D} = \frac{1}{\lambda}\left[\frac{\partial K_i}{\partial P_i} + X_i \frac{\partial K_i}{\partial Y}\right] \quad . \quad . \quad \text{(A.2)}$$

whereupon, by substitution, (A.1) becomes

$$\frac{\partial X_i}{\partial a_i} = K_i \frac{D_{ii}}{D} + \frac{X_i}{\lambda}\left[\frac{\partial K_i}{\partial P_i} + X_i \frac{\partial K_i}{\partial Y}\right] \quad . \quad . \quad \text{(A.3)}$$

Differentiating K_i/λ with respect to P_i and Y, we get

$$\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right) = \frac{1}{\lambda}\frac{\partial K_i}{\partial P_i} - \frac{K_i}{\lambda^2}\frac{\partial \lambda}{\partial P_i} \quad . \quad . \quad . \quad . \quad \text{(A.4)}$$

and

$$\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right) = \frac{1}{\lambda}\frac{\partial K_i}{\partial Y} - \frac{K_i}{\lambda^2}\frac{\partial \lambda}{\partial Y} \quad . \quad . \quad . \quad . \quad \text{(A.5)}$$

Multiplying through (A.5) by X_i, adding (A.4) to (A.5), and rearranging, we get

$$\frac{1}{\lambda}\left[\frac{\partial K_i}{\partial P_i}+X_i\frac{\partial K_i}{\partial Y}\right]=\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right)+X_i\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right)+\frac{K_i}{\lambda^2}\left[\frac{\partial\lambda}{\partial P_i}+X_i\frac{\partial\lambda}{\partial Y}\right] \quad \text{(A.6)}$$

From (2.7), (2.10) and (2.12) in the text, it follows that

$$\frac{\partial\lambda}{\partial P_i}+X_i\frac{\partial\lambda}{\partial Y}=\lambda\frac{D_{i\,2n+1}}{D}=\lambda\frac{D_{2n+1\,i}}{D}=-\lambda\frac{\partial X_i}{\partial Y}. \quad \text{(A.7)}$$

Combining (A.6) and (A.7) we find that

$$\frac{1}{\lambda}\left[\frac{\partial K_i}{\partial P_i}+X_i\frac{\partial K_i}{\partial Y}\right]=\frac{\partial}{\partial P_i}\left(\frac{K_i}{\lambda}\right)+X_i\frac{\partial}{\partial Y}\left(\frac{K_i}{\lambda}\right)-\frac{K_i}{\lambda}\frac{\partial X_i}{\partial Y}. \quad \text{(A.8)}$$

Substituting (A.8) into (A.3) produces (2.19) in the text.

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