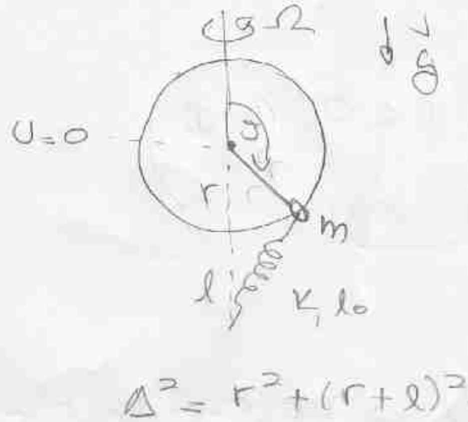
Pauta P3 - Examen 2007/02



$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\,\mathrm{sen}\,\theta\,\hat{\phi}$ (con $\dot{r} = 0$)

$$E = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\Omega^2\,\mathrm{sen}^2\theta) + mgr\cos\theta + \frac{1}{2}K(\Delta - l_0)^2 \quad (1)$$

$$\Delta^2 = r^2 + (r+l)^2 - 2r(r+l)\cos(\pi - \theta)$$
$$= 2r^2 + l^2 + 2rl + 2r(r+l)\cos\theta$$

$$E = \frac{1}{2}mr^2\dot{\theta}^2 + V_{eff}(\theta) \quad (1)$$

$$V_{eff}(\theta) = \frac{1}{2}mr^2\Omega^2\,\mathrm{sen}^2\theta + mgr\cos\theta + \frac{1}{2}K\left[\sqrt{r^2 + (r+l)^2 + 2r(r+l)\cos\theta} - l_0\right]^2$$

$$\frac{\partial V_{eff}}{\partial\theta} = mr^2\Omega^2\,\mathrm{sen}\,\theta\cos\theta - mgr\,\mathrm{sen}\,\theta + K\left[\sqrt{r^2 + (r+l)^2 + 2r(r+l)\cos\theta} - l_0\right]$$
$$\times \frac{1}{2}\left(r^2 + (r+l)^2 + 2r(r+l)\cos\theta\right)^{-1/2} \times -2r(r+l)\,\mathrm{sen}\,\theta$$

$$= \mathrm{sen}\,\theta\left\{mr^2\Omega^2\cos\theta - mgr - Kr(r+l)\left[1 - l_0\left(r^2 + (r+l)^2 + 2r(r+l)\cos\theta\right)^{-1/2}\right]\right\} \quad (1)$$

$$\frac{\partial V_{eff}}{\partial\theta}(\theta = \pi) = 0$$

$$\frac{\partial^2 V_{eff}}{\partial\theta^2} = \cos\theta\{\%\} + \mathrm{sen}\,\theta\left\{-mr^2\Omega^2\,\mathrm{sen}\,\theta + Kr(r+l)\frac{l_0}{2}\left[r^2 + (r+l)^2 + 2r(r+l)\cos\theta\right]^{-3/2} \times 2r(r+l)\,\mathrm{sen}\,\theta\right\} \quad (1)$$

$$\frac{\partial^2 V_{eff}}{\partial\theta^2}(\theta = \pi) = -\left\{-mr^2\Omega^2 - mgr - Kr(r+l)\left[1 - l_0\left(r^2 + (r+l)^2 - 2r(r+l)\right)^{-1/2}\right]\right\}$$
$$= mr^2\Omega^2 + mgr + Kr(r+l)\left[1 - l_0\left((r - (r+l))^2\right)^{-1/2}\right]$$
$$= mr^2\Omega^2 + mgr + Kr(r+l)\left[1 - l_0/l\right]$$

Inestable

$\Rightarrow \frac{\partial^2 V_{eff}}{\partial \vartheta^2}(\vartheta = \pi) < 0$

$\Rightarrow mr^2\Omega^2 + mgr + kr(r+l)(1 - l_0/l) < 0$ ①

$$\Omega^2 < \frac{(l_0/l - 1)kr(r+l) - mgr}{mr^2}$$

$$-\frac{(l_0/l - 1)kr(r+l) + mgr}{mr^2} < \Omega < \frac{(l_0/l - 1)kr(r+l) - mgr}{mr^2}$$

$$\boxed{\Omega_{min} = \frac{mgr - (l_0/l - 1)kr(k+l)}{mr^2}}$$ ①

De otra forma...

$$\mathcal{L} = \frac{1}{2} m (r^2\dot{\vartheta}^2 + r^2\Omega^2 \operatorname{sen}^2\vartheta) - mgr\cos\vartheta - \frac{1}{2}K(\Delta - l_0)^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} = m r^2 \dot{\vartheta}$$

$$\frac{\partial \mathcal{L}}{\partial \vartheta} = mr^2\Omega^2 \operatorname{sen}\vartheta\cos\vartheta + mgr\operatorname{sen}\vartheta - K(\Delta - l_0)\frac{\partial \Delta}{\partial \vartheta}$$

$$\frac{\partial \Delta}{\partial \vartheta} = \frac{+2r(r+l)\operatorname{sen}\vartheta}{2\sqrt{r^2 + (r+l)^2 - 2r(r+l)\cos(\pi-\vartheta)}}$$

$$\Rightarrow mr^2\ddot{\vartheta} - mr^2\Omega^2\operatorname{sen}\vartheta\cos\vartheta + mgr\operatorname{sen}\vartheta + K(\Delta - l_0)\frac{r(r+l)\operatorname{sen}\vartheta}{\Delta} = 0$$

$$\operatorname{sen}(\pi-\vartheta) = \operatorname{sen}\vartheta \simeq (\pi-\vartheta)$$
$$\cos(\pi-\vartheta) = -\cos\vartheta \simeq 1 - \frac{(\pi-\vartheta)^2}{2}$$

$$\Rightarrow mr^2\ddot{\vartheta} + mr^2\Omega^2(\pi-\vartheta) + mgr(\pi-\vartheta) + \frac{K(\hat{\Delta}-l_0)}{\Delta_*} r(r+l)(\pi-\vartheta) = 0$$

$$\vartheta_* = \pi - \vartheta$$

$$\ddot{\vartheta}_* + \underbrace{\left[\frac{mr^2\Omega^2 + mgr + K\left(1 - \frac{l_0}{\Delta_*}\right)r(r+l)}{mr^2}\right]}_{\omega^2}\vartheta_* = 0$$

$$\omega^2 = \frac{mr^2\Omega^2 + mgr + K\left(1 - \frac{l_0}{l}\right)r(r+l)}{mr^2} < 0$$

$$\Rightarrow \Omega^2 < \frac{\left(\frac{l_0}{l} - 1\right)Kr(r+l) - mgr}{mr^2}$$