1 Problem 1

Consider the game with the following payoff matrix.

	A	B
A	$0, 0$	$1, x_2$
B	$x_1, 1$	$0, 0$

The value of x_i is observable to player $i \in \{1, 2\}$ only. Suppose that x_1 and x_2 are independently and uniformly distributed in the interval $[-1, 1]$.

1. Show that there is no Bayesian equilibrium where one of the players always plays B; i. e. where, for some $i \in \{1, 2\}$, $s_i(x_i) = B$ for all $x_i \in [-1, 1]$.

2. Show that the following pair of strategies is a Bayesian equilibrium, (s_1, s_2) with $s_1(x_1) = A$ and

$$s_2(x_2) = \begin{cases} A & \text{if } x_2 \leq 0 \\ B & \text{otherwhise} \end{cases}$$

3. Find at least another Bayesian equilibrium.

2 Problem 2, Open Rule in US Congress

A policy $a \in R$ has to be decided. An informed Committee makes a recommendation, but the Congress might make amendments and, in practice, just decide any other policy ('Open Rule'). This is modelled as follows.
There are two players, the 'Committee' and the 'Floor' (who stands for the Congress median voter). The committee moves first and proposes a policy b. The Floor then decides the actual policy a. Further, the actual outcome is $x = a + \psi$, where ψ is a random variable uniformly distributed on the interval $[0, 1]$. The Committee knows ψ, but the Floor does not. Payoffs are $-(x - x_c)^2$ for the Committee (i. e. the Committees bliss point is $x = xc$) and $-x^2$ for the Floor (i. e. the Floors bliss point is $x = 0$).

1. Which kind of game is this?

2. Show that there always exists an equilibrium in which b is uninformative and $a = -\frac{1}{2}$. Is it a Perfect Bayesian equilibrium?

3. Look for an informative Perfect Bayesian Equilibria where the Committee 'reports low' when $\psi \in [0, k]$ and 'reports high' when $\psi \in]k, 1]$.

3 Problem 3

Sotheby's of London wants to sell a Rembrandt painting. Sotheby's knows that Bill Gates is interested. Sotheby's will make a 'take-it-or-leave-it' offer. Bill Gates' willingness to pay is unknown. Sotheby's estimates that with probability $\frac{1}{4}$ he is willing to spend 30 million dollars. With probability $\frac{3}{4}$ he is only willing to spend 10 million dollars.
Sotheby's contacts Bill Gates and asks whether he would be willing to pay 30 million dollars. Bill Gates can either say Yes or No (but he can lie, of course). After that, Sotheby's makes an offer of either 10 or 30 million dollars. Bill Gates will then purchase the picture if and only iff he is willing to accept the offer. Sotheby's payoffs are the received payments. Bill Gates' payoffs are his willingness to pay minus the price of the picture, whenever he purchases the picture. His payoffs are -1 million if he does not purchase the picture (these are the taxes he could have saved if he had purchased the Rembrandt).

1. How can you interpret this situation as a signalling game? Draw the game tree.

2. Find a Pooling Equilibrium, i.e. a Perfect Bayesian Equilibrium in which both types of Bill Gates give the same (pure) answer (Y or N).

3. Find a Separating Equilibrium, i.e. a Perfect Bayesian Equilibrium in which the two types of Bill Gates give different (pure) answers.

4 Problem 4

Consider a (horizontally) differentiated market environment with 2 firms. Market demand is implicitly defined by a system of linear and symmetric inverse demand functions

$$p_i = \alpha - x_i - \theta x_j$$

where x_i is the production of firm $i \in \{1, 2\}$, $\alpha > 0$ is a market size and the parameter θ describes the sensitivity of the firm's demand on the rival's production quantity. For simplicity, we suppose that the firms have no costs, $c = 0$.

1. Take $\alpha = 15$, $\theta = \frac{1}{2}$. Determine for each firm the best response function. Draw in a system of coordinates the best response functions of the firms where the quantity of firm 1 is depicted on the abscissa and the quantity of firm 2 is depicted on the ordinate. Find algebraically and graphically the Nash equilibrium $\tilde{x} = (\tilde{x_1}, \tilde{x_2})$ of this game! What quantities are produced by the firms in the Nash equilibrium? What profits are made by the firms?

2. Compare the Nash equilibrium you found in 2. with production profile $\hat{x} = (\hat{x_1}, \hat{x_2}) = (5, 5)$. Is the strategy profile $\hat{x} = (5, 5)$ a Nash equilibrium? Which production level $x \in \{\hat{x}, \tilde{x}\}$ gives higher profit to the firms? Is your finding in accordance with the theory? Explain your answer!

3. Suppose the firms choose their production levels sequentially. At first, firm 1 selects its production level, and then firm 2. When choosing its production level, firm 2 is informed about the choice of firm 1. Show that in the subgame perfect Nash equilibrium of this sequential game player 1 produces $\frac{3}{7}$ units more than in the NE you found in 2.!

   Suppose now that the sensitivity of demand parameter θ is a randomly distributed variable. The random variable θ has with probability $\frac{1}{2}$ the value 1 and with probability $\frac{1}{2}$ the value $\frac{1}{2}$. The realization of θ is not commonly known. Firm 1 observes the realization of the technology variable, but firm 2 doesnt.The timing of the game is identical to that described in part 3. For simplicity, we suppose that both firms are restricted to choose either production level 5 (low level) or production level 6 (high level).

4. Does a perfect Bayesian equilibrium exist at which firm 1 chooses the same action at every realization of θ? If such a (pooling) equilibrium exists, describe it precisely! Explain your answer!

5. Does a perfect Bayesian equilibrium exist at which firm 1 chooses different actions at different realizations of θ? If such a (separating) equilibrium exists, describe it precisely! Explain your answer!

5 Problem 5

Consider a firm employing two workers who produce an output according to the production function

$$f(e_1, e_2) = e_1 + e_2 + 2e_1e_2$$

where $e_1, e_2 \geq 0$ denote their eort levels. The firm pays to each worker $i \in \{1, 2\}$ a wage

$$w(e_1, e_2) = \frac{1}{2}f(e_1, e_2)$$

and each worker i has an utility function

$$U_i(e_1, e_2) = w(e_1, e_2) - e_i^2$$

1. Suppose the two workers choose simultaneously their effort level. Determine for each worker the best response function, and find the Nash equilibrium. Is the strategy profile $e = (1, 1)$ a Nash equilibrium? Explain your answer!

2. Suppose the workers choose simultaneously their effort level, but now interact infinitely, where both workers discount their future wages with the discount factor $\delta \in (0, 1)$. It is assumed that the workers use a trigger strategies which induce the outcome that both workers choose an effort level of 1 at each period and which trigger an endless punishment in form of an effort level of $\frac{1}{2}$in case of a deviation from it. Describe formally the

trigger strate- gies! Calculate the discount factor $\overline{\delta}$ such that the trigger strategies constitute a subgame perfect Nash equilibrium, if and only if $\delta \geq \overline{\delta}$

3. Suppose the workers choose sequentially their effort level. At first, worker 1 selects his effort level, and then worker 2. When choosing his effort level, worker 2 is informed about the choice of worker 1. Determine the subgame perfect Nash equilibrium for this sequential game! Which method do you use in solving this game?

   Suppose now the production function takes the form

$$f(e_1, e_2) = \theta(e_1 + e_2 + 2e_1e_2)$$

   where θ is a randomly distributed technology variable. The random variable θ has with probability $\frac{1}{2}$the value 1 and with probability $\frac{1}{2}$ the value $\frac{1}{2}$. The realization of θ is not commonly known. Worker 1 observes the realization of the technology variable, but worker 2 doesn't. The timing of the game is identical to that described in part 3. For simplicity, we suppose that both workers are restricted to choose either the effort level 0 (not working) or the effort level 1 (working).

4. Does a perfect Bayesian equilibrium exist at which worker 1 chooses the same action at every realization of θ? If such a (pooling) equilibrium exists, describe it precisely! Explain your answer!

5. Does a perfect Bayesian equilibrium exist at which worker 1 chooses different actions at different realizations of θ? If such a (separating) equilibrium exists, describe it precisely! Explain your answer!