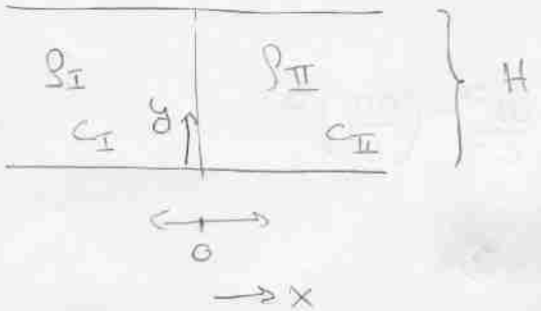
Pauta P1 - Examen 2007/02



$$\vec{v}(x,y,t) = \nabla\phi(x,y,t)$$

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

a) Tenemos $\phi(x,y,t) = f(x)\,g(y)\,\tau(t)$

$$\Rightarrow \underbrace{\frac{f''}{f}}_{-k_x^2} + \underbrace{\frac{g''}{g}}_{-k_y^2} = \frac{1}{c^2}\underbrace{\frac{\tau''}{\tau}}_{-\omega^2} \equiv -\frac{\omega^2}{c^2}$$

$$f(x) = Ae^{ik_x x} + Be^{-ik_x x}$$

$$g(y) = Ce^{ik_y y} + De^{-ik_y y} \qquad 0{,}5$$

$$\tau(t) = Ee^{-i\omega t}$$

Cond. de borde: $\nabla\phi\cdot\hat{n} = 0$ (en las paredes)

$$\Rightarrow \nabla\phi\cdot\hat{j} = 0 \quad \text{en } y=0,\ y=H$$

$$\Updownarrow$$

$$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right)\cdot\hat{j} = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 0 \qquad ik_y(C-D) = 0 \Rightarrow \boxed{C=D}$$

$$ik_y\left(Ce^{ik_yH} - De^{-ik_yH}\right) = 0$$

$$ik_yC\underbrace{\left(e^{ik_yH} - e^{-ik_yH}\right)}_{2i\,\mathrm{sen}(k_yH)} = 0$$

$$2i\,\mathrm{sen}(k_yH) = 0 \qquad 0{,}5$$

$$k_yH = n\pi \Rightarrow \boxed{k_y = \frac{n\pi}{H}}$$

$$g(y) = c\left(e^{ik_y y} + e^{-ik_y y}\right)$$
$$= \tilde{c}\cos(k_y y)$$

Luego $k_x^2 = \frac{\omega^2}{c^2} - k_y^2 = \frac{\omega^2}{c^2} - \left(\frac{n\pi}{H}\right)^2$

$$\boxed{k_n^2 = \frac{\omega^2}{c^2} - \left(\frac{n\pi}{H}\right)^2} \quad (1)$$

$$\Rightarrow \phi(x,y,t) = \left(A_n e^{ik_n x} + B_n e^{-ik_n x}\right)\cos\left(\frac{n\pi y}{H}\right)e^{-i\omega t}$$

b) Tenemos la ec. de Euler

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{\rho}\nabla p + \cancel{f}$$

Consideremos una pequeña perturbación

$$\rho = \rho_0 + \rho' \qquad p = p_0 + p' \qquad v' = v$$

↳ en el equilibrio

$$\Rightarrow \frac{\partial v'}{\partial t} + \cancel{(v'\cdot\nabla)v'} = -\frac{1}{\rho_0}\nabla p'$$

Despreciamos términos de mayor orden

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0}\nabla p'$$

Como $v = \nabla\phi \Rightarrow \frac{\partial \nabla\phi}{\partial t} = -\frac{1}{\rho_0}\nabla p'$

$$\frac{\partial \nabla\phi}{\partial t} = \nabla\left(\frac{\partial\phi}{\partial t}\right) \Rightarrow \boxed{p' = -\rho_0\frac{\partial\phi}{\partial t}}$$

Condiciones de borde en $x = 0$

$$\vec{v}(x=0^-)\cdot\hat{n} = \vec{v}(x=0^+)\cdot\hat{n} \quad \text{donde } \hat{n} = \hat{x}$$

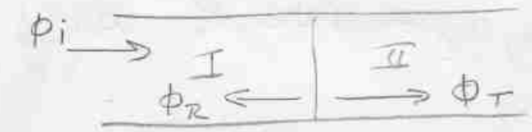
$$\Rightarrow \boxed{\frac{\partial\phi}{\partial x}(x=0^-) = \frac{\partial\phi}{\partial x}(x=0^+)} \quad 0{,}5$$

Además la presión debe ser igual en $x=0$

$$p(x=0^-) = p(x=0^+)$$

$$\Rightarrow \boxed{\rho_I \frac{\partial\phi}{\partial t}(x=0^-) = \rho_{II}\frac{\partial\phi}{\partial t}(x=0^+)} \quad 0{,}5$$

c)



Tenemos
$$\phi_I = \phi_i + \phi_R$$
$$\phi_{II} = \phi_T$$

$$\phi_i(x,y,t) = A_1 e^{i(k_I x - \omega t)}\cos\left(\frac{\pi y}{H}\right) \quad \leftarrow \text{enunciado}$$
$$\phi_R(x,y,t) = B_1 e^{-i(k_I x + \omega t)}\cos\left(\frac{\pi y}{H}\right)$$
$$\phi_T(x,y,t) = C_1 e^{i(k_{II} x - \omega t)}\cos\left(\frac{\pi y}{H}\right)$$
con $n = 1$!

Pues de la parte

$$\phi_I = \left[A_1 e^{i(k_I x - \omega t)} + B_1 e^{-i(k_I x + \omega t)}\right]\cos\left(\frac{\pi y}{H}\right)$$

$$\phi_{II} = C_1 e^{i(k_{II} x - \omega t)}\cos\left(\frac{\pi y}{H}\right) \quad \leftarrow \text{onda en 1 sola dirección}$$

$\Rightarrow$ Todas tienen la dependencia en y de $\cos\left(\frac{\pi y}{H}\right)$

1

Imponemos las cond. de borde de la parte b

$$\frac{\partial \Phi_I}{\partial x}(x=0) = \frac{\partial \Phi_{II}}{\partial x}(x=0)$$

$$ik_I\left[A_1 - B_1\right]\cancel{e^{-i\omega t}}\cancel{\cos\left(\frac{\pi y}{H}\right)} = ik_{II}C_1\cancel{e^{-i\omega t}}\cancel{\cos\left(\frac{\pi y}{H}\right)}$$

$$\boxed{1 - \frac{B_1}{A_1} = \frac{k_{II}}{k_I}\frac{C_1}{C_A}} \quad 1$$

$$\rho_I\frac{\partial \Phi_I}{\partial t}(x=0) = \rho_{II}\frac{\partial \Phi_{II}}{\partial t}(x=0)$$

$$-i\omega\left[A_1 + B_1\right]\cancel{e^{-i\omega t}}\cancel{\cos\left(\frac{\pi y}{H}\right)}\rho_I = -i\omega C_1\cancel{e^{-i\omega t}}\cancel{\cos\left(\frac{\pi y}{H}\right)}\rho_{II}$$

$$\boxed{1 + \frac{B_1}{A_1} = \frac{C_1}{A_1}\frac{\rho_{II}}{\rho_I}} \quad 2$$

$$1+2 \Rightarrow 2 = \frac{C_1}{A_1}\left(\frac{\rho_{II}}{\rho_I} + \frac{k_{II}}{k_I}\right),$$

$$y \quad \frac{k_{II}^2}{k_I^2} = \left(\frac{\omega^2}{C_{II}^2} - \frac{\pi^2}{H^2}\right) \Big/ \left(\frac{\omega^2}{C_I^2} - \frac{\pi^2}{H^2}\right)$$

$$= \frac{\omega^2H^2 - \pi^2C_{II}^2}{C_{II}^2\cancel{H^2}} \quad \frac{C_I^2\cancel{H^2}}{\omega^2H^2 - \pi^2C_I^2}$$

$$= \left(\frac{C_I}{C_{II}}\right)^2 \frac{\omega^2H^2 - \pi^2C_{II}^2}{\omega^2H^2 - \pi^2C_I^2}$$

$$\Rightarrow \boxed{\frac{C_1}{A_1} = 2\frac{\rho_I k_I}{\rho_{II}k_I + \rho_I k_{II}}} \qquad 0,5$$

$$(1)\cdot\frac{\rho_{II}}{\rho_I} - (2)\,\frac{k_{II}}{k_I} \Rightarrow \frac{\rho_{II}}{\rho_I} - \frac{k_{II}}{k_I} - \frac{B_1}{A_1}\left(\frac{\rho_{II}}{\rho_I} + \frac{k_{II}}{k_I}\right) = 0$$

$$\frac{B_1}{A_1} = \left(\frac{\rho_{II}}{\rho_I} - \frac{k_{II}}{k_I}\right) \Big/ \left(\frac{\rho_{II}}{\rho_I} + \frac{k_{II}}{k_I}\right)$$

$$= \frac{\rho_{II} k_I - k_{II}\rho_I}{\cancel{\rho_I k_I}} \; \frac{\cancel{\rho_I k_I}}{\rho_{II} k_I + k_{II}\rho_I}$$

$$\boxed{\frac{B_1}{A_1} = \frac{\rho_{II} k_I - k_{II}\rho_I}{\rho_{II} k_I + k_{II}\rho_I}} \quad 0,5$$

No era necesario hacer esto!

Luego tenemos la potencia transmitida

$$\vec{P} = -\rho_0 \nabla\phi \frac{\partial\phi}{\partial t} = \vec{v}\,p' \quad \rightarrow \text{vector}$$

En general

$$\phi = \phi_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \omega = c|\vec{k}|$$

$$\vec{v} = \phi_0 i\vec{k} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad p' = +\rho_0 i\omega\phi_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\langle\vec{P}\rangle = Re\{\vec{v}\}\cdot Re\{p'\}$$

$$\langle\vec{P}\rangle = \frac{1}{2} Re\{\vec{v}\cdot p'^{*}\} \quad \rightarrow \text{conjugado}$$

$$= \frac{1}{2} Re\{+\phi_0^2 \rho_0 \omega \vec{k}\}$$

$$= \frac{1}{2}\rho_0 \omega k |\phi_0|^2 \hat{k}$$

$$\Rightarrow \quad R = \frac{\langle\vec{P}_R\rangle}{\langle\vec{P}_i\rangle} = \frac{|B_1|^2}{|A_1|^2} \qquad T = \frac{\langle\vec{P}_T\rangle}{\langle\vec{P}_i\rangle} = \frac{\rho_{II} k_{II}}{\rho_I k_I}\frac{|C_1|^2}{|A_1|^2}$$

$$\Rightarrow \quad R = \left(\frac{\rho_{II} k_I - \rho_I k_{II}}{\rho_{II} k_I + \rho_I k_{II}}\right)^2 \qquad T = \frac{2\rho_I k_I \, \rho_{II} k_{II}}{(\rho_{II} k_I + \rho_I k_{II})^2}$$

Notar: $R + T = 1$ ✓✓

d) Si $\frac{\omega^2}{C_{II}^2} < \frac{\pi^2}{H^2} < \frac{\omega^2}{C_I^2} \iff \frac{\omega^2}{C_{II}^2} - \frac{\pi^2}{H^2} < 0 < \frac{\omega^2}{C_I^2} - \frac{\pi^2}{H^2}$

$\iff k_{II}^2 < 0 < k_I^2$

$\Rightarrow$ k_{II} es imaginario puro
k_I es real

$$\Rightarrow \left|\frac{B}{A}\right|^2 = \left|\frac{\rho_{II} k_I - \rho_I k_{II}}{\rho_{II} k_I + \rho_I k_{II}}\right|^2$$

$$= \left[\frac{\rho_{II} k_I - \rho_I k_{II}}{(\rho_{II} k_I - \rho_I k_{II})^*}\right]^2 \quad \text{complejo conjugado}$$

$$= 1$$