Defining Inflation Targets, the Policy Horizon and the Output-Inflation Tradeoff

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Abstract

This paper shows the equivalence between different approaches to defining a central bank's inflation objective. Defining a range and the percentage of time that inflation is expected to lie within that range is the same as defining a target for projected inflation within a given horizon. Both these definitions are in turn similar to defining the target in terms of the expected value and desired variance of inflation. All these definitions are connected by the actual process of inflation. A more volatile or persistent inflation increases the policy horizon. The paper also presents evidence on how inflation targets are actually defined in many countries and compares these stated targets with that implied by the actual process of inflation. To interpret these results, the paper presents a simple model, generalized in the appendices, to show how the central bank's tolerance of deviations from the target, as well as the policy horizon, depend on the cost output deviations from full-employment, and on the slope and the degree of backward-lookingness of the Phillips curve.

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1 Introduction

The main objective of the great majority of central banks in the world is controlling inflation. Some add to this the objective of financial stability; others include employment or economic development objectives. When financial stability is an objective, two dimensions are involved. The first is stability of the domestic financial system, which in simple terms may be described as preventing financial crises. The second is the normal functioning of monetary transactions with the rest of the world, which in simple terms means preventing balance of payments crises.

Regarding the inflationary objective, there is a growing trend toward inflation targeting schemes, in which the central bank sets and publicly announces a numerical objective for the rate of inflation, either a single number or a range. Although an objective thus stated is quite unambiguous, the percentage of time during which inflation is expected to lie within the range is usually not made explicit, since it is understood that inflationary or deflationary shocks can lead to deviations from the target.[1] But the horizon within which the target is to be met, and any deviations are to be corrected, usually is made explicit. Much imprecision in inflation targeting is due precisely to uncertainty regarding the economy's behavior and the shocks to which it is exposed.

It is important to emphasize that a target range refers to an inflationary objective pursued in the current period. Expressed in terms of an inflation projection, the objective is not a range but rather a specific value. Projection is not exact either, and this also introduces a certain variability, but of a much lesser magnitude, and that is why no range for the projection is specified.

The logic for setting a time horizon for the inflation target is that it is recognized that inflation cannot be controlled in the short term, since monetary policies act with lags. Furthermore, and as discussed below, central banks allow for a gradual correction of inflation when it deviates from the target, to avoid unnecessary costs in terms of lost output. In other words, even if monetary policy did not operate with lags, it would be desirable for monetary policy to make its adjustments gradually.[2] Moreover, in general, when the target is specified in the projection horizon, explicit reference is made to a precise point, which always corresponds to the center of the target range.

This paper will try to clarify some issues surrounding the definition of inflationary objectives and the conduct of monetary policy under inflation targeting. It presents a simple model as a basis for discussing these issues and for examining some empirical evidence on inflation targeting. The paper will show that:

[1] These schemes are known as "flexible inflation targets", as opposed to "strict inflation targets," because deviations of the conditional forecast of inflation adjust gradually to the target, because the loss function of he policymaker gives a weight to output stabilization. See Svensson (1999).

[2] This is valid for supply shocks, which are what is analyzed here. A more general model should admit demand shocks, which require a different policy response. For the sake of simplicity, this paper omits demand shocks because they do not change the conclusions at all.

- The inflation objective can be described in terms of a desired distribution for inflation, with an average value and a variability (variance). But in practice the target is always defined by a mean value or a range.

- Setting the target in terms of a mean and a variance is equivalent to setting it in terms of a range and a percentage of time that inflation is expected it to lie within that range. This is comparable to setting the target around an inflation projection, where the future time frame is known as the "policy horizon" and depends on the variance of the inflation target.[3] The greater the fraction of time that inflation is sought to lie within the range, the shorter the necessary policy horizon.

- A flexible inflation targeting scheme, where the target is defined with a time horizon, reflects the objective function of a central bank that values both price and output stability. In particular, a direct relationship also exists between the policy horizon and tolerance of inflation deviations from the target, on one hand, and the importance attributed by the authorities to deviations from full-employment output, on the other. The more inflation averse the policymaker, the shorter the policy horizon and the less volatility in inflation will be tolerated. In addition, the policy horizon increases with the degree to which price setting in the economy is backward looking, and decreases with the slope of the Phillips curve.[4]

Section 2 of this paper compares the definition on an inflation objective in terms of a target range and with a definition based on projected inflation in the policy horizon. Section 3 presents empirical evidence showing the implications of the actual stochastic process of inflation for defining the parameters of the inflation target. In section 4 this objective is rationalized as the result of a minimization of losses, which depends on unemployment and inflation deviations. Section 5 presents some concluding remarks. Finally, three appendices to the paper discuss the results in more detail and extend the analysis to more general settings, which, broadly speaking, do not change the main conclusions reached with the simple model.

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[3] Here it is assumed that the central bank controls the process of inflation; therefore, actual inflation adjusts to target inflation. Consequently, the variance of target inflation is equal to the variance of actual inflation, and both expressions are used indistinctly in addition to simply inflation variance.

[4] The optimal policy horizon has been discussed by Smets (2003), who estimates a model for the euro area, which allows for the calibration of the optimal horizon. His results are similar to those found here regarding the lengthening of the policy horizon with a stronger inflation aversion, a steeper Phillips curve, and a more indexed economy, discussed in section 4. But, the implied horizons derived by Smets (2003) are somewhat shorter than those derived from the actual times series process of inflation in section 2. In addition, in this paper I use a very simple tractable structure to derive the results and to provide empirical implications. Generalizations are provided in Appendix-A and Appendix-B

2 Defining the range for inflation target and the policy horizon

In this section the central bank is assumed to take inflation as given, and some equivalences are shown in the definition of the inflation target that are useful for understanding its formulation. The next section complements this discussion with an analysis of empirical evidence, and later by adding structure and endogenizing the inflation process.

Consider a central bank whose target for inflation is defined as a range between $\overline{\pi}$ and $\underline{\pi}$, with its center equal to $\pi^* = (\overline{\pi} + \underline{\pi})/2$. Some central banks define the target this way; for example, in Canada, Israel, and New Zealand this range is from 1 to 3 percent, in South Africa it is from 3 to 6 percent, and in Chile the range is from 2 to 4 percent.[5] Other countries define the target as a single number, without specifying a range; for example, the United Kingdom sets a target of 2 percent and Norway and Iceland 2.5 percent. Table 1 presents the stated inflation targets of 21 countries that have formally adopted such targets, as reported in IMF (2005) and Batini and Laxton (2005).

Most inflation-targeting countries set their target in terms of the consumer price index (CPI); only a few use some measure of core inflation. Except in Norway and the United Kingdom, where the government decides the inflation target, the central bank decides the target alone or in coordination with the government. In most inflation-targeting countries, the law establishes the mandate of price stability, and the central bank and/or the government interprets this mandate in terms of a specific target.

One can think of the inflation target as corresponding to a probability distribution for inflation, with the objective understood as an expected value and a variance. In what follows I assume here that inflation has a known symmetric distribution—specifically, a normal distribution, which is fully defined by its expected value and its variance. However, in reality most central banks define a range rather than a variance, because a range is more easily understood by the general public. Also, defining a distribution rather than a range requires much more information and certitude, which do not exist in practice. As this section should make clear, in order for equivalence to exist, not only must the range be defined, but also the fraction of time inflation is expected to be within the range, or, in other words, its probability of being within the range at a given time. I will denote this probability by x. In practice the value of x is not defined, although when a policy horizon is defined, information on the tolerated variability for inflation is implicitly provided.

Once a target range is known, the first question is what this range means. Central banks are reluctant to be specific, but it is useful to think that what the central bank wants is for inflation to lie within the range x percent of the time. It suffices to specify

[5] The exact phrasing of the target may differ across countries, for example sometimes the stated target is the range while others is the center of the target $\pm$ a deviation.

the target range and the percentage of the time one intends for inflation to lie within that range to establish the center of the range and the variance.

As shown in figure 1, given values for $\overline{\pi}$ and $\underline{\pi}$ and for x, the distribution must be such that the area below the curve between $\overline{\pi}$ and $\underline{\pi}$ equals x. That defines the variance of the probability distribution function. Alternatively, if the central bank fixes π^* and the variance, for any x there will be only one pair of $\overline{\pi}$ and $\underline{\pi}$ that define the range. Hence, for a central bank to define a target range and specify how strictly it intends to meet it is the same as setting an expected value and a variance for inflation. Therefore a central bank whose objective is to maintain inflation within a range between $\underline{\pi}$ and $\overline{\pi}$ x percent of the time is the same as a central bank that sets as its target an inflation rate that averages π^* with a variance of σ_π^2.

As shown by Svensson (1997), the inflation target may be operationalized by setting the objective in terms of an inflation projection over a given horizon. In practice, this period generally ranges between four and eight quarters. One reason for defining so long a horizon is that monetary policy affects inflation with a lag. A second is that adjusting inflation rapidly to its target entails undesired costs in terms of reduced economic activity and high unemployment, even if inflation is perfectly controllable. In other words, inflation targeting models do take unemployment into account. In fact, in the following section it is assumed, for the sake of simplicity, that the central bank determines inflation instantaneously and adjustment is still gradual.

Another relevant equivalence is that between the variance of inflation from its target and the policy horizon. Suppose that inflation follows the first-order autoregressive, or AR(1), process given by:[6]

$$\pi_t - \pi^* = \rho(\pi_{t-1} - \pi^*) + \epsilon_t, \tag{1}$$

where ϵ_t is an i.i.d. random shock with zero mean and variance σ_ϵ^2, and ρ the autocorrelation coefficient, which is between zero and one. The expected value of inflation is π^* and its unconditional variance is:[7]

$$\sigma_\pi^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}. \tag{2}$$

In making its monetary policy decisions, the central bank projects inflation into the future. The central bank observes ε_t, but from $t+1$ forward the best it can do is to assume that this shock is zero. The inflation projection, conditional on all of the information available at time t, one period ahead will be $\rho\pi_t + (1-\rho)\pi^*$, and the

[6] I assume that ρ is known, and in the next section it is treated as endogenous. However, if ρ is uncertain, the value estimated by the central bank will affect the variability of inflation, and this effect will depend on whether ρ is under- or overestimated (Amano, 2007).

[7] It suffices to take the variance of both sides of equation (1), where the unconditional variances of inflation and past inflation are the same and equal to σ_π^2.

projection T periods ahead is

$$\mathrm{E}_t \pi_{t+T} = \rho^T \pi_t + (1 - \rho^T)\pi^*. \tag{3}$$

As the horizon lengthens (that is, as T rises), ρ^T approaches zero and the projection approaches π^*. Consider then the case where the central bank announces that it wishes inflation to be around π^* in period T. More precisely, it wants the forecast of inflation to converge to π^*. In this respect, the objective of the central bank is the convergence of $\mathrm{E}_t \pi_{t+T}$, where the relevant information set contains π_t. Therefore the conditional forecast, $\rho^T \pi_t + (1 - \rho^T)\pi^*$ is the variable in which the operational objective of the central bank is based.

Given that only as T goes to infinity does the projection converge to π^*, it is assumed that a tolerance margin is allowed, expressed as the variance of the conditional forecast s. As a consequence, the variance of the projected inflation that is obtained from equation (3) is:

$$T = \frac{\log s - \log \sigma_\pi^2}{2 \log \rho} \tag{4}$$

$$= \frac{\log s - \log \sigma_\varepsilon^2 + \log(1 - \rho^2)}{2 \log \rho}. \tag{5}$$

In the latter expression it should be noted that, given that $s < \sigma_\pi^2$ and $\rho < 1$, both the numerator and the denominator are negative; accordingly, T is well defined, since it is necessarily positive. It follows that the greater the variance of target inflation, σ_π^2, or in other words the greater the range for a given x, the longer the policy horizon over which conditional forecast is expected to converge around π^*.

A change in the persistence of inflation has a direct effect on the convergence of the forecast, and hence on T, but it also has an effect on the variance of inflation, given σ_ε^2. Both effects result in an increase in T when persistence rises. Given the persistence parameter, which I derive in section 4, and given the variance of the shock to inflation, the inflation target is fully defined. It is possible to define it either in terms of π^* and σ_π^2, or in terms of $\overline{\pi}$, $\underline{\pi}$, and x, or in terms of π^*, s, and T. An increase in ρ or an increase in σ_ε^2 increases the policy horizon, T, and the variance of inflation, σ_π^2, and consequently reduces x.

In other words, any of the above three sets of variables—representing the distribution of inflation, the range of inflation and the percentage of time within the range, or a time horizon for the inflation forecast and the variance of the conditional forecast—is sufficient to define the parameters of the inflation target. Indeed, if one knew the economy's behavior exactly, it would be impossible to separate the inflation targeting formulation from the policy horizon. However, in reality this is not known with accuracy, and this explains the lack of numerical precision in all the parameters of the objective function. Moreover, one could argue that specifying these parameters with precision might lead to inconsistencies, precisely due to the uncertainty regarding the actual structure of the economy. For example, one might define the target as a range and a value for x, but this might be inconsistent with the policy horizon, and with s, given the actual persistence of inflation.

Rather than give exact definitions of all of the inflation target's parameters, central banks have moved toward increasing transparency and providing public explanations of their deviations from the target in their regular inflation reports (also called monetary policy reports). For example, the governor of the Bank of England must write a formal letter to the U.K. Chancellor of the Exchequer to account for deviations of more than one percentage point from the target.[8] All these forms substitute a public and transparent rendering of accounts for a more mechanical and explicit approach to the inflation target, as is appropriate in a world with much more uncertainty than the standard models assume. Certain risks and contingencies cannot be predicted with central banks' projection models, nor can all policy responses to more complex scenarios than simple deviation of inflation from its target—particularly those scenarios relevant to maintaining financial stability—be anticipated. All this suggests the need to balance a good definition of the rule, on the basis of which the central bank's performance may be evaluated, with due flexibility in a highly uncertain real world.[9]

3 Empirical evidence on the parameters of the inflation target

The equivalences described in section 2 were derived with a minimal structure, based only on the univariate process of inflation, without the need for additional specifications. But they can be used to derive the implicit values of the inflation target for countries that have formally adopted inflation targeting regimes. This section looks at the implications of these equivalences for the actual definition of inflation targets across countries. The next section puts these results in the context of a simple macroeconomic model.

I proceed by estimating an AR(1) process for inflation, as defined in equation (1), for all inflation targeting countries since the beginning of the regime, using each country's declared target. Given the estimated persistence parameter (ρ) and the variance of the shock (σ_ε^2), one can compute the implicit fraction of time x that inflation is expected to be within the target range, as presented in figure 1. In addition, using equation (5), one can compute the implicit horizon (T). The only unknown variable is s, the deviation of the forecast from the center of the band. In practice, discussions of inflation forecasts center on how far the forecast is away from the target, or from the midpoint of the

[8]This was a requirement imposed when the Bank of England was granted operationally independence in 1997. In April 2007, for the first time since independence was granted, a letter was written, since CPI inflation reached 3.1% in the year ending March.

[9]In this regard, it suffices to recall the difficulties central banks have in forecasting the course of GDP.

target range. For this reason I assume three alternative values for s starting from the forecast distribution and assuming that, with 90 percent probability, the central bank wants the forecast to be $\pm.1$, $\pm.2$ and $\pm.3$ from the midpoint level of inflation.

The persistence of inflation and the problems of measurement have been widely discussed, and therefore the results presented here are intended only as illustrations of the implicit parameters of the inflation target. In particular, traditional measures of persistence use more than one lag, and the persistence parameter is the sum of coefficients of the lagged variable. In this paper, however, and to be more consistent with the simple framework I have presented, I instead estimate AR(1) processes. An important issue regarding measures of persistence is the potential for structural breaks in the intercept. As known from Perron (1990), ignoring such breaks could induce an upward bias in the persistence parameter. This could be particularly important in the case of inflation, since the objective of central banks may have been changing, without any change in inflation persistence.[10] Most of the evidence shows that a structural break does occur around the adoption of the inflation target. Because the estimations in this paper start with the date when the target is supposed to be reached, this should not be a serious problem, except for the reduction in the length of the time series.

Another important issue is the measurement of inflation. In general, the measure used is month-to-month or quarter-to-quarter inflation. Here, however, I use monthly measures of 12-months inflation. It is well known that this procedure may result in higher measured persistence. Indeed, it may lead to acceptance of the hypothesis of a unit root of inflation, when actually the process is I(0). The reason for using 12-months inflation is to be consistent with the definition used by inflation targeting central banks, which is usually 12-months inflation on a monthly or quarterly basis. Hence the measure I use is the relevant one regarding the actual definition of the target.[11]

Table 2 presents the estimations and computations of the implicit parameters of the inflation target for each inflation targeting country. The first two columns indicate the quarter in which the inflation target was first implemented and the current range for the target. Several countries define the target for periods of approximately one year. I refer to these as nonstable targets (lower panel of table 2). In contrast, a stable inflation target is defined as one in which the objective is set without an ending date. Targets of this type are more straightforwardly handled by the approach described in the previous section; for the nonstable targets I explicitly consider changes in the target over time in making the estimations.

[10] For recent discussions of this issue in the context of industrial countries, in particular in the Euro area, see the website of the Inflation Persistence Network of the European Central Bank at www.ecb.int/home/html/researcher_ipn.en.html. For aggregate evidence estimating aggregate parameters of inflation persistence and structural breaks, see Gadzinski and Orlandi (2004), Levin and Piger (2004), and O'Reilly and Whelan (2004). A summary is presented in Altissimo et al. (2006).

[11] Indeed, it is possible that quarter-to-quarter inflation displays very low persistence, although 12-months measure are actually highly persistent. See, for example, García and Valdés (2005) for the Chilean case in comparison with other countries, or Benati (2006) for a discussion of persistence across monetary regimes for some industrial countries.

The AR(1) processes are estimated from the beginning of the inflation targeting regime until the last available observation, using quarterly data from International Financial Statistics. The next two columns present estimates of the autocorrelation coefficient (ρ) and the variance of the shock (σ_{ε}^{2}). The next column reports estimates of x, the percentage of the time that inflation is expected to be within the target range, given the estimated AR(1) process. Iceland, Norway, and the United Kingdom specify a single number for the inflation target, so for these countries x cannot be defined.

Among the remaining countries, there is wide dispersion in the implicit value of x. For Australia, Brazil, Colombia, Israel, the Philippines, Poland, and South Africa, x is between 10 and 35 percent, meaning that inflation is within the target range less than a majority of the time. The reason is that the variance of ε is relatively large. However, three out of these seven countries (Australia, Colombia, and Philippines) are, together with Korea, the only ones in the sample where the target range is less than 1 percentage point. This suggests that a range of about 2 percentage points would ensure that inflation will be within the range most of the time.

In the remaining 13 countries, with the exception of the Czech Republic, which is borderline, inflation is expected to be within the range more than 50 percent of the time. Switzerland, the country with the least volatile inflation and relatively low persistence, is within its target range (0 to 2 percent) almost all of the time. The median estimate of x is about 50 percent; the estimates tend to be somewhat higher for countries with stable inflation targets.

As discussed above, the implicit horizon varies with the central bank's tolerance of forecast inflation being away from the midpoint of the range. The greater the tolerance, the shorter the horizon. There is wide variety in the implicit horizon, which depends crucially on the persistence of inflation: the more persistence, the longer the horizon. The median horizon for $\pm.3$ is about two years, and for a tolerance of $\pm.1$ it lengthens to somewhat more than three years. Countries with the largest persistence would have an implicit horizon, for a tolerance of $\pm.3$, of seven to eight years, which is much longer than the stated horizon, implying that, given the persistence and variance of the inflation process, actual tolerance must be greater. This is the case for Brazil, the Philippines, and Sweden. The results are very sensitive to the persistence parameter, and the parameters reported here are somewhat higher than those found in other studies of industrialized countries.

Other estimates of the optimal horizon are discussed in Smets (2003), who finds it to be about three to four years.[12] He obtains his results by calibrating a dynamic general equilibrium model with sticky prices. The results reported in table 2 suggest horizons between two and three years, depending on the estimate of inflation persistence.

Again, to obtain these values I have simply used the actual definition of the inflation target range and the autocorrelation coefficient from an AR(1) process fitted to

[12]Previous studies found shorter horizons. For example, Batini and Haldane (1999) suggest a horizon of between three and six quarters. For further discussion and references see Smets (2003).

inflation, without adding more structure. In the next section I provide a structural interpretation using a simplified macroeconomic model.

4 The output-inflation tradeoff

Whereas the previous section assumed that the central bank takes the inflation process as given, this section goes further and adds structure to the economy, to understand where inflation comes from and how it relates to the output gap. This is done by deriving expression (1) above from the fundamental parameters of the economy, which in this case are given by preferences between unemployment and inflation along a Phillips curve. The value of ρ is determined by the monetary authorities, who gradually adjust inflation so as to reduce the cost in terms of output. The possibility of demand shocks is ignored.

Here I will use the model presented in De Gregorio (1995), which allows the optimal course of inflation to be derived from a function of social loss from inflation and output gaps, plus a Phillips curve that incorporates indexation. I will assume that there is an optimal inflation rate π^*, but that the central bank adjusts inflation gradually toward this rate in order to reduce welfare losses.

The social loss function is given by:[13]

$$\mathrm{L} = a(y - \bar{y})^2 + (\pi - \pi^*)^2, \tag{6}$$

where y is GDP and $\bar{y}$ its full-employment level. It should be noted that here there is no inflationary bias as in Barro and Gordon (1983), since the central bank's preferences are socially optimal (Rogoff, 1985).[14]

Inflation is determined by the following Phillips curve:[15]

$$\pi_t = \alpha\pi_{t-1} + (1 - \alpha)\mathrm{E}_{t-1}\pi_t + \delta(y - \bar{y}) + \nu. \tag{7}$$

The term ν corresponds to an i.i.d inflationary shock with zero mean and variance σ_ν^2. This Phillips curve incorporates persistence via the term $\alpha\pi_{t-1}$, which may be interpreted as the result of indexation of prices and salaries. A simple case is that of

[13]This is a simplification of a more general loss function that could be more formally derived following Woodford (2003, chap. 6)

[14]Strictly speaking, from a welfare point of view the relevant objective is to minimize the present value of losses rather than the value in each period. The solution of that problem is significantly more complex; the details are presented in Appendix-A. The assumption of a static loss function implicitly assumes that the central bank has no ability to commit to future policies, and so so it optimizes period by period.

[15]A simplification of this model is one in which this Phillips curve is not forward looking. In the case of a forward-looking Phillips curve and indexation, it would be the hybrid Phillips curve of Galí and Gertler (1999). The solution is also more complicated, since, as in the case of a infinite horizon, the solution entails a second-order difference equation. This problem is solved in Appendix-B.

certain regulated utility prices, which are indexed to past inflation. The persistence term could also represent the outcome of overlapping decisions on prices and salaries as in the extension of Taylor (1980) proposed by Fuhrer and Moore (1995).[16] The parameter α, which takes a value between zero and one, represents the degree of indexation. The Phillips curve's slope is δ and, to simplify the notation, its inverse is defined as θ.

Solving for the output gap in the Phillips curve and replacing it in the objective function, we have that the first-order condition for the central bank's optimization is given by the following expression (subscript t is eliminated, and instead subscript -1 is used for a one-period lag):

$$\pi - \pi^* = \frac{1}{1+a\theta^2}[\alpha a\theta^2(\pi_{-1} - \pi^*) + (1-\alpha)a\theta^2(\mathrm{E}_{t-1}\pi - \pi^*) + a\theta^2\nu]. \tag{8}$$

Taking expectations from the above equation to solve for rational expectations of inflation, and replacing this expression in the same first-order condition, the following optimal inflation is obtained:

$$\pi - \pi^* = \frac{1}{1+\phi}(\pi_{-1} - \pi^*) + \frac{\nu}{1+\phi\alpha}, \tag{9}$$

where

$$\phi \equiv \frac{1}{a\theta^2\alpha}. \tag{10}$$

Optimal inflation has the same form assumed in equation (1), where the autocorrelation coefficient and the error depend on the fundamental parameters of the model and on the inflationary shock. That is,

$$\rho = \frac{1}{1+\phi} = \frac{a\theta^2\alpha}{1+a\theta^2\alpha} \quad \text{and} \quad \varepsilon = \frac{\nu}{1+\phi\alpha}. \tag{11}$$

It should be noted that expected inflation is equal to the central value of the target, π^*, and the variance is

$$\sigma_\pi^2 = \left(\frac{a\theta^2}{1+a\theta^2}\right)\frac{\sigma_\nu^2}{1-\rho^2}, \tag{12}$$

From these equations it can be easily verified that ρ and σ_π^2 are increasing functions of a, α, and θ, and that σ_π^2 is increasing in the variance of the inflationary shock (σ_ν^2). In section 2 it was shown that increasing the variance of target inflation is similar to extending the policy horizon or widening the target range, all else equal. An increase in the variance of inflation produces the same results. Since an increase in any of the three parameters (a, α, θ) increases both σ_π^2 and ρ, one can conclude that increases in those parameters also lengthen the policy horizon T, as can be seen from equation (5).

These results can be interpreted as follows:

[16] For more details see Walsh (2003, chapter 5.3).

- When the central bank is not concerned about unemployment (that is, $a = 0$), the value of ρ would be zero and expected inflation would adjust to π^* in each period. Therefore the policy horizon collapses to zero: the central bank attempts to meet the inflation projection in each period. In this case inflation would equal π^*, because monetary policy would fully offset the effect of any inflationary shock. As a increases, the policy horizon lengthens, or, similarly, the inflation target variance increases.

- The greater the volatility of inflationary shocks, the greater the variance of target inflation, which, in turn, generates a longer policy horizon.

- Something similar occurs when the degree of backward-lookingness, measured by α, increases, as this also produces a slower adjustment and greater variability of the inflation target: the target range increases.

- When the slope of the Phillips curve decreases (δ falls and θ rises), the output gap has a smaller impact on inflation. Therefore the central bank will accept a greater inflation variance, or a longer policy horizon, because it does not want to vary the output gap too much to offset inflationary shocks.

Why does all this happen? Because even though the central bank defines its objective in terms of an inflation target, it also considers the costs, in terms of unemployment, of attaining the target. In other words, having an inflation target does not mean that unemployment costs are disregarded.

It should be noted that, in this exercise, monetary policy operates without lags, and the inflation target is not intended to be met in the short term. The authorities could always try to attain the target in the short term, in which case $\pi = \pi^* + \varepsilon$ and expected inflation would be π^*. However, the authorities will not do this, because given that a fraction of prices mechanically follow past prices, the expected value of inflation will deviate from $\alpha\pi_{t-1} + (1-\alpha)\mathrm{E}_{t-1}\pi_t = \alpha\pi_{t-1} + (1-\alpha)\pi^*$, giving rise to output fluctuations that, to the extent that $a > 0$, result in welfare losses. Although expectations are rational, the Phillips curve has a built-in persistence that leads to deviations from full employment.

To sum up, inflation persistence declines, and the policy horizon shortens, in response to a decline in a, that is, an increase in inflation aversion; or to a decline in α, the degree of indexation; or to an increase in the slope δ of the Phillips curve $\delta = 1/\theta$, that is, a decline in θ. The decline in inflation persistence should also be accompanied by a decline in the variance of inflation, as long as the variance of the inflation shock (σ_ν^2) remains constant.

As pointed out above, I have made two important assumptions for the sake of simplicity. The first is that the central bank cannot commit to a path of future policies, and therefore it optimizes on a period-by-period basis. The case of a central bank that can so commit, where the loss is minimized on an infinite horizon, is solved in Appendix-A, and the effects of changes in a and θ have the same sign as those found in this section.

The only effects for which the sign cannot be determined are changes in α. The other simplifying assumption was to assume a non-forward-looking Phillips curve, like that implied by New Keynesian models. Using a hybrid New Keynesian Phillips curve like that of Galí and Gertler (1999), the sign on the partial derivatives of the persistence parameter is the same as those in this section, as shown in Appendix-B.

The results in this section and the appendices are summarized in table 3, which shows that in almost all cases the partial derivatives obtained with this simple model are robust to more complicated specifications. In general, the results indicate that when the monetary authority is more averse to a larger output gap, or when the Phillips curve is flatter, the persistence and the variance of inflation increase, which leads, via equation (5), to an increase in the policy horizon. Similarly, an increase in the degree of backward-lookingness of the Phillips curve, for example because of increased indexation, also increases the persistence and variance of inflation, leading to a longer policy horizon. However, this result cannot be generalized in the case of an infinite horizon. Finally, more volatile inflationary shocks (higher σ_ν^2) have no effect on persistence but do increase the policy horizon.

The model can also be used to analyze the volatility of output. The unconditional variances of inflation and output in this model can be easily computed, and the impact of changes in parameters on these variances can be examined. The results can be used to explain how monetary policy may have contributed to the decline in the variance of both inflation and output observed in many countries since the 1980s, or what has been called the "great moderation".[17]

I have already computed the variance of inflation, which is directly related to persistence and is presented in equation (12). Solving for the evolution of output as a function of the inflationary shock and past inflation, we have that

$$y - \bar{y} = \theta \left[\frac{\alpha}{1 + a\alpha\theta^2}\pi^* - \frac{\alpha}{1 + a\alpha\theta^2}\pi_{-1} - \frac{\nu}{1 + a\theta^2} \right]. \tag{13}$$

Using this expression and the variance of inflation, it can be shown that the variance of output (σ_y^2) is

$$\sigma_y^2 = \left(\frac{1 + a\theta^2\alpha}{1 + a\theta^2} \right)^2 \frac{\theta^2\sigma_\nu^2}{1 + 2a\theta^2\alpha}. \tag{14}$$

The following results can be verified after some tedious, but straightforward, algebra:

$$\partial\sigma_\pi^2/\partial a > 0, \quad \partial\sigma_\pi^2/\partial\alpha > 0, \quad \partial\sigma_\pi^2/\partial\theta > 0, \quad \text{and} \partial\sigma_\pi^2/\partial\sigma_\nu^2 > 0$$

$$\partial\sigma_y^2/\partial a < 0, \quad \partial\sigma_y^2/\partial\alpha > 0, \quad \partial\sigma_y^2/\partial\theta ? 0, \quad \text{and} \partial\sigma_y^2/\partial\sigma_\nu^2 > 0.$$

The term $\partial\sigma_y^2/\partial\theta$ it is not uniquely signed but depends mainly on the persistence

[17]For references and a general discussion see Stock and Watson (2003). For a recent empirical analysis of the great moderation, highlighting the role of changes in the conduct of monetary policy, and references of recent work, see Gali and Gambetti (2007)

parameter α. If this variable is large (small) enough, the output variance increases (decreases) as the inverse of the slope of the Phillips curve, θ, increases.[18]

A number of explanations have been offered for the increased stability of inflation and output in recent decades. Several of these lie outside the realm of monetary policy, such as the impact of better inventory management, sectoral shifts toward sectors less sensitive to the business cycle, or financial deepening (Stock and Watson, 2003). But it has also been argued that improved macroeconomic policy may have contributed. This may be particularly important in those emerging economies that have seen reduced volatility and improved macroeconomic management in recent years. In this model a reduction in α, the degree of indexation or backward-looking price setting, reduces the variance of both inflation and output. We can also interpret this as an increase in credibility, since a smaller proportion of prices would be set based on past inflation. In addition, as the experience with high inflation in many countries suggests, a reduction in inflation also reduces the degree of indexation, as the need for protection from the erosive effects of inflation diminishes. Further, a reduction in the variance of inflationary shocks, σ_ν^2, would increase the stability of output and inflation. In contrast, as expected, an increase in a, the degree of preference for output versus inflation stability, reduces the variance of output but increases the variance of inflation, and therefore changes in a, by itself, cannot account for the great moderation. Finally, with an increase in the slope of the Phillips curve (θ declines), the variance of inflation declines, and the effect on the variance of output is uncertain and will depend on the configuration of the parameter. Therefore, only with a reduction in α or σ_ν^2 can the greater stability of output and inflation be unambiguously explained. In the case where the Phillips curve becomes steeper, for example as the result of increased trade openness[19], the results in Appendix-C indicate that it only reduces unambiguously the variance of output and inflation in the case where α is relatively low.

5 Concluding remarks

In general, targeting inflation on the basis of a range within which one expects inflation to lie most of the time is similar to fixing an objective for projected inflation in a given policy horizon, or indicating an expected value and a variance for target inflation. In any case, the definition is not quite accurate, since the structure of the whole economy is not known with sufficient certainty to define the parameters of the inflation targeting rule. Likewise, some flexibility must be retained to address situations that are impossible to anticipate.

On the other hand, defining the monetary authorities' objective in terms of an inflation target does not mean that the business cycle, particularly unemployment, becomes irrelevant. This is reflected in the fact that the target is not intended to be

[18]See Appendix-C for a detailed characterization.

[19]For further discussion see Romer (1993), Temple (2002) and Bowdler (2005).

met always. It is also well reflected in the fact that the target is established in the context of a policy horizon generally of one to two years.

Since inflation targeting is a way of organizing monetary policy that takes into account the trade-offs between inflation and unemployment, one might think that the target of monetary policy could instead be defined in terms of output (or unemployment). One advantage of specifying the macroeconomic stability objective in terms of an inflation target is that it avoids the inconvenience of defining two objectives that might be mutually inconsistent. For example, defining a limit for output variation along with an explicit inflation target could make both objectives incompatible with the economy's structure. There are additional complications when the target is specified in terms of the level of economic activity: since the full-employment output level is not known precisely, it could lead to the well-known acceleration of inflation if very low unemployment is pursued or, similarly, to a deceleration of inflation in the event that full-employment output is underestimated.[20] This has However, the essential reason for choosing an inflation targeting regime is that inflation would be otherwise undetermined. Monetary policy deals with prices and inflation. Defining an inflation target allows the monetary authority to anchor inflation. In terms of the model presented in this paper, there would be no definition of $\bar{\pi}$.

This paper has used a simple analytical model that assumes that the central bank minimizes both losses due to output deviations from full employment and deviations of inflation from the target. The model also assumes that the central bank chooses inflation given a Phillips curve that has a backward-looking component. It can be shown that an increase in the aversion to output fluctuations, an increase in the extent of backward-lookingness in the Phillips curve, or a decline in the slope of the Phillips curve increases inflation volatility and lengthens the policy horizon. Most of these results hold in more general specifications, presented in the appendices, that include intertemporal optimization by the central bank or in the context of a forward-looking hybrid New Keynesian Phillips curve like that of Galí and Gertler (1999).

The model omits some relevant aspects of monetary policy practice, but incorporating these should not change the main conclusions of this paper. The economy is subject to many other shocks than merely inflationary ones. The credibility of the central bank has not been considered, but the decisions made by the central bank and the formulation of its objective reveal information about its ability to contain inflation, as well as its commitment to the target. Incorporating these aspects adds much more complexity, but generally they work in the direction of more rigorously meeting the target, since added credibility makes any subsequent adjustment less costly. In terms of the model in section 4, credibility can be thought of as reducing inertia and the degree of indexation, thus enabling a more rapid return of inflation to the target range when deviations occur.

[20] It is possible that following an interest rate rule could also lead to excessive inflation if full employment output is overestimated. As recently discussed in Woddford (2007), a price-level rule could avoid this problem.

The need for gradualism has been justified by the persistence of inflation. But excessive activism, in the sense of a very short policy horizon, can be viewed as leading to volatility of interest rates and asset prices, which could provoke undesired instability in the financial system. In a more general dynamic stochastic model, one could conceive of an optimal monetary policy having a variable horizon depending on the nature and magnitude of shocks. In practice, however, the same objectives may be achieved with escape clauses that allow for deviations from the target in exceptional situations, for example when financial stability is threatened. In such cases extending the policy horizon to better accommodate the potential financial risks may prove optimal.

The analysis has been presented in the context of a closed economy. Extension to an open economy, including interactions with the exchange rate, should not change the main conclusions, but they would certainly add new sources of fluctuation caused by shocks in the international environment. In any event, incorporating elements of an open economy could provide an additional reason for the central bank to adopt a medium-term horizon. In the case of a very short horizon or a very narrow target range, the principal mechanism of monetary policy pass-through to inflation would be the exchange rate rather than aggregate demand effects. This, in turn, could generate deviations of the exchange rate that might affect the external equilibrium of the economy—a very relevant consideration for emerging economies subject to strong fluctuations in external financing.

A policy of flexible inflation targeting does consider full employment among its objectives. However, it is still preferable to conduct monetary policy around such a target. This permits monetary policymakers both to organize the decision-making process and to communicate to the public their commitment to meeting the inflationary objective. These, in turn, strengthen monetary policy credibility, which is crucial to minimizing the costs of attaining macroeconomic stability.

References

Altissimo, F., M. Ehrmann and F. Smets (2006), "Inflation Persistence and Price-Setting Behaviour in the Euro Area. A Summary of the IPN Evidence," ECB Occasional Paper, No. 46. Frankfurt: European Central Bank

Amano, R. (2007), "Inflation Persistence and Monetary Policy: A Simple Result," *Economic Letters*, Vol. 94, pp. 26-31.

Barro, R. and D. Gordon (1983), "A Positive Theory of Monetary Policy in a Natural-Rate Model," *Journal of Political Economy*, Vol. 91, No. 4, pp. 589-610.

Batini, N. and A. Haldane (1999), "Forward-Looking Rules for Monetary Policy," in J.B. Taylor (ed.), *Monetary Policy Rules*, Chicago: The University of Chicago Press.

Batini, N. and D. Laxton (2005), "Under What Conditions Can Inflation Targeting Be Adopted? The Experience of Emerging Markets," forthcoming in K. Schmidt-Hebbel and F. Mishkin (eds.), *Monetary Policy Under Inflation Targeting*, Santiago: Banco Central de Chile.

Benati, L. (2006), "Investigating Inflation Persistence Across Monetary Regimes," mimeo, London: Bank of England.

Bowdler, C. (2005), "Openness, Exchange Rate Regime and the Phillips Curve," Economics Papers 2005-W25, Nuffield College, University of Oxford.

De Gregorio, J. (1995), "Policy Accommodation and Gradual Stabilizations", *Journal of Money, Credit and Banking,* Vol. 27, No. 3, pp. 727-741.

Fuhrer, J. and G. Moore (1995), "Inflation Persistence", *Quarterly Journal of Economics*, 110, pp. 127-160.

Gadzinski, G. and F. Orlandi (2004), "Inflation Persistence in the European Union, the Euro Area, and the United States", mimeo, Frankfurt: Eurosystem Inflation Persistence Network (IPN), European Central Bank.

Galí, J. and L. Gambetti (2007), "On the Sources of the Great Moderation," mimeo, Universitat Pompeu Fabra.

Galí, J. and M. Gertler (1999), "Inflation Dynamics: A Structural Econometric Analysis," *Journal of Monetary Economics*, Vol. 44, No. 2, pp. 195–222.

García, P. and R. Valdés (2005), "The Inflation Process in Chile: Changes and Stability," mimeo, Santiago: Central Bank of Chile.

International Monetary Fund (2005), *World Economic Outlook*, Washington DC: International Monetary Fund, October.

Levine, A. and J. Piger (2004), "Is Inflation Persistence Intrinsic in Industrial Countries?", ECB Working Paper No. 334.

Mishkin, F. and K. Schmidt-Hebbel (2005), "Does Inflation Targeting Make a Difference?," forthcoming in K. Schmidt-Hebbel and F. Mishkin (eds.), *Monetary Policy Under Inflation Targeting*, Santiago: Banco Central de Chile.

O'Reilly, G. and K. Whelan (2004), "Has Euro-Area Inflation Persistence Changed over Time?," ECB Working Paper No. 335. Frankfurt: European Central Bank.

Perron, P. (1990), "Testing for a Unit Root in a Time Series with a Changing Mean," *Journal of Economics and Business Statistics*, Vol. 8, pp. 153-162.

Rogoff, K. (1985), "The Optimal Commitment to an Intermediate Monetary Target", *Quarterly Journal of Economics*, Vol. 100, No. 4, pp. 1169-89.

Romer, D. (1993), "Openness and Inflation: Theory and Evidence," *Quarterly Journal of Economics*, Vol. 108, No. 4, pp. 969-03.

Smets, F. (2003), "Maintaining Prices Stability: How Long is the Medium Term?," *Journal of Monetary Economics*, Vol. 50, pp. 1293-1309.

Stock, J., and M. Watson (2003), "Has the Business Cycle Changed? Evidence and Explanations," in *Monetary Policy and Uncertainty: Adapting to a Changing Economy*, pp. 9-56. Federal Reserve Bank of Kansas City.

Svensson, L. (1997), "Inflation Forecast Targeting: Implementing and Monitoring Inflation Targets", *European Economic Review*, Vol. 41, pp. 1111-46.

Svensson, L. (1999), "Inflation Targeting: Some Extensions," *The Scandinavian Journal of economics*, Vol. 101, No. 3, pp. 337-361.

Taylor, J. (1980),"Aggregate Dynamics and Staggered Contracts", *Journal of Political Economy*, Vol. 88, No. 1, pp. 1-23.

Temple, J. (2002), "Openness, Inflation, and the Phillps Curve," *Journal of Money, Credit and Banking,* Vol. 34, No. 2, pp. 450-468.

Tuladhar, A. (2005) "Governance Structures and Decision-Making Roles in Inflation Targeting Central Bank" IMF Working Paper WP/05/183., Washington: International Monetary Fund.

Walsh, C. (2003), *Monetary Policy and Theory*, 2nd ed., Cambridge, Mass.: MIT Press.

Woodford, M. (2003), *Interest and Prices*, Princeton, N.J.: Princeton University Press.

Woodford, M (2007), "How Important is Money in the Conduct of Monetary Policy?," CEPR Discussion Paper No. 6211.

Table 1: **Inflation Targets**

Country	Beginning (1)	Current Range (%) (2)	Target (3)	Current Horizon (quarters) (4)	Selects Target (5)
Australia	1993Q1	2-3	CPI	open	CB–G
Brazil	1999Q2	2.5-6.5	CPI	4	CB–G
Canada	1991Q1	1-3	CPI	6-8	CB–G
Chile	1999Q3	2-4	CPI	8	CB
Colombia	1999Q3	4.5-5.5	CPI	4	CB–G
Czech Republic	1998Q1	2-4	CPI	6-8	CB
Hungary	2001Q3	2.5-4.5	CPI	4	CB
Iceland	2001Q1	2.5	CPI	all times	CB–G
Israel	1997Q2	1-3	CPI	year end	G
Korea	1998Q2	2.5-3.5	Core CPI	4	CB–G
Mexico	2002Q1	2-4	CPI	4	CB
New Zealand	1990Q1	1 - 3	CPI	6-8	CB–G
Norway	2001Q1	2.5	Core CPI	8	G
Peru	2002Q1	1.5-3.5	CPI	4	CB
Philippines	2002Q1	5-6	CPI	4	CB–G
Poland	1999Q1	1.5-3.5	CPI	5-7	CB
South Africa	2000Q1	3-6	CPIX	4	CB
Sweden	1993Q1	1-3	CPI	4-8	CB
Switzerland	2000Q1	0-2	CPI	open	CB
Thailand	2000Q2	0-3.5	Core CPI	4	CB
United Kingdom	1992Q4	2	CPI	all times	G

Source: Columns (1) and (2) from Batini and Laxton (2005), (3) and (5) from Tuladhar (2005) and column (4) from individual country sources.
Notes: CB: central bank; G: government; CPI: consumer price index.

Table 2: **Implicit Parameters of the Inflation Target**

	Beginning	Current Range (%)	ρ	$\sigma_\epsilon^2 \times 1000$	x (%)	Current Horizon	T $\pm.1$	T $\pm.2$ (months)	T $\pm.3$
Stable Inflation Targets									
Australia	1994Q3	2-3	0.84	0.09	22.4	-	60	48	41
Canada	1995Q1	1-3	0.66	0.04	77.7	24	20	15	12
Chile	2001Q1	2-4	0.82	0.05	59.4	24	47	37	31
Iceland	2002Q1	2.5	0.54	0.06	-	-	14	11	9
Korea	1999Q1	2.5-3.5	0.79	0.04	63.6	12	40	31	26
New Zealand	1993Q1	1 - 3	0.89	0.04	53.3	24	88	70	59
Norway	2001Q1	2.5	0.62	0.13	-	24	21	17	14
Peru	2002Q1	1.5-3.5	0.84	0.08	45.9	12	60	48	41
South Africa	2001Q1	3-6	0.85	0.31	34.8	12	78	65	57
Sweden	1995Q1	1-3	0.93	0.03	50.6	24	134	106	90
Switzerland	2000Q1	0-2	0.6	0.01	97.9	-	12	8	6
United Kingdom	1992Q4	2	0.81	0.02	-	-	40	30	24
Median			*0.82*	*0.05*	*56.4*	*24*	*40*	*31*	*26*
Unstable Inflation Targets									
Brazil	1999Q2	2.5-6.5	0.9	0.41	33.3	12	128	108	97
Colombia	1999Q3	4.5-5.5	0.73	0.19	19.5	12	36	29	25
Czech Republic	1998Q1	2-4	0.76	0.11	46.1	24	38	30	26
Hungary	2001Q3	2.5-4.5	0.79	0.06	56.3	12	41	32	27
Israel	1997Q2	1-3	0.73	0.37	27.7	6	39	32	28
Mexico	2002Q1	2-4	0.72	0.07	59.4	12	29	23	19
Philippines	2002Q1	5-6	0.89	0.29	10.5	12	113	95	84
Poland	1999Q1	1.5-3.5	0.85	0.15	33.3	21	71	58	51
Median			*0.78*	*0.17*	*33.3*	*12*	*40*	*32*	*27.5*
Median All Countries			*0.82*	*0.07*	*50.6*	*12*	*40*	*31.5*	*26.5*

Source: Table 1, Mishkin and Schmidt-Hebbel (2005) and author's calculations.
Notes: Stable inflation targets are those which have not recently changed while unstable inflation target regimes are periodically revising their inflation objective. Note that the initial stable date does not necessarily coincide with the start date in table 1, since that table shows the date when the inflation target was adopted, and in this case it refers to the date when the target is expected to be reached.
x corresponds to the percentage of the time inflation is within the target given its distribution, and T the length of the horizon given the tolerance to deviations of the inflation forecast, where $\pm.\alpha$ indicates the range in which inflation forecast should be 90% of the times.
Thailand is excluded since the estimated ρ is greater than one.

Table 3: **Summary of Results**

	ρ	σ_π^2	T
Base Model - Section 4			
a	+	+	+
α	+	+	+
θ	+	+	+
σ_ν^2	0	+	+
Infinite Horizon - App. A			
a	+	+	+
α	?	?	?
θ	+	+	+
σ_ν^2	0	+	+
Hybrid Phillips Curve - App. B			
a	+	+	+
α	+	+	+
θ	+	+	+
σ_ν^2	0	+	+

Notes:
ρ is the autocorrelation coefficient of inflation.
σ_π^2 is the variance of inflation.
T is the policy horizon.
a is the aversion to output fluctuations vis-a-vis inflation deviations in the loss of the central banker.
α is the degree of backaward lookingness in the Phillips curve (coefficient on π_{t-1}).
θ is the inverse of the slope of the Phillips curve. Greater θ is a flatter Phillips curve.
σ_ν^2 is the variance of the inflationary shock to the Phillips curve.

Appendix-A The case of a central bank with an infinite horizon

This appendix extends the results presented in section 4 by incorporating an infinite horizon in the monetary authorities loss function.

$$\begin{aligned} \mathrm{L}_t &= \mathrm{E}_t\left[\sum_{i=0}^{\infty}\beta^i\Big\{a(y_{t+i}-\overline{y})^2+(\pi_{t+i}-\pi^*)^2\Big\}\right] && (A.1)\\ y_t-\overline{y} &= \theta\big[\pi_t-(1-\alpha)\mathrm{E}_{t-1}\pi_t-\alpha\pi_{t-1}-\nu_t\big] && (A.2)\end{aligned}$$

The monetary authority minimizes (A.1) subject to (A.2). Expanding the objective function for term in π_t we have that:

$$\begin{aligned} \mathrm{L}_t = \Bigg\{ & a(\theta\big[\pi_t-(1-\alpha)\mathrm{E}_{t-1}\pi_t-\alpha\pi_{t-1}-\nu_t\big])^2+(\pi_t-\pi^*)^2\dots\\ & +\beta\mathrm{E}_t\Big\{a(\theta\big[\pi_{t+1}-(1-\alpha)\mathrm{E}_t\pi_{t+1}-\alpha\pi_t-\nu_{t+1}\big])^2+(\pi_{t+1}-\pi^*)^2\Big\}+\dots\Bigg\},\end{aligned}$$

and the first order condition is:

$$\frac{\partial \mathrm{L}}{\partial \pi_t} = a\theta^2\Big[\pi_t-(1-\alpha)\mathrm{E}_{t-1}\pi_t-\alpha\pi_{t-1}-\nu_t-\alpha^2\beta\big(\mathrm{E}_t\pi_{t+1}-\pi_t\big)\Big]+(\pi_t-\pi^*)=0, \quad (A.3)$$

which simplifies to

$$\pi_t-\pi^*=\frac{a\theta^2}{a\theta^2(1+\alpha^2\beta)+1}\Bigg[(1-\alpha)\mathrm{E}_{t-1}(\pi_t-\pi^*)+\alpha(\pi_{t-1}-\pi^*)+\alpha^2\beta\mathrm{E}_t(\pi_{t+1}-\pi^*)+\nu_t\Bigg] \quad (A.4)$$

Note that if $\beta=0$ as in the finite horizon case, equation (A.4) simplifies to equation (8).

Taking expectations as of $t-1$ we have

$$\begin{aligned} \mathrm{E}_{t-1}(\pi_t-\pi^*) &= \frac{a\theta^2}{a\theta^2(1+\alpha^2\beta)+1}\Big[(1-\alpha)\mathrm{E}_{t-1}(\pi_t-\pi^*)+\alpha(\pi_{t-1}-\pi^*)+\alpha^2\beta\mathrm{E}_{t-1}(\pi_{t+1}-\pi^*)\Big]\\ \mathrm{E}_{t-1}(\pi_t-\pi^*) &= \frac{a\theta^2\alpha}{a\theta^2\alpha(1+\alpha\beta)+1}\Big[(\pi_{t-1}-\pi^*)+\alpha\beta\mathrm{E}_{t-1}(\pi_{t+1}-\pi^*)\Big] && (A.5)\end{aligned}$$

We can use the method of undetermined coefficients to solve for the following, which is presumes to be a first order autoregressive process of order 1:

$$\mathrm{E}_{t-1}(\pi_t-\pi^*)=\rho(\pi_{t-1}-\pi^*) \quad (A.6)$$

where the autocorrelation coefficient depends on $\{\alpha,\beta,\theta,a\}$. Substituting this AR(1) process into the first order condition (A.5) yields the following equation for the solution for ρ:

$$a\alpha^2\beta\theta^2\rho^2-[a\theta^2\alpha(1+\alpha\beta)+1]\rho+a\alpha\theta^2=0. \quad (A.7)$$

The solutions for ρ of this equation, denoted by r_1 and r_2, are:

$$r_{1,2}=\frac{-\phi_2\pm\sqrt{\phi_2^2-4\phi_1\phi_3}}{2\phi_3} \quad (A.8)$$

With $\phi_1 = \frac{a\theta^2\alpha}{a\theta^2\alpha(1+\alpha\beta)+1}$, $\phi_2 = -1$ and $\phi_3 = \phi_1\alpha\beta$. Now we show that the roots of equation (A.8) are real and distinct:

Result 1:

$$\phi_2^2 - 4\phi_1\phi_3 > 0$$

Proof:

$$\begin{aligned} 0 &< \phi_2^2 - 4\phi_1\phi_3 \\ &= 1 - 4\phi_1^2\alpha\beta \\ &= 1 - 4\left[\frac{a\theta^2\alpha}{a\theta^2\alpha(1+\alpha\beta)+1}\right]^2\alpha\beta \\ &= \frac{\Big[a\theta^2\alpha(1+\alpha\beta)+1\Big]^2 - 4\big(a\theta^2\alpha\big)^2\alpha\beta}{\Big[a\theta^2\alpha(1+\alpha\beta)+1\Big]^2} \end{aligned} \tag{A.9}$$

The denominator is squared and therefore positive. For the numerator, it is easy to verify that this expression corresponds to:

$$\underbrace{\big(a\theta^2\alpha\big)^2\big(1-\alpha\beta\big)^2}_{+} + \underbrace{2a\theta^2\alpha(1+\alpha\beta)+1}_{+} > 0$$

and therefore, the roots of equation (A.8) are real and distinct. ‖

Given the previous results, we can find the values of $r_{1,2}$.

Define Φ as

$$\Phi = \phi_2^2 - 4\phi_1\phi_3 = \frac{\big(a\theta^2\alpha\big)^2\big(1-\alpha\beta\big)^2 + 2a\theta^2\alpha(1+\alpha\beta)+1}{\Big(a\theta^2\alpha(1+\alpha\beta)+1\Big)^2} \tag{A.10}$$

Replacing in equation (A.8):

$$\begin{aligned} r_1 &= \frac{-\phi_2 + \sqrt{\phi_2^2 - 4\phi_1\phi_3}}{2\phi_3} \\ &= \frac{1+\sqrt{\Phi}}{2a\theta^2\alpha^2\beta}\Big(a\theta^2\alpha(1+\alpha\beta)+1\Big) \\ &= \frac{a\theta^2\alpha + a\theta^2\alpha^2\beta) + 1 + \sqrt{\big(a\theta^2\alpha\big)^2\big(1-\alpha\beta\big)^2 + 2a\theta^2\alpha(1+\alpha\beta)+1}}{2a\theta^2\alpha^2\beta} \end{aligned}$$

Denote $\Lambda = \big(a\theta^2\alpha\big)^2\big(1-\alpha\beta\big)^2 + 2a\theta^2\alpha(1+\alpha\beta)+1$. Because α and β are smaller than 1, we have $a\theta^2\alpha > a\theta^2\alpha^2\beta$ and as a consequence:

$$r_1 = \frac{a\theta^2\alpha + a\theta^2\alpha^2\beta + 1 + \sqrt{\Lambda}}{2a\theta^2\alpha^2\beta} > \frac{2a\theta^2\alpha^2\beta + 1 + \sqrt{\Lambda}}{2a\theta^2\alpha^2\beta} > 1$$

This shows that $r_1 > 1$, and hence we can rule out this explosive solution as not possible in a rational expectations equilibrium. We must verify that $r_2 < 1$ and whether we must impose restrictions on the parameter space.

$$\begin{aligned} r_2 &= \frac{-\phi_2 - \sqrt{\phi_2^2 - 4\phi_1\phi_3}}{2\phi_3} \\ &= \frac{1-\sqrt{\Phi}}{2a\theta^2\alpha^2\beta}\Big(a\theta^2\alpha(1+\alpha\beta)+1\Big) \end{aligned}$$

$$r_2 = \frac{a\theta^2\alpha(1+\alpha\beta)+1-\sqrt{\left(a\theta^2\alpha\right)^2\left(1-\alpha\beta\right)^2+2a\theta^2\alpha(1+\alpha\beta)+1}}{2a\theta^2\alpha^2\beta} < 1 \tag{A.11}$$

since, after straightforward algebra, condition

$$a\theta^2\alpha(1+\alpha\beta)+1-\sqrt{\left(a\theta^2\alpha\right)^2\left(1-\alpha\beta\right)^2+2a\theta^2\alpha(1+\alpha\beta)+1} < 2a\theta^2\alpha^2\beta$$

is equivalent to

$$1+\alpha\beta > 1-\alpha\beta$$

Therefore the solution is:

$$\mathrm{E}_{t-1}(\pi_{t-1}-\pi^*) = r_2(\pi_{t-1}-\pi^*) \tag{A.12}$$

where:

$$\rho = r_2, \tag{A.13}$$

given by (A.11).

Finally, to characterize the process of inflation we need to find the shock to the AR(1) process. Using (A.11) in equation (A.4) we find, after some algebra, that equilibrium inflation is

$$\begin{aligned} \pi_t - \pi^* &= \frac{a\theta^2}{a\theta^2(1+\alpha^2\beta)+1}\Big[(1-\alpha)\mathrm{E}_{t-1}(\pi_t-\pi^*)+\alpha(\pi_{t-1}-\pi^*)+\alpha^2\beta\mathrm{E}_t(\pi_{t+1}-\pi^*)+\nu_t\Big] \\ \pi_t - \pi^* &= \frac{a\theta^2}{a\theta^2(1+\alpha^2\beta)+1}\Big[(1-\alpha)(\pi_{t-1}-\pi^*)r_2+\alpha(\pi_{t-1}-\pi^*)+\alpha^2\beta(\pi_t-\pi^*)r_2+\nu_t\Big], \end{aligned}$$

which yields:

$$\pi_t - \pi^* = \underbrace{\frac{a\theta^2\big\{(1-\alpha)r_2+\alpha\big\}}{a\theta^2(1+\alpha^2\beta(1-r_2))+1}}_{\rho}(\pi_{t-1}-\pi^*) + \underbrace{\frac{a\theta^2}{a\theta^2(1+\alpha^2\beta(1-r_2))+1}\nu_t}_{\varepsilon_t}. \tag{A.14}$$

Note that there is no inconsistency in defining ρ as above. Replacing $r_2 = \rho$ in the above-mentioned expression and solving for this last variable, shows that ρ satisfies the same second order equation derived under the first order autoregressive process for expectations defined by (A.6). We conclude that,

$$\pi_t - \pi^* = \rho(\pi_{t-1}-\pi^*) + \varepsilon, \tag{A.15}$$

with

$$\rho = \frac{a\theta^2\alpha(1+\alpha\beta)+1-\sqrt{(a\theta^2\alpha)^2(1-\alpha\beta)^2+2a\theta^2\alpha(1+\alpha\beta)+1}}{2a\theta^2\alpha^2\beta}, \tag{A.16}$$

and

$$\varepsilon_t = \frac{a\theta^2}{a\theta^2(1+\alpha^2\beta(1-r_2))+1}\nu_t. \tag{A.17}$$

where we used $\rho = r_2$.

Equation (A.15) is very similar to (9), and it can be easily shown that the latter is a particular case of $\beta = 0$. Given the AR(1) nature of the process, the sacrifice ratio is again independent of the persistence parameter, although this is not the proper welfare measure in this model since it would require discounting, and the evaluation of the inflationary losses.

RESULT 2: $\partial\rho/\partial a$ and $\partial\rho/\partial\theta$ are positive. If $\beta < \frac{1}{4}$ or $\beta > \frac{1}{4}$ and $a\theta^2 < \frac{8}{8\beta+1}$, $\partial\rho/\partial\alpha > 0$. Otherwise, $\partial\rho/\partial\alpha > 0$ in $[0, \alpha_2]$ with

$$\alpha_2 = \frac{a\theta^2 + \sqrt{a^2\theta^4 + 16a\theta^2\beta}}{4a\theta^2\beta} < 1$$

if $a\theta^2 > \frac{8}{8\beta+1}$ holds.

PROOF: ρ can be written as:

$$\rho = \frac{1+\alpha\beta}{2\alpha\beta} + \frac{1}{2a\theta^2\alpha^2\beta} - \frac{1}{2\alpha\beta}\sqrt{(1-\alpha\beta)^2 + \frac{2(1+\alpha\beta)}{a\theta^2\alpha} + \frac{1}{(a\theta^2\alpha)^2}}.$$

Differentiating with respect to a we have that

$$\frac{\partial\rho}{\partial a} = -\frac{1}{2a^2\theta^2\alpha^2\beta} - \frac{1}{4\alpha\beta}(\psi)^{-\frac{1}{2}}\left(-\frac{2(1+\alpha\beta)}{a^2\theta^2\alpha} - \frac{2}{a^3(\theta^2\alpha)^2}\right),$$

where

$$\psi = (1-\alpha\beta)^2 + \frac{2(1+\alpha\beta)}{a\theta^2\alpha} + \frac{1}{(a\theta^2\alpha)^2} = \frac{1 + 2a\theta^2\alpha(1+\alpha\beta) + (a\theta^2\alpha)^2(1-\alpha\beta)^2}{(a\theta^2\alpha)^2},$$

Collecting terms,

$$\begin{aligned}\frac{\partial\rho}{\partial a} &= \frac{1}{2a^2\theta^2\alpha^2\beta}\left[\psi^{-\frac{1}{2}}\left(1+\alpha\beta+\frac{1}{a\theta^2\alpha}\right) - 1\right] \\ &= \frac{1}{2a^2\theta^2\alpha^2\beta}\left[\psi^{-\frac{1}{2}}\frac{(1+\alpha\beta)a\theta^2\alpha + 1}{a\theta^2\alpha} - 1\right],\end{aligned}$$

and replacing ψ, it is easy to see that the first term in the square bracket is greater than one, and hence it is proven that $\partial\rho/\partial a$ is greater than zero for all positive values of a and θ, and for all α and β between zero and one.

For θ we have that

$$\frac{\partial\rho}{\partial\theta} = -\frac{2}{2a\theta^3\alpha^2\beta} - \frac{1}{4\alpha\beta}(\psi)^{-\frac{1}{2}}\left(-\frac{4(1+\alpha\beta)}{a\theta^3\alpha} - \frac{4}{\theta^5(a\alpha)^2}\right),$$

and collecting terms we arrive at:

$$\frac{\partial\rho}{\partial\theta} = \frac{1}{a\theta^3\alpha^2\beta}\left[\psi^{-\frac{1}{2}}\left(1+\alpha\beta+\frac{1}{a\theta^2\alpha}\right) - 1\right],$$

which, for the same reasons as in the case of a, is greater than zero for all positive values of a and θ, and for all α and β between zero and one.

Finally for α, we have that:

$$\frac{\partial \rho}{\partial \alpha} = -\frac{1}{2\beta\alpha^2} - \frac{1}{a\theta^2\beta\alpha^3} + \frac{1}{2\alpha\beta}(\psi^{-\frac{1}{2}})\left((1-\alpha\beta)\beta + \frac{1}{a(\theta\alpha)^2} + \frac{1}{(a\theta^2)^2\alpha^3}\right) + \frac{1}{2\beta\alpha^2}(\psi)^{\frac{1}{2}}$$

Collecting terms,

$$\frac{\partial \rho}{\partial \alpha} = \frac{1}{2\beta\alpha^2}(\psi^{\frac{1}{2}} - 1) + \frac{1}{\alpha^3\beta\ a\theta^2}\left(\frac{\psi^{-\frac{1}{2}}}{2}\left(1 + (1-\alpha\beta)\alpha^2\beta a\theta^2 + \frac{1}{a\theta^2\alpha}\right) - 1\right).$$

Define $\phi = 1 + 2a\theta^2\alpha(1+\alpha\beta) + (a\theta^2\alpha)^2(1-\alpha\beta)^2$. Simple calculus leads to

$$\frac{\partial \rho}{\partial \alpha} = \frac{1}{2\beta\alpha^3\beta a\theta^2\phi^{\frac{1}{2}}}\left[\phi + a\theta^2\alpha + (1-\alpha\beta)\alpha^2\beta a^2\theta^4 + 1 - \phi^{\frac{1}{2}}[2 + a\theta^2\alpha]\right]$$

so $\frac{\partial \rho}{\partial \alpha} > 0$ is equivalent to

$$\phi + a\theta^2\alpha + (1-\alpha\beta)\alpha^2\beta a^2\theta^4 + 1 > \phi^{\frac{1}{2}}[2 + a\theta^2\alpha]$$

Squaring both sides of the above inequality and eliminating terms, we obtain the condition

$$2a\theta^2\beta\alpha^2 - a\theta^2\alpha - 2 < 0$$

which corresponds to a quadratic function in the variable α. Because the coefficient $2a\theta^2\beta$ is positive, this function is negative between the roots α_1 and α_2 (they exist because $a^2\theta^4 - 4(2a\theta^2\beta)(-2) > 0$), with

$$\alpha_1 = \frac{a\theta^2 - \sqrt{a^2\theta^4 + 16a\theta^2\beta}}{4a\theta^2\beta}$$

$$\alpha_2 = \frac{a\theta^2 + \sqrt{a^2\theta^4 + 16a\theta^2\beta}}{4a\theta^2\beta}$$

It is clear that $\alpha_1 < 0$ and $\alpha_2 > 0$, but whether $\alpha_2 > 1$ or not, depends on the model parameters β, a and θ. Condition $\alpha_2 > 1$ is equivalent to

$$\sqrt{a^2\theta^4 + 16a\theta^2\beta} > a\theta^2(4\beta - 1) \tag{A.18}$$

If $\beta < 1/4$, (A.18) always holds, and as a consequence, $\frac{\partial \rho}{\partial \alpha} > 0$ for all $\alpha \in [0,1]$. Assuming that β greater that 1/4, this condition is reduced to

$$a\theta^2 < \frac{8}{8\beta + 1}$$

and therefore $\frac{\partial \rho}{\partial \alpha} > 0$ for all $\alpha \in [0,1]$. If this last inequality does not holds, $\alpha_2 < 1$ and $\frac{\partial \rho}{\partial \alpha}$ will be positive for $\alpha \in [0, \alpha_2]$. ||

Using L'Hopital it is easy to show that ρ goes to the original ρ defined in (10) and (11) when β goes to zero.

Furthermore, the following can be established:

Result 3: $\partial\sigma_\pi^2/\partial a$ and $\partial\sigma_\pi^2/\partial\theta$ are positive, and $\partial\sigma_\pi^2/\partial\alpha$ cannot be unambiguously signed. Therefore, $\partial T/\partial a$ and $\partial T/\partial\theta$ are also positive, and $\partial T/\partial\alpha$ cannot be signed.

PROOF: The error of the inflation process is the same as the error of the Phillips curve up to a multiplicative constant (see (A.17)). This constant is :

$$\frac{a\theta^2}{a\theta^2(1+\alpha^2\beta(1-\rho))+1}, \tag{A.19}$$

which is increasing in ρ. In addition the above expression is increasing in a and θ for a given ρ. Then, the variance of the error of the inflation process, σ_ε^2, is increasing in a and θ. But, we are interested in:

$$\sigma_\pi^2 = \frac{1}{1-\rho^2}\sigma_\varepsilon^2 = \frac{1}{1-\rho^2}\left(\frac{a\theta^2}{a\theta^2(1+\alpha^2\beta(1-\rho))+1}\right)^2\sigma_\nu^2. \tag{A.20}$$

Since $\frac{1}{1-\rho^2}$ is increasing in ρ, and σ_ε^2 is increasing in a and θ, the results is direct.

This cannot be for α, since $\partial\rho/\partial\alpha$ can be either positive of negative, and, furthermore, the partial derivative of σ_π^2 with respect to α cannot be signed.

The effects on T are direct using equation (5), which establishes that when ρ and σ_ν^2 rise, T lengthen. $\|$

Finally, the effects of increasing the variance of the inflationary shock are easy to verify, since σ_ν^2 does not affect the persistence parameter, but does increase the variance of inflation, and hence it lengthen the policy horizon.

Appendix-B Results under a New Keynesian-hybrid Phillips curve

The problem of the central bank is to minimize the loss function given by equation (6) subject to the following hybrid Phillips curve:

$$\pi_t = \alpha\pi_{t-1} + (1-\alpha)E_t\pi_{t+1} + \delta(y_t - \overline{y}) + \nu_t. \tag{B.1}$$

Substituting into the loss function we have that the objective to be minimized is:

$$\mathrm{L} = a(\theta(\pi_t - \alpha\pi_{t-1} - (1-\alpha)E_t\pi_{t+1} - \nu_t))^2 + (\pi - \pi^*)^2.$$

Differentiating with respect to π we have the following first order conditions:

$$2a(\theta^2(\pi_t - \alpha\pi_{t-1} - (1-\alpha)E_t\pi_{t+1} - \nu_t)) + 2(\pi_t - \pi^*) = 0$$

Solving for the inflation deviation with respect to the target:

$$\pi - \pi^* = \frac{1}{1+a\theta^2}\left(\alpha a\theta^2(\pi_{t-1} - \pi^*) - a\theta^2\pi^* + \alpha a\theta^2\pi^* + (1-\alpha)a\theta^2 E_t\pi_{t+1} + a\theta^2\nu_t\right) \tag{B.2}$$

Collecting terms we have the following difference equation for inflation:

$$\pi_t - \pi^* = \frac{1}{1+a\theta^2}\left(\alpha a\theta^2(\pi_{t-1} - \pi^*) + (1-\alpha)a\theta^2(E_t\pi_{t+1} - \pi^*) + a\theta^2 v\right).$$

Taking expectations conditional on the information at $t-1$ we have that:

$$E_{t-1}(\pi_t - \pi^*) = \frac{1}{1+a\theta^2}\left(\alpha a\theta^2(\pi_{t-1} - \pi^*) + (1-\alpha)a\theta^2(E_{t-1}(\pi_{t+1} - \pi^*))\right) \tag{B.3}$$

I use again undetermined coefficients to solve this equation assuming that:

$$E_{t-1}(\pi_t - \pi^*) = \rho(\pi_{t-1} - \pi^*). \tag{B.4}$$

where the autocorrelation coefficient depends on $\{\alpha, a, \theta\}$. Regrouping terms, the autocorrelation coefficient must solve the following second order difference equation:

$$a\theta^2(1-\alpha)E_{t-1}(\pi_{t+1} - \pi^*) - (1 + a\theta^2)E_{t-1}(\pi_t - \pi^*) + a\theta^2\alpha(\pi_{t-1} - \pi^*) = 0$$

Replacing (B.3) in the previous equation leads to the following second degree equation for the persistence parameter ρ:

$$a\theta^2(1-\alpha)\rho^2 - (1 + a\theta^2)\rho + a\theta^2\alpha = 0. \tag{B.5}$$

Considering only the root less than one, we have that ρ is given by:

$$\rho = \frac{1 + a\theta^2 - \sqrt{(1 + a\theta^2)^2 - 4a\theta^2(1-\alpha)a\theta^2\alpha}}{2a\theta^2(1-\alpha)},$$

which is clearly positive and can be written as:

$$\rho = \frac{1}{2a\theta^2(1-\alpha)} + \frac{1}{2(1-\alpha)} - \sqrt{\frac{(1 + a\theta^2)^2}{4a^2\theta^4(1-\alpha)^2} - \frac{\alpha}{1-\alpha}} \tag{B.6}$$

Result 4: $\frac{\partial\rho}{\partial a}$, $\frac{\partial\rho}{\partial\theta}$ and $\frac{\partial\rho}{\partial\alpha}$ are all positive.

Proof: Taking the partial derivative with respect to a we have that:

$$\frac{\partial\rho}{\partial a} = -\frac{1}{2a^2\theta^2(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left(\frac{2(1 + a\theta^2)\theta^2 4a^2\theta^4(1-\alpha)^2 - 8a\theta^4(1-\alpha)^2(1 + a\theta^2)^2}{16a^4\theta^8(1-\alpha)^4}\right),$$

where $\varphi = \frac{(1+a\theta^2)^2}{4a^2\theta^4(1-\alpha)^2} - \frac{\alpha}{1-\alpha}$, hence,

$$\begin{aligned}
\frac{\partial\rho}{\partial a} &= -\frac{1}{2a^2\theta^2(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left(\frac{8a\theta^4(1-\alpha)^2(1 + a\theta^2)[a\theta^2 - (1 + a\theta^2)]}{16a^4\theta^8(1-\alpha)^4}\right) \\
\frac{\partial\rho}{\partial a} &= -\frac{1}{2a^2\theta^2(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left(-\frac{1 + a\theta^2}{2a^3\theta^4(1-\alpha)^2}\right)
\end{aligned}$$

Collecting terms:

$$\frac{\partial\rho}{\partial a} = \frac{1}{2a^2\theta^2(1-\alpha)}\left[\frac{1}{2}\frac{(1 + a\theta^2)}{a\theta^2(1-\alpha)}\varphi^{-\frac{1}{2}} - 1\right] \tag{B.7}$$

Simplifying the term $\frac{1}{2}\frac{(1+a\theta^2)}{a\theta^2(1-\alpha)}\varphi^{-\frac{1}{2}}$, we have:

$$\begin{aligned}
&= \frac{1}{2}\frac{(1 + a\theta^2)}{a\theta^2(1-\alpha)}\left(\frac{(1 + a\theta^2)^2}{4a^2\theta^4(1-\alpha)^2} - \frac{\alpha}{1-\alpha}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}\frac{(1 + a\theta^2)}{a\theta^2(1-\alpha)}\left(\frac{(1 + a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)}{4a^2\theta^4(1-\alpha)^2}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}\frac{(1 + a\theta^2)}{a\theta^2(1-\alpha)}2a\theta^2(1-\alpha)\left[(1 + a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)\right]^{-\frac{1}{2}} \\
&= (1 + a\theta^2)\left[(1 + a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)\right]^{-\frac{1}{2}}
\end{aligned}$$

Finally, the partial derivative of ρ with respect to a is:

$$\frac{\partial\rho}{\partial a} = \frac{1}{2a^2\theta^2(1-\alpha)}\left[(1+a\theta^2)\left[(1+a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)\right]^{-\frac{1}{2}} - 1\right] \tag{B.8}$$

Then, $\frac{\partial\rho}{\partial a} > 0$ if and only if

$$(1+a\theta^2)^2 > (1+a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)$$

which is certainly true.

Now, taking the partial derivative with respect to θ, we have:

$$\begin{aligned}
\frac{\partial\rho}{\partial\theta} &= -\frac{1}{a\theta^3(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left(\frac{2(1+a\theta^2)2a\theta 4a^2\theta^4(1-\alpha)^2 - 16a^2\theta^3(1-\alpha)^2(1+a\theta^2)^2}{16a^4\theta^8(1-\alpha)^4}\right) \\
&= -\frac{1}{a\theta^3(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\frac{16(1+a\theta^2)a^2\theta^3(1-\alpha)^2\left[a\theta^2 - (1+a\theta^2)\right]}{16a^4\theta^8(1-\alpha)^4} \\
&= -\frac{1}{a\theta^3(1-\alpha)} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left(-\frac{1+a\theta^2}{a^2\theta^5(1-\alpha)^2}\right)
\end{aligned}$$

Collecting terms we have that:

$$\frac{\partial\rho}{\partial\theta} = \frac{1}{a\theta^3(1-\alpha)}\left[\frac{1}{2}\frac{(1+a\theta^2)}{a\theta^2(1-\alpha)}\varphi^{-\frac{1}{2}} - 1\right],$$

which finally reduces to:

$$\frac{\partial\rho}{\partial\theta} = \frac{1}{a\theta^3(1-\alpha)}\left[(1+a\theta^2)\left[(1+a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)\right]^{-\frac{1}{2}} - 1\right]. \tag{B.9}$$

Note that the expression in parentheses is the same as that which defines the sign in the partial derivative of ρ with respect to a. Therefore, we conclude that $\frac{\partial\rho}{\partial\theta} > 0$ as well.

Finally, taking the partial derivative with respect to α we have that:

$$\frac{\partial\rho}{\partial\alpha} = \frac{1}{2a\theta^2(1-\alpha)^2} + \frac{1}{2(1-\alpha)^2} - \frac{1}{2}\varphi^{-\frac{1}{2}}\left[\frac{(1+a\theta^2)^2 2(1-\alpha)}{4a^2\theta^4(1-\alpha)^4} - \frac{1-\alpha+\alpha}{(1-\alpha)^2}\right]$$

Define

$$\mu = \frac{(1+a\theta^2)^2 2(1-\alpha)}{4a^2\theta^4(1-\alpha)^4} - \frac{1-\alpha+\alpha}{(1-\alpha)^2} = \frac{(1+a\theta^2)^2 - 2a^2\theta^4(1-\alpha)}{2a^2\theta^4(1-\alpha)^3}$$

and recall that

$$\varphi = \frac{(1+a\theta^2)^2}{4a^2\theta^4(1-\alpha)^2} - \frac{\alpha}{1-\alpha}$$

Then,

$$\varphi^{-\frac{1}{2}}\mu = \frac{1}{a\theta^2(1-\alpha)^2}\left[\frac{(1+a\theta^2)^2 - 2a^2\theta^4(1-\alpha)}{\left[(1+a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)\right]^{\frac{1}{2}}}\right]$$

Defining $\eta = [(1+a\theta^2)^2 - 4a^2\theta^4\alpha(1-\alpha)]$ and replacing this expression in $\frac{\partial\rho}{\partial\alpha}$ we conclude that

$$\frac{\partial\rho}{\partial\alpha} = \frac{1}{2a\theta^2(1-\alpha)^2\eta^{\frac{1}{2}}}[\eta^{\frac{1}{2}}(1+a\theta^2) - (1+a\theta^2)^2 + 2a^2\theta^4(1-\alpha)]$$

If $(1+a\theta^2)^2 - 2a^2\theta^4(1-\alpha) < 0$, $\frac{\partial\rho}{\partial\alpha} > 0$. If not, $\frac{\partial\rho}{\partial\alpha} > 0$ if and only if

$$\eta(1+a\theta^2)^2 > [(1+a\theta^2)^2 - 2a^2\theta^4(1-\alpha)]^2$$

Replacing η, this above expression is equivalent to

$$(1+a\theta^2)^4 - 4a^2\theta^4\alpha(1-\alpha)(1+a\theta^2)^2 > (1+a\theta^2)^4 - 4a^2\theta^4(1-\alpha)(1+a\theta^2)^2 + 4a^4\theta^8(1-\alpha)^2$$

and simple algebra leads to

$$(1+a\theta^2)^2 > a^2\theta^4$$

which is true. Thus, $\frac{\partial\rho}{\partial\alpha} > 0$. $\|$

Finally, the following result holds:

Result 5: $\frac{\partial\sigma_\pi^2}{\partial a}$, $\frac{\partial\sigma_\pi^2}{\partial\alpha}$, $\frac{\partial\sigma_\pi^2}{\partial\theta} > 0$. All together with Result 4 implies that $\frac{\partial T}{\partial a}$, $\frac{\partial T}{\partial\alpha}$ and $\frac{\partial T}{\partial\theta} > 0$.

Proof: Recall that

$$\pi_t - \pi^* = \frac{1}{1+a\theta^2}\left(\alpha a\theta^2(\pi_{t-1} - \pi^*) + (1-\alpha)a\theta^2(E_{t-1}\pi_{t+1} - \pi^*) + a\theta^2 v\right).$$

The first order autoregressive process for expectations yields

$$\pi_t - \pi^* = \frac{1}{1+a\theta^2}\left([\alpha a\theta^2 + (1-\alpha)a\theta^2\rho^2](\pi_{t-1} - \pi^*) + a\theta^2 v\right).$$

Setting $\rho = \frac{1}{1+a\theta^2}\left[\alpha a\theta^2 + (1-\alpha)a\theta^2\rho^2\right]$ we obtain that this variable satisfies

$$a\theta^2(1-\alpha)\rho^2 - (1+a\theta^2)\rho + a\theta^2\alpha = 0.$$

which is exactly the same second order equation satisfied by the persistence parameter. This ensures that no inconsistency arises in setting ρ as above. As before, we define ρ as the unique root less than one. Therefore,

$$\pi_t - \pi^* = \rho(\pi_{t-1} - \pi^*) + \epsilon_t$$

with ρ as before and

$$\epsilon_t = \frac{1}{1+\frac{1}{a\theta^2}}\sigma_\nu^2$$

As $\sigma_\pi^2 = \frac{1}{1-\rho^2}\sigma_\epsilon^2$, we obtain

$$\sigma_\pi^2 = \frac{1}{1-\rho^2}\left(\frac{1}{1+\frac{1}{a\theta^2}}\right)^2\sigma_\nu^2$$

Since $\frac{\partial\rho}{\partial\eta} > 0$ for $\eta = \alpha, \theta, a$, it is obvious that $\frac{\partial\sigma_\pi^2}{\partial\eta} > 0$ for $\eta = \alpha, \theta, a$. The reasoning for T is the same as the one in Result 3.$\|$

Appendix-C Output Variance and the Inverse of the Phillips Curve Slope

Result 5: $\frac{\partial \sigma_y^2}{\partial \theta}$, which is not uniquely signed, and satisfies:

$$\frac{\partial \sigma_y^2}{\partial \theta} = \begin{cases} > 0 & 3/4 \leq \alpha \leq 1 \\ < 0 & 0 \leq \alpha \leq 1/3 \\ > 0 & \alpha \in (1/3, 3/4) \ \wedge \ \theta^2 \in [r_-, r_+] \\ < 0 & \alpha \in (1/3, 3/4) \ \wedge \ \theta^2 \in [r_-, r_+]^c \end{cases}$$

with

$$r_\pm = \frac{1 - 3\alpha \pm \sqrt{1 - 6\alpha - 7\alpha^2}}{2a\alpha(4\alpha - 3)}$$

Proof: Recalling that

$$\sigma_y^2 = \left(\frac{1 + a\theta^2\alpha}{1 + a\theta^2}\right)^2 \frac{\theta^2 \sigma_\nu^2}{1 + 2a\theta^2\alpha}$$

then, we obtain

$$\frac{\partial \sigma_y^2}{\partial \theta} = 2\left(\frac{1 + a\theta^2\alpha}{1 + a\theta^2}\right) \frac{2a\theta\alpha(1 + a\theta^2) - 2a\theta(1 + a\theta^2\alpha)}{(1 + a\theta^2)^2} \frac{\theta^2\sigma_\nu^2}{1 + 2a\theta^2\alpha}$$
$$+ \left(\frac{1 + a\theta^2\alpha}{1 + a\theta^2}\right)^2 \frac{2\theta\sigma_\nu^2(1 + 2a\theta^2\alpha) - \theta^2\sigma_\nu^2 4a\theta\alpha}{(1 + 2a\theta^2\alpha)^2}$$

which reduces to

$$\frac{\partial \sigma_y^2}{\partial \theta} = \frac{2\theta\sigma_\nu^2(1 + a\theta^2\alpha)}{(1 + a\theta^2)^2(1 + 2a\theta^2\alpha)} \left[\frac{2a\theta^2(\alpha - 1)}{1 + a\theta^2} + \frac{1 + a\theta^2\alpha}{1 + 2a\theta^2\alpha}\right]$$

Therefore, the term that defines the sign in the above expression is

$$\frac{2a\theta^2(\alpha - 1)}{1 + a\theta^2} + \frac{1 + a\theta^2\alpha}{1 + 2a\theta^2\alpha}$$

Because both denominators are positive we only need to analyze the numerator, which is defined by

$$\theta^4(a^2\alpha)(4\alpha - 3) + \theta^2 a(3\alpha - 1) + 1$$

and corresponds to a quadratic function in θ^2. Define $\eta = \theta^2$ and consider the function

$$f(\eta) = \eta^2(a^2\alpha)(4\alpha - 3) + \eta a(3\alpha - 1) + 1 \tag{C.1}$$

It is clear that when $\alpha = 3/4$, $f(\cdot)$ reduces to

$$f(\eta) = \eta a(\frac{9}{4} - 1) > 0$$

if $\eta > 0$. Because in this case, $\eta = \theta^2 > 0$, $\frac{\partial \sigma_y^2}{\partial \theta} > 0$.

Suppose now that $\alpha \neq 3/4$. The features of $f(\cdot)$ now depend on the persistence parameter α. Define

$$\Delta = a^2(3\alpha - 1)^2 - 4(a^2\alpha)(4\alpha - 3)$$

which determines whether $f(\cdot)$ has roots or not. After some straightforward algebra,

$$\Delta = a^2[1 - 6\alpha - 7\alpha^2] \tag{C.2}$$

and because α belongs to [0,1], $1 - 6\alpha - 7\alpha^2 > 0$. Therefore, if $\alpha \neq 3/4$, $f(\cdot)$ always has at least one root (moreover, only when $\alpha = 1$, does f have only one root). The roots are defined by

$$r_{\pm} = \frac{1 - 3\alpha \pm \sqrt{1 - 6\alpha - 7\alpha^2}}{2a\alpha(4\alpha - 3)}$$

Consider the following cases:

(i) $\alpha > 3/4$: In this case it is easy to verify that both roots are negative. Because the quadratic coefficient in $f(\cdot)$ is positive, for any $\eta = \theta^2 > 0$, $f(\eta) > 0$ and, as a consequence, $\frac{\partial \sigma_y^2}{\partial \theta} > 0$.

(ii) $1/3 < \alpha < 3/4$: In this region, since the denominator in $r_{\pm}$ is negative and $\sqrt{1 - 6\alpha - 7\alpha^2} < |1 - 3\alpha|$, both roots are positive. Hence, $\frac{\partial \sigma_y^2}{\partial \theta} > 0$ only when $\theta^2 \in [r_-, r_+]$, since the quadratic coefficient in $f(\cdot)$ is negative. Otherwise, $\frac{\partial \sigma_y^2}{\partial \theta} < 0$.

(iii) $\alpha < 1/3$: It is straightforward that $r_+ < 0$. As in (ii), since $\sqrt{1 - 6\alpha - 7\alpha^2} < |1 - 3\alpha| = 1 - 3\alpha$, $r_- < 0$, it follows that for any $\eta = \theta^2 > 0$ $f(\eta) < 0$ (note that in this case f is decreasing for $\eta > 0$). Therefore, $\frac{\partial \sigma_y^2}{\partial \theta} < 0$.

In other words, if the persistence parameter α is large\ small enough, the output variance increases\decreases as the inverse of the slope of the Phillips's curves θ becomes larger. When $1/3 < \alpha < 3/4$, the sign is not clear, but it is interesting to note the effect of the loss function's output weight a: as this variable increases, both r_- and r_+ decrease, and so the range of θ in which $\frac{\partial \sigma_y^2}{\partial \theta}$ is positive becomes smaller. $\|$

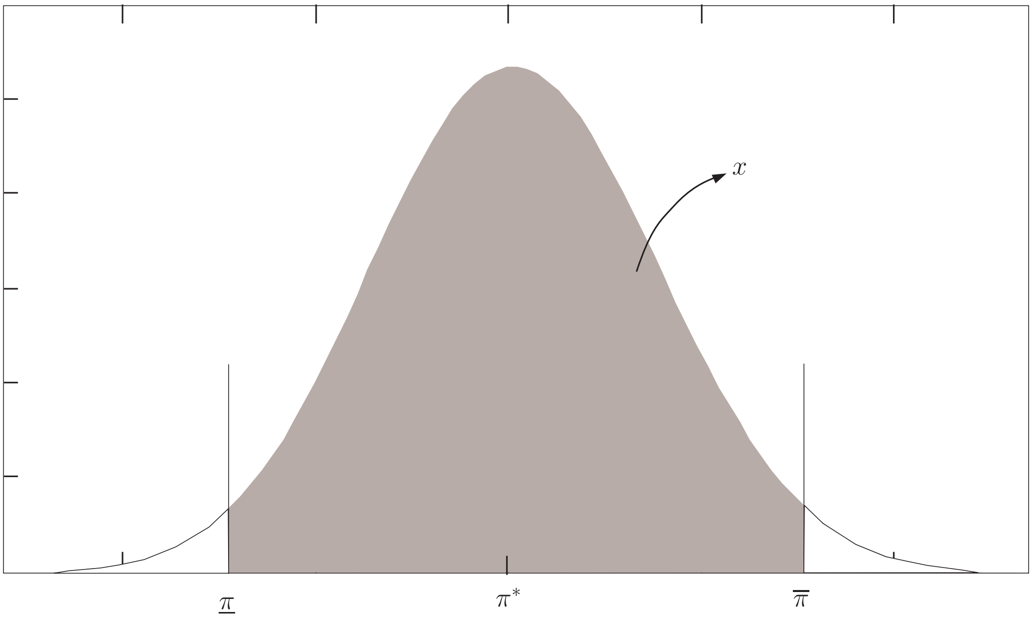

Figure 1: Distribution of Inflation

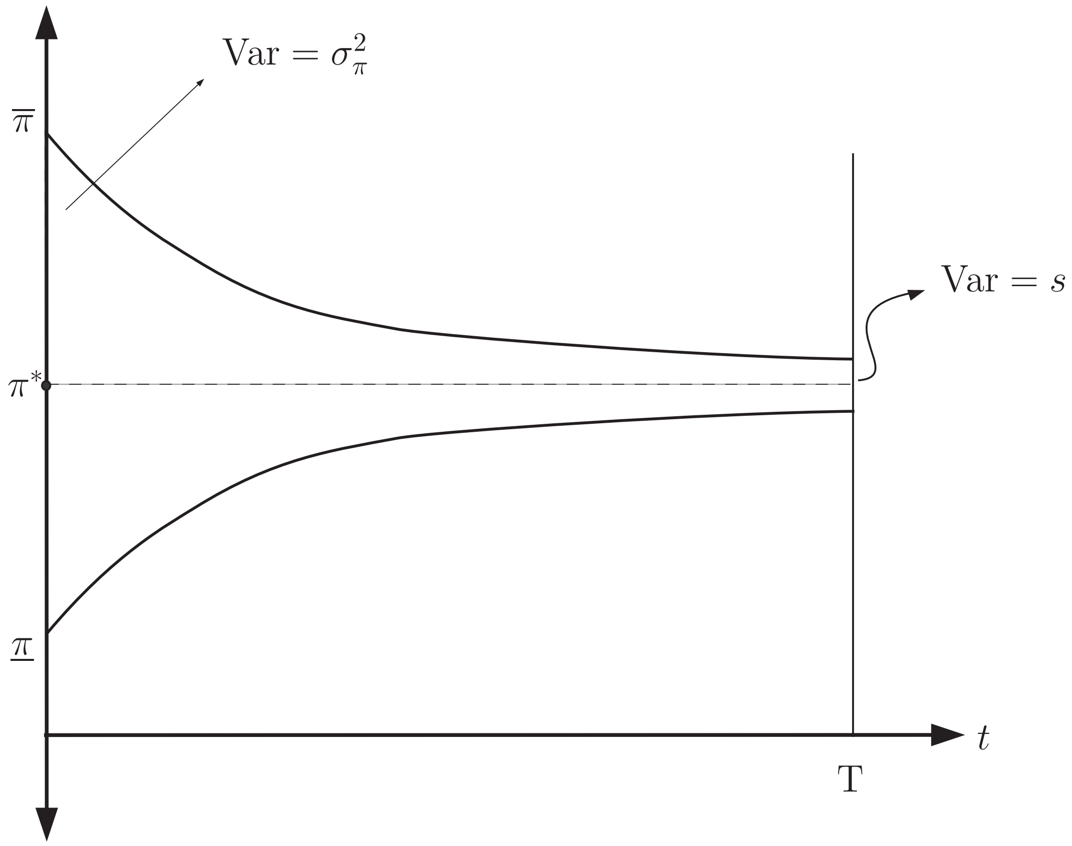

Figure 2: Inflation Target and Policy Horizon