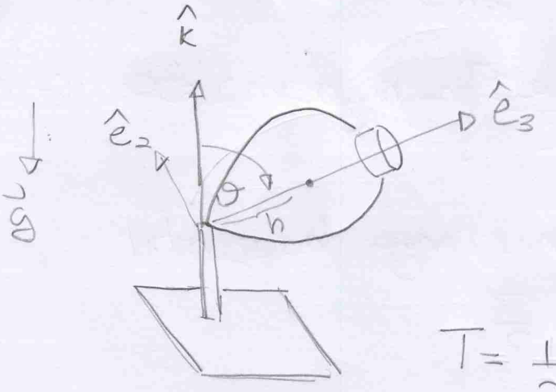
Pauta C2 - P1




$$\vec{\omega} = \dot{\theta}\hat{e}_1 + \dot{\phi}\hat{k} + \dot{\psi}\hat{e}_3$$

$$\hat{k} = \operatorname{sen}\theta\,\hat{e}_2 + \cos\theta\,\hat{e}_3$$

$$\Rightarrow \vec{\omega} = \dot{\theta}\hat{e}_1 + \dot{\phi}\operatorname{sen}\theta\,\hat{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3$$

$$T = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \operatorname{sen}^2\theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \operatorname{sen}^2\theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = cte \equiv \boxed{P_\phi = I_1 \dot{\phi}\operatorname{sen}^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta} \quad (1)$$

$$\frac{\partial L}{\partial \psi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\psi}} = cte \equiv \boxed{P_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta)} \quad (2)$$

$$\Rightarrow P_\phi = I_1 \dot{\phi}\operatorname{sen}^2\theta + P_\psi \cos\theta$$

$$\boxed{\dot{\phi} = \frac{P_\phi - P_\psi\cos\theta}{I_1 \operatorname{sen}^2\theta}} \quad (2 \text{ ptos})$$

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \left(\frac{P_\phi - P_\psi\cos\theta}{I_1\operatorname{sen}^2\theta}\right)^2 \operatorname{sen}^2\theta + \frac{1}{2}\frac{P_\psi^2}{I_3} + mgh\cos\theta$$

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2}\frac{(P_\phi - P_\psi\cos\theta)^2}{I_1\operatorname{sen}^2\theta} + \frac{1}{2}\frac{P_\psi^2}{I_3} + mgh\cos\theta$$

Potencial efectivo $\quad V_{eff} = \dfrac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\operatorname{sen}^2\theta} + \dfrac{P_\psi^2}{2I_3} + mgh\cos\theta$

Ptos de equilibrio $\rightarrow \left.\dfrac{\partial V_{eff}}{\partial \theta}\right|_{\theta^*} = 0$

$$\Rightarrow \frac{\cancel{2}(P_\phi - P_\psi \cos\theta) P_\psi \operatorname{sen}\theta}{\cancel{2} I_1 \operatorname{sen}^2\theta} - \frac{(P_\phi - P_\psi \cos\theta)^2}{\cancel{2} I_1} \frac{\cancel{2}\cos\theta}{\operatorname{sen}^3\theta} - mgh \operatorname{sen}\theta = 0$$

$$\frac{(P_\phi - P_\psi \cos\theta)}{I_1 \operatorname{sen}\theta} \left[ P_\psi - \frac{(P_\phi - P_\psi \cos\theta)\cos\theta}{\operatorname{sen}^2\theta} \right] - mgh \operatorname{sen}\theta = 0 \quad (2 \text{ ptos})$$

Para que el eje de simetría se mantenga horizontal el punto de equilibrio debe ser $\theta = \frac{\pi}{2}$

$$\Rightarrow \frac{P_\phi}{I_1} P_\psi - mgh = 0$$

$$\boxed{P_\phi P_\psi = I_1 mgh}$$

$$\text{y} \quad \left.\begin{aligned} P_\phi &= I_1 \dot\phi \\ P_\psi &= I_3 \dot\psi \end{aligned}\right\} \text{ de 1 y 2 en } \theta = \frac{\pi}{2}$$

$$\Rightarrow I_1 I_3 \dot\phi \dot\psi = I_1 mgh$$

$$\boxed{\dot\phi \dot\psi = \frac{mgh}{I_3}}$$

condición para que se mantenga horizontal (2 ptos)

# $P_1$ - OTRA FORMA !

$$L = \frac{1}{2} I_1 (\dot{\vartheta}^2 + \dot{\phi}^2 \operatorname{sen}^2\vartheta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\vartheta)^2 - mgh\cos\vartheta$$

$$\dot{\phi} = \frac{P_\phi - P_\psi \cos\vartheta}{I_1 \operatorname{sen}^2\vartheta}$$

$$P_\phi = I_1 \dot{\phi} \operatorname{sen}^2\vartheta + I_3 P_\psi \cos\vartheta$$

$$P_\psi = I_3 (\dot{\psi} + \dot{\phi}\cos\vartheta)$$

$$\frac{\partial L}{\partial \vartheta} = I_1 \dot{\phi}^2 \operatorname{sen}\vartheta\cos\vartheta + I_3(\dot{\psi} + \dot{\phi}\cos\vartheta)(-\dot{\phi}\operatorname{sen}\vartheta) + mgh\operatorname{sen}\vartheta$$

$$\Rightarrow I_1\ddot{\vartheta} - I_1\dot{\phi}^2\operatorname{sen}\vartheta\cos\vartheta + I_3\dot{\phi}\operatorname{sen}\vartheta\overbrace{(\dot{\psi} + \dot{\phi}\cos\vartheta)}^{P_\psi/I_3} - mgh\operatorname{sen}\vartheta = 0$$

Para que el trompo se mantenga horizontal

$$\dot{\vartheta} = \ddot{\vartheta} = 0 \quad y \quad \vartheta = \frac{\pi}{2}$$

$$\Rightarrow \quad I_3 \dot{\phi} \cancel{\operatorname{sen}\frac{\pi}{2}} \frac{P_\psi}{I_3} - mgh \cancel{\operatorname{sen}\frac{\pi}{2}} = 0$$

$$\dot{\phi} P_\psi = mgh$$

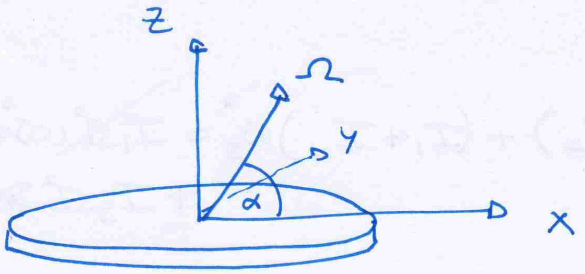
Pero $\quad P_\psi = I_3 \dot{\psi}$

$$\Rightarrow \quad I_3 \dot{\phi} \dot{\psi} = mgh$$

$$\boxed{\dot{\phi}\dot{\psi} = \frac{mgh}{I_3}}$$

Pauta C2 – P2

Ejes $(X, Y, Z)$ solidarios al plato




$$\vec{w} = w_1 \hat{\imath} + w_2 \hat{\jmath} + w_3 \hat{k}$$

En $t=0$ $\quad \vec{w} = \vec{\Omega} = \Omega(\cos\alpha\,\hat{\imath} + \operatorname{sen}\alpha\,\hat{k})$

$$\Rightarrow \quad w_1 = \Omega\cos\alpha \qquad w_2 = 0 \qquad w_3\,\Omega\operatorname{sen}\alpha$$

Ecuaciones de euler:

$$I_1\dot{w}_1 - (I_2 - I_3)w_2w_3 = 0 \qquad (1)$$
$$I_2\dot{w}_2 - (I_3 - I_1)w_1w_3 = 0 \qquad (2)$$
$$I_3\dot{w}_3 - (I_1 - I_2)w_1w_2 = 0 \qquad (3)$$

Como $\quad I_3 = I_1 + I_2$

$$(1) \Rightarrow \quad \dot{w}_1 + w_2w_3 = 0 \qquad / w_1$$
$$(2) \Rightarrow \quad \dot{w}_2 - w_1w_3 = 0 \qquad / w_2$$

$$(+) \quad w_1\dot{w}_1 + w_2\dot{w}_2 = 0$$

$$\frac{d}{dt}\left(\frac{w_1^2}{2} + \frac{w_2^2}{2}\right) = 0$$

$$\Rightarrow \quad w_1^2 + w_2^2 = cte \underset{(t=0)}{=} \Omega^2\cos^2\alpha + 0$$

$$\boxed{w_1^2 + w_2^2 = \Omega^2\cos^2\alpha} \quad *$$

(2)

$$(1)\cdot w_1 + (2)\,w_2 + (3)\cdot w_3$$

$$\Rightarrow \quad I_1w_1\dot{w}_1 + I_2w_2\dot{w}_2 + I_3w_3\dot{w}_3 = 0$$

$$\frac{1}{2}\frac{d}{dt}\left(I_1w_1^2 + I_2w_2^2 + I_3w_3^2\right) = 0$$

$$I_1w_1^2 + I_2w_2^2 + I_3w_3^2 = cte$$

En t=0

$$\Rightarrow I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = I_1\Omega^2\cos^2\alpha + I_3\Omega^2\sin^2\alpha$$

$$\star \Rightarrow \omega_1^2 = \Omega^2\cos^2\alpha - \omega_2^2$$

$$\Rightarrow \cancel{I_1\Omega^2\cos^2\alpha} - \omega_2^2(I_1 - I_2) + (I_1+I_2)\omega_3^2 = \cancel{I_1\Omega^2\cos^2\alpha} + I_3\Omega^2\sin^2\alpha$$

$$\omega_2^2(I_2-I_1) + \omega_3^2(I_1+I_2) = (I_1+I_2)\Omega^2\sin^2\alpha$$

$$\boxed{\omega_3^2 = \Omega^2\sin^2\alpha - \frac{(I_2-I_1)\,\omega_2^2}{I_1+I_2}} \quad \star\star \quad (2)$$

Teníamos $\dot\omega_2 = \omega_1\omega_3$

$$(\star) y (\star\star) \Rightarrow \dot\omega_2 = \sqrt{\Omega^2\cos^2\alpha - \omega_2^2}\sqrt{\Omega^2\sin^2\alpha - \frac{(I_2-I_1)\,\omega_2^2}{I_1+I_2}}$$

usando $\frac{I_1}{I_2} = \cos 2\alpha$

$$\frac{I_2-I_1}{I_1+I_2} = \frac{1 - I_1/I_2}{I_1/I_2 + 1} = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} = \frac{1-(\cos^2\alpha - \sin^2\alpha)}{1+(\cos^2\alpha - \sin^2\alpha)}$$

$$= \frac{\cancel{\cos^2\alpha} + \sin^2\alpha - (\cancel{\cos^2\alpha} - \sin^2\alpha)}{\cos^2\alpha + \cancel{\sin^2\alpha} + (\cos^2\alpha - \cancel{\sin^2\alpha})} = \frac{\cancel{2}\sin^2\alpha}{\cancel{2}\cos^2\alpha}$$

$$\Rightarrow \dot\omega_2 = \sqrt{\underbrace{\Omega^2\cos^2\alpha - \omega_2^2}_{\Omega^2\cos^2\alpha\left(1 - \frac{\omega_2^2}{\Omega^2\cos^2\alpha}\right)}}\sqrt{\underbrace{\Omega^2\sin^2\alpha - \frac{\sin^2\alpha\,\omega_2^2}{\cos^2\alpha}}_{\Omega^2\sin^2\alpha\left(1 - \frac{\omega_2^2}{\Omega^2\cos^2\alpha}\right)}}$$

$$\Rightarrow \dot\omega_2 = \Omega^2\sin\alpha\cos\alpha\left(1 - \frac{\omega_2^2}{\Omega^2\cos^2\alpha}\right)$$

$$u \equiv \frac{\omega_2}{\Omega \cos\alpha} \qquad \dot{u} = \frac{\dot{\omega}_2}{\Omega \cos\alpha}$$

$$\Rightarrow \quad \dot{u}\, \Omega \cos\alpha = \Omega^2 \operatorname{sen}\alpha \cos\alpha \,(1 - u^2)$$

$$\frac{du}{dt} = \Omega \operatorname{sen}\alpha\,(1 - u^2)$$

$$\Omega \operatorname{sen}\alpha \int_0^t dt = \int_0^u \frac{du}{1-u^2}$$

$$\Omega \operatorname{sen}\alpha\, t = \operatorname{arctanh} u$$

$$u = \tanh(\Omega t \operatorname{sen}\alpha)$$

$$\Rightarrow \quad \omega_2(t) = u(t)\, \Omega \cos\alpha$$

$$\boxed{\omega_2(t) = \Omega \cos\alpha \tanh(\Omega t \operatorname{sen}\alpha)}$$

2