7 Vibrations of Coupled Mass Points

As the first and most simple system of vibrating mass points, we consider the free vibration of two mass points, fixed to two walls by springs of equal spring constant, as is shown in the Figure 7.1.

The two mass points shall have equal masses. The displacements from the rest positions are denoted by x_1 and x_2, respectively. We consider only vibrations along the line connecting the mass points.

When displacing the mass 1 from the rest position, there acts the force $-kx_1$ by the spring fixed to the wall, and the force $+k(x_2 - x_1)$ by the spring connecting the two mass points. Thus, the mass point 1 obeys the equation of motion

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1). \tag{7.1a}$$

Analogously, for the mass point 2 we have

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1). \tag{7.1b}$$

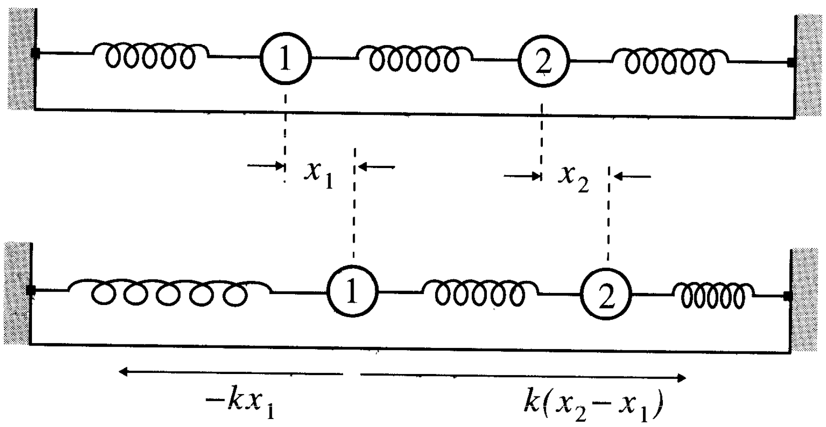

Figure 7.1. Mass points coupled by springs.

We first determine the possible frequencies of common vibration of the two particles. *The frequencies that are equal for all particles are called eigenfrequencies.* The related vibrational states are called eigen- or normal vibrations. These definitions are correspondingly generalized for a N-particle system. We use the *ansatz*

$$x_1 = A_1 \cos \omega t, \qquad x_2 = A_2 \cos \omega t, \tag{7.2}$$

i.e., both particles shall vibrate with the same frequency ω. The specific type of the *ansatz,* be it a sine or cosine function or a superposition of both, is not essential. We would always get the same condition for the frequency, as can be seen from the following calculation.

Insertion of the *ansatz* into the equations of motion yields two linear homogeneous equations for the amplitudes:

$$\begin{aligned} A_1(-m\omega^2 + 2k) - A_2 k &= 0, \\ -A_1 k + A_2(-m\omega^2 + 2k) &= 0. \end{aligned} \tag{7.3}$$

The system of equations has nontrivial solutions for the amplitudes only if the determinant of coefficients D vanishes:

$$D = \begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{vmatrix} = (-m\omega^2 + 2k)^2 - k^2 = 0.$$

We thus obtain an equation for determining the frequencies:

$$\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k^2}{m^2} = 0.$$

The positive solutions of the equation are the frequencies

$$\omega_1 = \sqrt{\frac{3k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k}{m}}\,.$$

These frequencies are called *eigenfrequencies* of the system; the corresponding vibrations are called *eigenvibrations* or *normal vibrations.* To get an idea about the type of the normal vibrations, we insert the eigenfrequency into the system (7.3). For the amplitudes, we find

$$A_1 = -A_2 \qquad \text{for} \quad \omega_1 = \sqrt{\frac{3k}{m}}$$

and

$$A_1 = A_2 \qquad \text{for} \quad \omega_2 = \sqrt{\frac{k}{m}}\,.$$

The two mass points vibrate in-phase with the lower frequency ω_2, and with the higher frequency ω_1 against each other. The two vibration modes are illustrated by Figure 7.2.

The number of normal vibrations equals the number of coordinates (degrees of freedom) which are necessary for a complete description of the system. This is a consequence of the fact that for N degrees of freedom there appear N equations of the kind (7.2) and N equations of motion of the kind (7.1a),(7.1b). This leads to a determinant of rank N for ω^2, and therefore in general to N normal frequencies. Since we have restricted ourselves in the

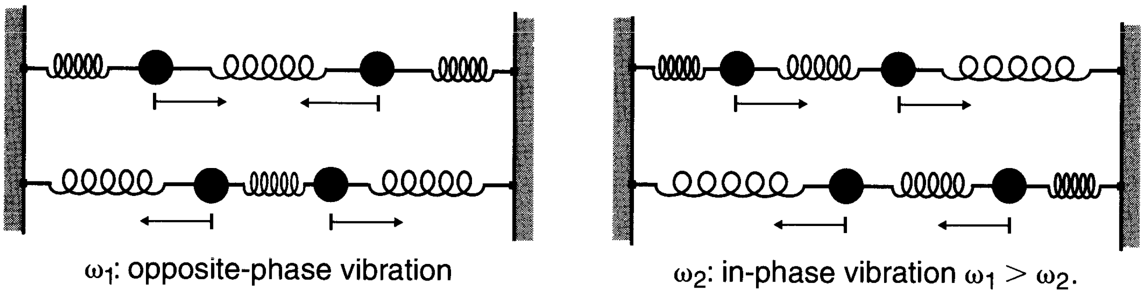

Figure 7.2.

example to the vibrations along the x-axis, the two coordinates x_1 and x_2 are sufficient to describe the system, and we obtain the two eigenvibrations with the frequencies ω_1, ω_2.

In our example, the normal vibrations mean in-phase or opposite-phase (= in-phase with different sign of the amplitudes) oscillations of the mass points. The amplitudes of equal size are related to the equality of masses ($m_1 = m_2$). The general motion of the mass points corresponds to a superposition of the normal modes with different phase and amplitude.

The differential equations (7.1a),(7.1b) are linear. The general form of the vibration is therefore the superposition of the normal modes. It reads

$$\begin{aligned} x_1(t) &= C_1\cos(\omega_1 t+\varphi_1)+C_2\cos(\omega_2 t+\varphi_2),\\ x_2(t) &= -C_1\cos(\omega_1 t+\varphi_1)+C_2\cos(\omega_2+\varphi_2). \end{aligned} \tag{7.4}$$

Here, we already utilized the result that x_1 and x_2 have opposite-equal amplitudes for a pure ω_1-vibration, and equal amplitudes for pure ω_2-vibrations. This ensures that the special cases of the pure normal vibrations with $C_2 = 0,\ C_1 \neq 0$ and $C_1 = 0,\ C_2 \neq 0$ are included in the *ansatz* (7.4). Equation (7.4) is the most general *ansatz* since it involves 4 free constants. Thus one can incorporate any initial values for $x_1(0), x_2(0), \dot{x}_1(0), \dot{x}_2(0)$.

For example, the initial conditions are

$$x_1(0) = 0, \qquad x_2(0) = a, \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

To determine the 4 free constants $C_1,\ C_2,\ \varphi_1,\ \varphi_2$, we insert the initial conditions into the equations (7.4) and their derivatives:

$$x_1(0) = C_1\cos\varphi_1 + C_2\cos\varphi_2 = 0, \tag{7.5}$$

$$x_2(0) = -C_1\cos\varphi_1 + C_2\cos\varphi_2 = a, \tag{7.6}$$

$$\dot{x}_1(0) = -C_1\omega_1\sin\varphi_1 - C_2\omega_2\sin\varphi_2 = 0, \tag{7.7}$$

$$\dot{x}_2(0) = C_1\omega_1\sin\varphi_1 - C_2\omega_2\sin\varphi_2 = 0. \tag{7.8}$$

Addition of (7.7) and (7.8) yields

$$C_2\sin\varphi_2 = 0.$$

Subtraction of (7.7) and (7.8) yields

$$C_1\sin\varphi_1 = 0.$$

From addition and subtraction of (7.5) and (7.6), it follows that

$$2C_2\cos\varphi_2 = a \quad \text{and} \quad 2C_1\cos\varphi_1 = -a.$$

Thus, one obtains

$$\varphi_1 = \varphi_2 = 0, \qquad C_1 = -\frac{a}{2}, \qquad C_2 = \frac{a}{2}.$$

The overall solution therefore reads

$$x_1(t) = \frac{a}{2}(-\cos\omega_1 t + \cos\omega_2 t) = a\sin\left(\frac{\omega_1-\omega_2}{2}\right)t\,\sin\left(\frac{\omega_1+\omega_2}{2}\right)t,$$

$$x_2(t) = \frac{a}{2}(\cos\omega_1 t + \cos\omega_2 t) = a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\,\cos\left(\frac{\omega_1+\omega_2}{2}\right)t.$$

For $t = 0 : x_1(0) = 0, x_2(0) = a$, as required. The second mass plucks at the first one and causes it to vibrate. These are *beat vibrations* (see Example 7.2).

Example 7.1: Exercise: Two equal masses coupled by two equal springs

Two equal masses move without friction on a plate. They are connected to each other and to the wall by two springs, as is indicated by Figure 7.3. The two spring constants are equal, and the motion shall be restricted to a straight line (one-dimensional motion).

Find

(a) the equations of motion,

(b) the normal frequencies, and

(c) the amplitude ratios of the normal vibrations and the general solution.

Solution (a) Let x_1 and x_2 be the displacements from the rest positions. The equations of motion then read

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1), \tag{7.9}$$

$$m\ddot{x}_2 = -k(x_2 - x_1). \tag{7.10}$$

(b) For determining the normal frequencies, we use the *ansatz*

$$x_1 = A_1\cos\omega t, \qquad x_2 = A_2\cos\omega t$$

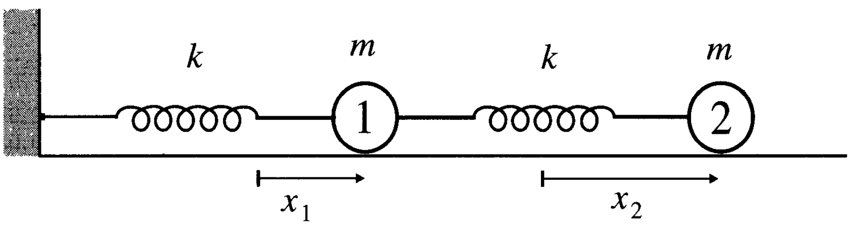

Figure 7.3. Two equal masses coupled by two equal springs.

and thereby get from (7.9) and (7.10) the equations

$$(2k - m\omega^2)A_1 - kA_2 = 0,$$
$$-kA_1 + (k - m\omega^2)A_2 = 0. \qquad \textbf{(7.11)}$$

From the requirement for nontrivial solutions of the system of equations, it follows that the determinant of coefficients vanishes:

$$D = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0.$$

From this follows the determining equation for the eigenfrequencies,

$$\omega^4 - 3\frac{k}{m}\omega^2 + \frac{k^2}{m^2} = 0,$$

with the positive solutions

$$\omega_1 = \frac{\sqrt{5}+1}{2}\sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \frac{\sqrt{5}-1}{2}\sqrt{\frac{k}{m}}, \qquad \omega_1 > \omega_2.$$

(c) By inserting the eigenfrequencies in (7.11) one sees that the higher frequency ω_1 corresponds to the opposite-phase mode, and the lower frequency ω_2 to the equal-phase normal vibration:

$$\text{with} \quad \omega_1^2 = \frac{1}{2}(3+\sqrt{5})\frac{k}{m}, \quad \text{it follows from (7.11) that} \quad A_2 = -\frac{\sqrt{5}-1}{2}A_1,$$
$$\text{with} \quad \omega_2^2 = \frac{1}{2}(3-\sqrt{5})\frac{k}{m}, \quad \text{it follows from (7.11) that} \quad A_2 = \frac{\sqrt{5}+1}{2}A_1.$$

Since the two mass points are fixed in different ways, we find amplitudes of different magnitudes.

The general solution is obtained as a superposition of the normal vibrations, using the calculated amplitude ratios:

$$x_1(t) = C_1\cos(\omega_1 t + \varphi_1) + C_2\cos(\omega_2 t + \varphi_2),$$
$$x_2(t) = -\frac{\sqrt{5}-1}{2}C_1\cos(\omega_1 t + \varphi_1) + \frac{\sqrt{5}+1}{2}C_2\cos(\omega_2 t + \varphi_2).$$

The 4 free constants are determined from the initial conditions of the specific case.

Example 7.2: Exercise: Coupled pendulums

Two pendulums of equal mass and length are connected by a spiral spring. They vibrate in a plane. The coupling is weak (i.e., the two eigenmodes are not very different). Find the motion with small amplitudes.

Solution The initial conditions are

$$x_1(0) = 0, \qquad x_2(0) = A, \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

We start from the vibrational equation of the simple pendulum:

$$ml\ddot{\alpha} = -mg\sin\alpha.$$

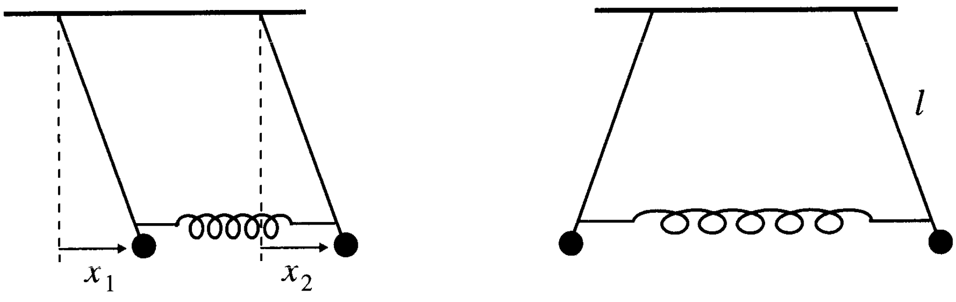

Figure 7.4.

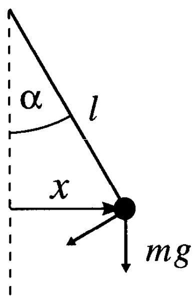

Figure 7.5.

For small amplitudes, we set $\sin\alpha = \alpha = x/l$ and obtain

$$m\ddot{x} = -m\frac{g}{l}x.$$

For the coupled pendulums, the force $\mp k(x_1 - x_2)$ caused by the spring still enters, which leads to the equations

$$\ddot{x}_1 = -\frac{g}{l}x_1 - \frac{k}{m}(x_1 - x_2),$$
$$\ddot{x}_2 = -\frac{g}{l}x_2 + \frac{k}{m}(x_1 - x_2). \tag{7.12}$$

This coupled set of differential equations can be decoupled by introducing the coordinates

$$u_1 = x_1 - x_2 \quad \text{and} \quad u_2 = x_1 + x_2.$$

Subtraction and addition of the equations (7.12) yield

$$\ddot{u}_1 = -\frac{g}{l}u_1 - 2\frac{k}{m}u_1 = -\left(\frac{g}{l} + 2\frac{k}{m}\right)u_1,$$
$$\ddot{u}_2 = -\frac{g}{l}u_2.$$

These two equations can be solved immediately:

$$u_1 = A_1\cos\omega_1 t + B_1\sin\omega_1 t,$$
$$u_2 = A_2\cos\omega_2 t + B_2\sin\omega_2 t, \tag{7.13}$$

where $\omega_1 = \sqrt{g/l + 2(k/m)}$, $\omega_2 = \sqrt{g/l}$ are the eigenfrequencies of the two vibrations. The coordinates u_1, u_2 are called *normal coordinates*. Normal coordinates are often introduced to decouple a coupled system of differential equations. The coordinate $u_1 = x_1 - x_2$ describes the opposite-phase and $u_2 = x_1 + x_2$ the equal-phase normal vibration. The equal-phase normal mode proceeds as if the coupling were absent.

For sake of simplicity, we incorporate the initial conditions in the system (7.13). For the normal coordinates we then have

$$u_1(0) = -A, \qquad u_2(0) = A, \qquad \dot{u}_1(0) = \dot{u}_2(0) = 0.$$

Insertion into (7.13) yields

$$A_1 = -A, \qquad A_2 = A, \qquad B_1 = B_2 = 0,$$

and thus,

$$u_1 = -A\cos\omega_1 t, \qquad u_2 = A\cos\omega_2 t.$$

Returning to the coordinates x_1 and x_2:

$$x_1 = \frac{1}{2}(u_1 + u_2) = \frac{A}{2}(-\cos\omega_1 t + \cos\omega_2 t),$$
$$x_2 = \frac{1}{2}(u_2 - u_1) = \frac{A}{2}(\cos\omega_1 t + \cos\omega_2 t).$$

After transforming the angular functions, one has

$$x_1 = A\sin\left(\frac{\omega_1 - \omega_2}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right),$$
$$x_2 = A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right).$$

We have presupposed the coupling of the two pendulums to be weak, i.e.,

$$\omega_2 = \sqrt{\frac{g}{l}} \approx \omega_1 = \sqrt{\frac{g}{l} + 2\frac{k}{m}},$$

hence, the frequency $\omega_1 - \omega_2$ is small. The vibrations $x_1(t)$ and $x_2(t)$ can then be interpreted as follows: The amplitude factor of the pendulum vibrating with the frequency $\omega_1 + \omega_2$ is slowly modulated by the frequency $\omega_1 - \omega_2$. This process is called *beat vibration*. Figure 7.6 illustrates the process. The two pendulums exchange their energy with the amplitude modulation frequency $\omega_1 - \omega_2$. If one pendulum reaches its maximum amplitude (energy), the other pendulum comes to rest. This complete energy transfer occurs only for identical pendulums. If the pendulums differ in mass or length, the energy transfer becomes incomplete; the pendulums vary in amplitudes but without coming to rest.

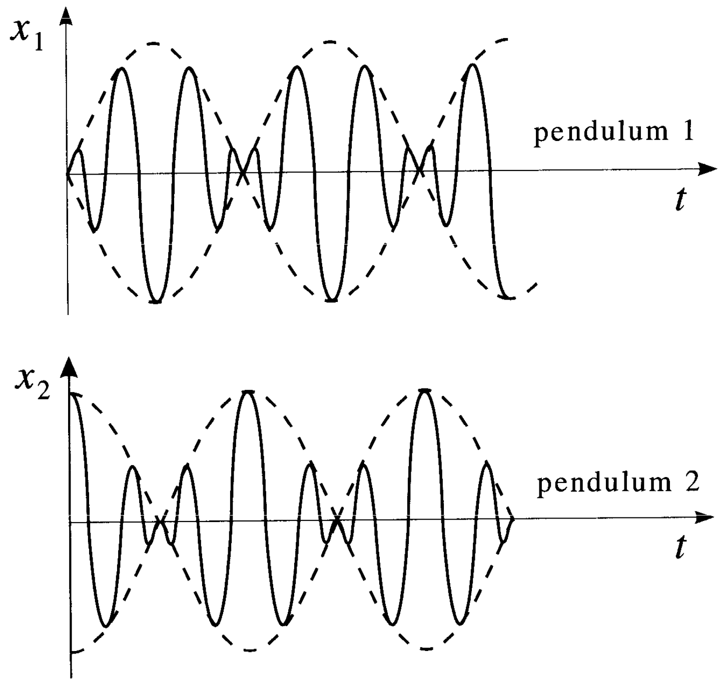

Figure 7.6.

The vibrating chain[1]

We consider another vibrating mass system: the vibrating chain. The "chain" is a massless thread set with N mass points. All mass points have the mass m and are fixed to the thread at equal distances a. The points 0 and $N+1$ at the ends of the thread are tightly fixed and do not participate in the vibration. The displacement from the rest position in y-direction is assumed to be relatively small, so that the minor displacement in x-direction is negligible. The total string tension T is only due to the clamping of the end points and is constant over the entire thread.

If one picks out the νth particle, the forces acting on this particle are due to the displacements of the particles $(\nu-1)$ and $(\nu+1)$. According to Figure 7.7 the backdriving forces are given by

$$\mathbf{F}_{\nu-1} = -(T\cdot\sin\alpha)\mathbf{e}_2,$$
$$\mathbf{F}_{\nu+1} = -(T\cdot\sin\beta)\mathbf{e}_2.$$

Since the displacement in y-direction is small by definition, α and β are small angles, and

[1] It is recommended that the reader go through Chapter 8 ("The Vibrating String") before studying this section. The concepts presented here will be more easily understood, and the mathematical approaches will be more transparent in their physical motivation.

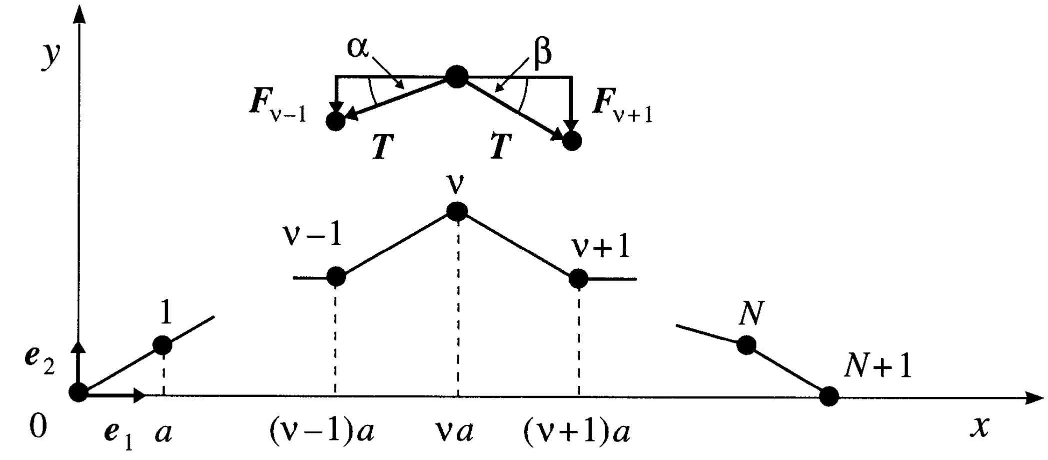

Figure 7.7.

hence, one has to a good approximation

$$\sin\alpha = \tan\alpha \quad \text{and} \quad \sin\beta = \tan\beta.$$

From Figure 7.7, one sees that

$$\tan\alpha = \frac{y_\nu - y_{\nu-1}}{a} \quad \text{and} \quad \tan\beta = \frac{y_\nu - y_{\nu+1}}{a}.$$

Hence, the forces are given by

$$\mathbf{F}_{\nu-1} = -T\left(\frac{y_\nu - y_{\nu-1}}{a}\right)\mathbf{e}_2,$$
$$\mathbf{F}_{\nu+1} = -T\left(\frac{y_\nu - y_{\nu+1}}{a}\right)\mathbf{e}_2.$$

The total backdriving force is the sum $\mathbf{F}_{\nu-1} + \mathbf{F}_{\nu+1}$, i.e., the equation of motion for the particle reads

$$m\frac{d^2y_\nu}{dt^2}\mathbf{e}_2 = -T\left(\frac{y_\nu - y_{\nu-1}}{a}\right)\mathbf{e}_2 - T\left(\frac{y_\nu - y_{\nu+1}}{a}\right)\mathbf{e}_2$$

or

$$\frac{d^2y_\nu}{dt^2} = \frac{T}{ma}(y_{\nu-1} - 2y_\nu + y_{\nu+1}). \qquad \textbf{(7.14)}$$

Since the index ν runs from $\nu = 1$ to $\nu = N$, one obtains a system of N coupled differential equations. Considering that the endpoints are fixed, by setting for the indices $\nu = 0$ and $\nu = N + 1$

$$y_0 = 0 \quad \text{and} \quad y_{N+1} = 0 \quad \text{(boundary condition)},$$

one obtains from the differential equation (7.14) with the indices $\nu = 1$ and $\nu = N$ the differential equation for the first and last particle that can participate in the vibration:

$$\begin{aligned} m\frac{d^2y_1}{dt^2} &= \frac{T}{a}(-2y_1 + y_2), \\ m\frac{d^2y_N}{dt^2} &= \frac{T}{a}(y_{N-1} - 2y_N). \end{aligned} \tag{7.15}$$

We now look for the eigenfrequencies of the particle system, i.e., the frequencies of vibration common to all particles. To get a determining equation for the eigenfrequency ω_n, we introduce in equation (7.14) the *ansatz*

$$y_\nu = A_\nu \cos\omega t. \tag{7.16}$$

We obtain

$$-m\omega^2 \cdot A_\nu \cdot \cos\omega t = \frac{T}{a}(A_{\nu-1} - 2A_\nu + A_{\nu+1})\cos\omega t,$$

and after rewriting,

$$-A_{\nu-1} + \left(2 - \frac{ma\omega^2}{T}\right)A_\nu - A_{\nu+1} = 0, \qquad \nu = 2, \dots, N-1. \tag{7.17a}$$

By insertion of (7.16) into (7.15), we get the equations for the first and the last vibrating particle:

$$\begin{aligned} \left(2 - \frac{ma\omega^2}{T}\right)A_1 - A_2 &= 0, \\ -A_{N-1} + \left(2 - \frac{ma\omega^2}{T}\right)A_N &= 0. \end{aligned} \tag{7.17b}$$

With the abbreviation

$$\frac{2T - ma\omega^2}{T} = c, \tag{7.18}$$

the equations (7.17a) and (7.17b) can be rewritten as follows:

$$\begin{aligned} cA_1 - A_2 \qquad\qquad &= 0, \\ -A_1 + cA_2 - A_3 \qquad &= 0, \\ - A_2 + cA_3 - A_4 &= 0, \\ \vdots \qquad\qquad &\;\;\vdots \\ - A_{N-1} + cA_N &= 0 \end{aligned}$$

This is a system of homogeneous linear equations for the coefficients A_ν. For any nontrivial solution of the equation system (not all $A_\nu = 0$) the determinant of coefficients must vanish. This determinant has the form

$$D_N = \begin{vmatrix} c & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & c & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & c & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & c & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & c \end{vmatrix} .$$

It has N rows and N columns. The eigenfrequencies are obtained as solution of the equation

$$D_N = 0.$$

Expanding D_N with respect to the first row, we get

$$D_N = c \cdot \begin{vmatrix} c & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & c & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & c & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & c & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & c & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & c \end{vmatrix}$$

$$+ \begin{vmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -c & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & c \end{vmatrix}$$

The left-hand determinant has exactly the same form as D_N, but is lower by one order ($N-1$ rows, $N-1$ columns). It would be the determinant of coefficients for a similar system with one mass point less, i.e., D_{N-1}. The right-hand determinant is now expanded with respect to the first column, which leads to

$$D_N = cD_{N-1} + (-1)\cdot \begin{vmatrix} c & -1 & 0 & \cdots & 0 & 0 \\ -1 & c & -1 & \cdots & 0 & 0 \\ 0 & -1 & c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & c & -1 \\ 0 & 0 & 0 & \cdots & -1 & c \end{vmatrix}.$$

The last determinant is just D_{N-2}. Hence we get the determinant recursion equation

$$D_N = cD_{N-1} - D_{N-2}, \qquad \text{if} \quad N \geq 2. \tag{7.19}$$

Moreover,

$$D_1 = |c| = c \quad \text{and} \quad D_2 = \begin{vmatrix} c & -1 \\ -1 & c \end{vmatrix} = c^2 - 1. \tag{7.20}$$

By setting $N = 2$ in (7.19), we recognize that (7.19) combined with (7.20) is satisfied only if we formally set

$$D_0 = 1. \tag{7.21}$$

Our problem is now to solve the determinant equation (7.19). We use the *ansatz*

$$D_N = p^N,$$

where the constant p must be determined. Insertion into (7.19) yields

$$p^N = cp^{N-1} - p^{N-2},$$

and after division by p^{N-2},

$$p^2 - cp + 1 = 0 \quad \text{or} \quad p = \frac{c \pm \sqrt{c^2 - 4}}{2}.$$

The mathematical possibility $p^{N-2} = 0$ that leads to $p \equiv 0$ does not obey the boundary condition $D_0 = 1$ and is therefore inapplicable. Substituting $c = 2\cos\Theta$, we obtain for p

$$p = \cos\Theta \pm \sqrt{\cos^2\Theta - 1} = \cos\Theta \pm i\sin\Theta = e^{\pm i\Theta}.$$

The solutions of equation (7.19) are then

$$D_N = p^N = (e^{i\Theta})^N = e^{iN\Theta} = \cos N\Theta + i\sin N\Theta$$

and

$$D_N = (e^{-i\Theta})^N = e^{-iN\Theta} = \cos N\Theta - i \sin N\Theta .$$

Since the equation system (7.19) is homogeneous and linear, the general solution is a linear combination of $\cos N\Theta$ and $\sin N\Theta$:

$$D_N = G \cos N\Theta + H \sin N\Theta . \tag{7.22}$$

Since $D_0 = 1$ and $D_1 = c = 2\cos\Theta$ (see above), G and H are determined as

$$G = 1, \qquad H = \cot\Theta ,$$

so that

$$D_N = \cos N\Theta + \frac{\sin N\Theta \cos\Theta}{\sin\Theta} = \frac{\sin(N+1)\Theta}{\sin\Theta},$$

because $\sin\Theta\cos N\Theta + \sin N\Theta\cos\Theta = \sin(N+1)\Theta$.

For any nontrivial solution of the equation system we must have $D_N = 0$, i.e., D_N must vanish for all N; it follows that

$$\sin((N+1)\Theta) = 0,$$

or

$$\Theta = \Theta_n = \frac{n\pi}{N+1}, \qquad n = 1, \ldots, N. \tag{7.23}$$

$n = 0$ drops out since it leads to the solution $\Theta_0 = 0$, and hence to $D_N = N + 1 \neq 0$, and thus does not lead to a solution of the equation $D_N = 0$. For c we then get according to (7.18):

$$c = 2 - \frac{\omega^2 ma}{T} = 2\cos\frac{n\pi}{N+1},$$

and ω is calculated from

$$\omega^2 = \omega^2_{(n)} = \frac{2T}{ma}\left(1 - \cos\frac{n\pi}{N+1}\right) \tag{7.24a}$$

as

$$\omega_{(n)} = \sqrt{\frac{2T}{ma}}\sqrt{1 - \cos\frac{n\pi}{N+1}}. \tag{7.24b}$$

These are the *eigenfrequencies of the system*; the fundamental frequency is obtained for $n = 1$ as the lowest eigenfrequency. There are exactly N eigenfrequencies, as is seen from (7.23): For $n \geq N + 1$, we set $n = (N + 1) + \tau$ and find

$$\Theta_n = \pi + \frac{\tau\pi}{N+1} .$$

If one inserts the above expression into equation (7.17a) and (7.17b) for ω and c, respectively, one obtains for the amplitudes of the normal vibration

$$-A^{(n)}_{\nu-1} + 2A^{(n)}_{\nu} \cos\frac{n\pi}{N+1} - A^{(n)}_{\nu+1} = 0,$$

$$2A^{(n)}_1 \cos\frac{n\pi}{N+1} = A^{(n)}_2, \qquad (7.25)$$

$$2A^{(n)}_N \cos\frac{n\pi}{N+1} = A^{(n)}_{N-1}$$

where the A_ν depend on n ($A_\nu = A^{(n)}_\nu$). *The system of equations (7.25) for the* A_ν *is the same as that for the determinants* D_N *(equation (7.19)), with the same coefficient* $c = 2\cos n\pi/(N+1) = 2\cos\Theta_n$. Only the boundary conditions (7.25) do not correspond to those for the D_N (see equations (7.20) and (7.21)). The general solution for the coefficients A_ν is therefore obtained from equation (7.22) with at first arbitrary coefficients $E^{(n)}$:

$$A^{(n)}_\nu = E^{(n)}_1 \cos\nu\Theta_n + E^{(n)}_2 \sin\nu\Theta_n,$$

or, in detail,

$$A^{(n)}_\nu = E^{(n)}_1 \cos\frac{n\pi\nu}{N+1} + E^{(n)}_2 \sin\frac{n\pi\nu}{N+1}. \qquad (7.26)$$

Since the points $\nu = 0$ and $\nu = N+1$ are tightly clamped, for all eigenmodes n we have $y_0 = y_{N+1} = 0$, or

$$A^{(n)}_0 = A^{(n)}_{N+1} = 0 \quad \text{(boundary condition)}.$$

Then one obtains for $\nu = 0$ in (7.26):

$$E^{(n)}_1 = 0, \quad \text{i.e.,} \quad A^{(n)}_\nu = E^{(n)}_2 \sin\frac{n\pi\nu}{N+1}.$$

After insertion into equation (7.16), one gets

$$y^{(n)}_\nu = E^{(n)}_2 \sin\frac{n\pi\nu}{N+1} \cos\omega_{(n)}t. \qquad (7.27)$$

If one inserts $y_\nu = B_\nu \sin\omega t$ instead of equation (7.16) into equation (7.14), one determines B_ν by the same method as A_ν and obtains

$$B^{(n)}_\nu = E^{(n)}_4 \sin\frac{n\pi\nu}{N+1}, \qquad (E^{(n)}_3 = 0);$$

hence, the solutions for the y_ν read

$$y^{(n)}_\nu = E^{(n)}_2 \sin\frac{n\pi\nu}{N+1} \cos\omega_{(n)}t \qquad (7.28a)$$

and

$$y^{(n)}_\nu = E^{(n)}_4 \sin\frac{n\pi\nu}{N+1} \sin\omega_{(n)}t\,. \qquad (7.28b)$$

The sum of these individual solutions yields the general solution, which therefore reads

$$y_\nu = \sum_{n=1}^{N} \sin\frac{n\pi\nu}{N+1}\left(E_4^{(n)}\sin\omega_{(n)}t + E_2^{(n)}\cos\omega_{(n)}t\right)$$
$$= \sum_{n=1}^{N} \sin\frac{n\pi\nu}{N+1}(a_n\sin\omega_{(n)}t + b_n\cos\omega_{(n)}t), \tag{7.29}$$

where the constants $E_2^{(n)}$ and $E_4^{(n)}$ were renamed b_n and a_n, respectively. They are determined from the initial conditions.

The equation of the vibrating chord must follow from the limit for $N \to \infty$ and $a \to 0$ (continuous mass distribution):

$$\sin\frac{n\pi\nu}{N+1} = \sin\frac{n\pi a\nu}{(N+1)a} \qquad (x_\nu = a\nu \text{ takes only discrete values})$$
$$= \sin\frac{\pi n(a\nu)}{l+a} \qquad (l = Na \text{ is the length of the chord})$$
$$\lim_{\substack{N\to\infty\\ a\to 0}}\left(\sin\frac{\pi nx}{l+a}\right) = \sin\frac{\pi nx}{l} \qquad (x \text{ continuous}).$$

$\omega_{(n)}^2$ becomes (expansion of the cosine in (7.24a) in a Taylor series):

$$\omega_{(n)}^2 = \frac{2T}{ma}\left(1 - 1 + \frac{1}{2}\left(\frac{n\pi}{N+1}\right)^2 - \cdots\right) \approx \frac{T(n\pi)^2}{(m/a)(N+1)^2a^2},$$

and with $\sigma = m/a$ = mass density of the chord,

$$\lim_{\substack{N\to\infty\\ a\to 0}}\left(\frac{T(n\pi)^2}{\sigma(N+1)^2a^2}\right) = \frac{T(n\pi)^2}{\sigma l^2},$$

i.e.,

$$\omega_{(n)} = \sqrt{\frac{T}{\sigma}}\frac{n\pi}{l}.$$

Hence, one has as a limit

$$y_n(x) = \sin\left(\frac{n\pi x}{l}\right)\left[a_n\sin\left(\sqrt{\frac{T}{\sigma}}\cdot\frac{n\pi}{l}t\right) + b_n\cos\left(\sqrt{\frac{T}{\sigma}}\frac{n\pi}{l}t\right)\right]. \tag{7.30}$$

This is the equation for the nth eigenmode of the vibrating chord (l is the chord length). It will be derived once again in the next chapter in a different way and will then be discussed in more detail.

Example 7.3: Exercise: Eigenfrequencies of the vibrating chain

When solving the determinant equation (7.19), we have made a mathematical restriction for c by setting $c = 2\cos\Theta$.

Show that for the cases

(a) $|c| = 2$,

(b) $c < -2$

the eigenvalue equation $D_N = 0$ cannot be satisfied. Clarify that thereby the special choice of the constant c is justified.

Solution (a)

$$D_n = cD_{n-1} - D_{n-2}, \qquad D_1 = c = \pm 2, \qquad D_0 = 1. \tag{7.31}$$

We assert and prove by induction

$$|D_n| \geq |D_{n-1}|. \tag{7.32}$$

Induction start: $n = 2$, $|D_0| = 1$, $|D_1| = 2$, $|D_2| = 3$.

Induction conclusion from $n-1$, $n-2$ to n:

$$\begin{aligned} |D_n|^2 &= 4|D_{n-1}|^2 \pm 4|D_{n-1}||D_{n-2}| + |D_{n-2}|^2 \\ &\geq 4|D_{n-1}|^2 + |D_{n-2}|^2 - 4|D_{n-1}||D_{n-2}| \end{aligned}$$

$$\Rightarrow \qquad |D_n|^2 - |D_{n-1}|^2 \geq 3|D_{n-1}|^2 + |D_{n-2}|^2 - 4|D_{n-1}||D_{n-2}|.$$

According to the induction condition,

$$|D_{n-1}| = |D_{n-2}| + \epsilon \quad \text{with} \quad \epsilon \geq 0.$$

From this, it follows that

$$\begin{aligned} |D_n|^2 - |D_{n-1}|^2 &\geq 4|D_{n-2}|^2 + 6\epsilon|D_{n-2}| + 3\epsilon^2 - 4\epsilon|D_{n-2}| - 4|D_{n-2}|^2 \\ &\geq 2\epsilon|D_{n-2}| \\ &\geq 0 \end{aligned}$$

$$\Rightarrow \qquad |D_n| \geq |D_{n-1}|. \tag{7.33}$$

Since $|D_n|$ monotonically increases in n, and $|D_1| = 2 > 0$, we have $|D_N| > 0$. Therefore $D_N = 0$ cannot be satisfied. $\omega = 0$ and $\omega = \sqrt{2T/ma}$ are not eigenfrequencies of the vibrating chain.

(b) By inserting the *ansatz* $D_n = Ap^n$, $p \neq 0$, we also find the solution of the recursion formula $D_n = cD_{n-1} - D_{n-2}$, $D_1 = c$, $D_0 = 1$:

$$\left.\begin{aligned} p_1 &= \tfrac{1}{2}\left(c + (c^2-4)^{1/2}\right) < 0 \\ p_2 &= \tfrac{1}{2}\left(c - (c^2-4)^{1/2}\right) < 0 \end{aligned}\right\} \quad 0 > p_1 > p_2. \tag{7.34}$$

The general solution for incorporating the boundary conditions $D_0 = 1$, $D_1 = c$ reads

$$D_n = A_1 p_1^n + A_2 p_2^n. \tag{7.35}$$

With $D_0 = 1,\ D_1 = c$, it follows that

$$A_1 + A_2 = 1,$$

$$\frac{A_1}{2}\left(c + (c^2-4)^{1/2}\right) + \frac{A_2}{2}\left(c - (c^2-4)^{1/2}\right) = c,$$

$$A_1 = \frac{c + (c^2-4)^{1/2}}{2(c^2-4)^{1/2}}, \quad \Leftrightarrow \quad A_2 = \frac{-c + (c^2-4)^{1/2}}{2(c^2-4)^{1/2}}. \tag{7.36}$$

One then has

$$\begin{aligned} D_n &= \frac{1}{2}\frac{c + (c^2-4)^{1/2}}{(c^2-4)^{1/2}} p_1^n + \frac{1}{2}\frac{(c^2-4)^{1/2} - c}{(c^2-4)^{1/2}} p_2^n \\ &= \frac{1}{(c^2-4)^{1/2}}\left(p_1^{n+1} - p_2^{n+1}\right). \end{aligned} \tag{7.37}$$

To determine the physically possible vibration modes, we had required that $D_N = 0$:

$$D_N = 0 \quad \Rightarrow \quad \left(\frac{p_2}{p_1}\right)^{N+1} = 1. \tag{7.38}$$

But now $0 > p_1 > p_2$, hence $(p_2/p_1)^{N+1} > 1$. Thus, for the case $c < -2$ eigenfrequencies do not exist too.

These supplementary investigations can be summarized as follows: The possible eigenfrequencies of the vibrating chain lie between 0 and $\sqrt{2T/ma}$:

$$0 < |\omega| < \sqrt{\frac{2T}{ma}}. \tag{7.39}$$

Example 7.4: Exercise: Vibration of two coupled mass points, two dimensional

Two mass points (equal mass m) lie on a frictionless horizontal plane and are fixed to each other and to two fixed points A and B by means of springs (spring tension T, length l).

(a) Establish the equation of motion.

(b) Find the normal vibrations and frequencies and describe the motions.

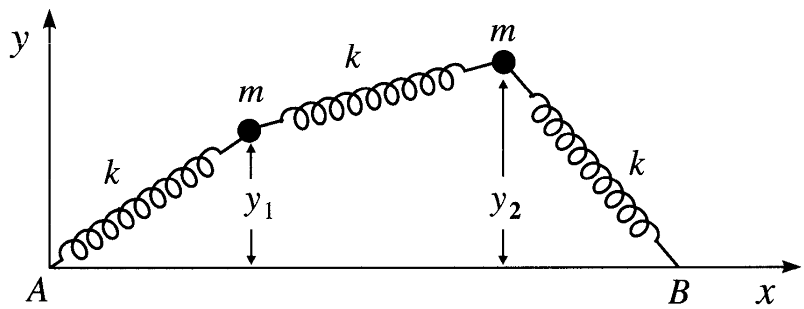

Figure 7.8.

Figure 7.9.

Solution (a) For the vibrating chain with n mass points, which are equally spaced by the distance l, the equations of motion

$$\frac{d^2y_N}{dt^2} = \frac{T}{ml}(y_{N-1} - 2y_N + y_{N+1}) \qquad (N = 1, \dots, n)$$

were established. For the first and second mass point, we have

$$\begin{aligned}\ddot{y}_1 &= k(y_0 - 2y_1 + y_2) = k(y_2 - 2y_1),\\ \ddot{y}_2 &= k(y_1 - 2y_2 + y_3) = k(y_1 - 2y_2)\end{aligned} \tag{7.40}$$

with $k = T/ml$; the chain is clamped at the points A and B, i.e., $y_0 = y_3 = 0$.

(b) Solution *ansatz*: $y_1 = A_1 \cos\omega t$, $y_2 = A_2 \cos\omega t$ (ω = eigenfrequency). Insertion into (7.40) yields

$$\begin{aligned}(2k - \omega^2)A_1 - kA_2 &= 0,\\ (2k - \omega^2)A_2 - kA_1 &= 0.\end{aligned} \tag{7.41}$$

To get the nontrivial solution, the determinant of coefficients must vanish, i.e.,

$$D = \begin{vmatrix} 2k - \omega^2 & -k \\ -k & 2k - \omega^2 \end{vmatrix} = 0;$$

i.e., $\omega^4 + 3k^2 - 4k\omega^2 = 0$, from which it follows that $\omega_1^2 = 3k$, $\omega_2^2 = k$.

Insertion in (7.41) yields $A_1 = A_2$ for ω_2 and $A_1 = -A_2$ for ω_1. This is an opposite-phase and an equal-phase vibration, respectively. We note that the vibration with the higher frequency has opposite phases and a "node," while the vibration with lower frequency has equal phases and a "vibration antinode."

Example 7.5: Exercise: Three masses on a string

Three mass points are fixed equidistantly on a string that is fixed at its endpoints.

(a) Determine the eigenfrequencies of this system if the string tension T can be considered constant (this holds for small amplitudes).

(b) Discuss the eigenvibrations of the system. *Hint*: Note Problems 8.1 and 8.2 in Chapter 8.

Solution (a) For the equations of motion of the system, one finds straightaway

$$m\ddot{x}_1 + \left(\frac{2T}{L}\right)x_1 - \left(\frac{T}{L}\right)x_2 = 0,$$

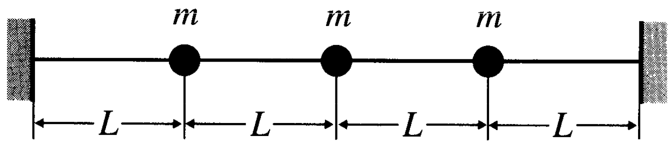

Figure 7.10.

$$m\ddot{x}_2 + \left(\frac{2T}{L}\right)x_2 - \left(\frac{T}{L}\right)x_3 - \left(\frac{T}{L}\right)x_1 = 0, \qquad (7.42)$$
$$m\ddot{x}_3 + \left(\frac{2T}{L}\right)x_3 - \left(\frac{T}{L}\right)x_2 = 0.$$

Assuming periodic oscillations, i.e., solutions of the form

$$x_1 = A\sin(\omega t + \psi) \qquad \ddot{x}_1 = -\omega^2 A\sin(\omega t + \psi),$$
$$x_2 = B\sin(\omega t + \psi) \qquad \ddot{x}_2 = -\omega^2 B\sin(\omega t + \psi),$$
$$x_3 = C\sin(\omega t + \psi) \qquad \ddot{x}_3 = -\omega^2 C\sin(\omega t + \psi),$$

we get after insertion into equation (7.42)

$$\left(\frac{2T}{L} - \omega^2 m\right)A - \left(\frac{T}{L}\right)B = 0,$$
$$-\left(\frac{T}{L}\right)A + \left(\frac{2T}{L} - \omega^2 m\right)B - \left(\frac{T}{L}\right)C = 0, \qquad (7.43)$$
$$-\left(\frac{T}{L}\right)B + \left(\frac{2T}{L} - \omega^2 m\right)C = 0.$$

As in Problem 8.2, one gets the equation for the frequencies of the system from the expansion of the determinant of coefficients:

$$\left(\frac{Lm}{T}\right)^3\omega^6 - 6\left(\frac{Lm}{T}\right)^2\omega^4 + \frac{10Lm}{T}\omega^2 - 4 = 0$$

or

$$\left(\frac{Lm}{T}\right)^3\Omega^3 - 6\left(\frac{Lm}{T}\right)^2\Omega^2 + \frac{10Lm}{T}\Omega - 4 = 0 \qquad (7.44)$$

with $\Omega \mathrel{\widehat{=}} \omega^2$. This cubic equation with the coefficients

$$a = \left(\frac{Lm}{T}\right)^3, \qquad b = -6\left(\frac{Lm}{T}\right)^2, \qquad c = \frac{10Lm}{T}, \qquad d = -4$$

can be solved by Cardano's method.

With the substitutions

$$y = \Omega + \frac{b}{3a}, \qquad 3p = -\frac{1}{3}\frac{b^2}{a^2} + \frac{c}{a} = -2\frac{T^2}{L^2m^2}, \qquad 2q = \frac{2}{27}\frac{b^3}{a^3} - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a} = 0$$

we get $q^2 + p^3 < 0$, i.e., there are three real solutions which by using the auxiliary quantities

$$\cos\varphi = -\frac{q}{\sqrt{-p^3}} = 0, \; y_1 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3} - \frac{\pi}{3}\right) = -\sqrt{2}\frac{T}{Lm},$$

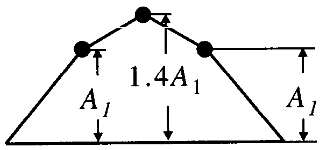

Figure 7.11.

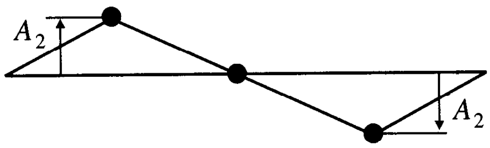

Figure 7.12.

$$y_2 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3}+\frac{\pi}{3}\right) = 0,$$

$$y_3 = 2\sqrt{-p}\cos\frac{\varphi}{3} = \sqrt{2}\frac{T}{Lm}$$

can be calculated as

$$\omega_1 = \sqrt{0.6\frac{T}{Lm}}, \qquad \omega_2 = \sqrt{\frac{2T}{Lm}}, \qquad \omega_3 = \sqrt{3.4\frac{T}{Lm}}.$$

(b) From the first and third equation of (7.43), one finds for the amplitude ratios

$$\frac{B}{A} = \frac{B}{C} = 2 - \frac{mL\omega^2}{T}. \qquad \textbf{(7.45)}$$

Discussion of the modes:

(1) $\omega = \omega_1 = (0.6T/Lm)^{1/2}$ inserted into (7.45) $\Rightarrow$ $B_1/A_1 = B_1/C_1 = 1.4$ or $B_1 = 1.4A_1 = 1.4C_1$.

All three masses are deflected in the same direction, where the first and third mass have equal amplitudes, and the second mass has a larger amplitude.

(2) $\omega = \omega_2 = (2T/Lm)^{1/2}$ inserted into (7.45) $\Rightarrow$ $B_2/A_2 = B_2/C_2 = 0$ and $A_2 = -C_2$ from the second equation of (7.43). The central mass is at rest, while the first and third mass are vibrating in opposite directions but with equal amplitude.

(3) $\omega = \omega_3 = (3.4T/Lm)^{1/2}$ inserted into (7.45) $\Rightarrow$ $B_3/A_3 = B_3/C_3 = -1.4$, i.e., $A_3 = C_3 = -1.4B_3$. The first and the last mass are deflected in the same direction, while the central mass vibrates with different amplitude in the opposite direction.

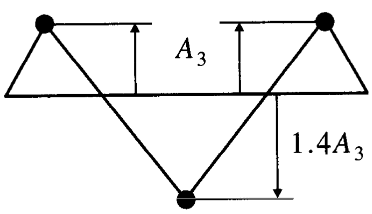

Figure 7.13.

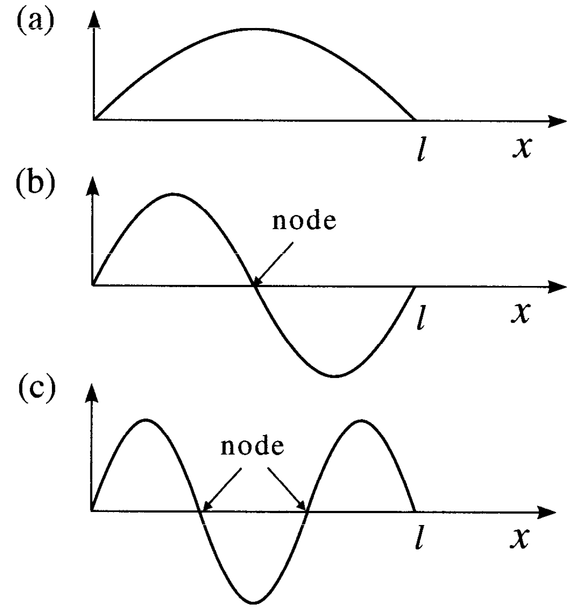

Figure 7.14.

The system discussed here has three vibration modes with 0, 1, and 2 nodes, respectively. For a system with n mass points, both the number of modes as well as the number of possible nodes $(n-1)$ increases. A system with $n \to \infty$ is called a "vibrating string."

A comparison of the figures clearly shows the approximation of the vibrating string by the system of three mass points.

Example 7.6: Exercise: Eigenvibrations of a three-atom molecule

Discuss the eigenvibrations of a three-atom molecule. In the equilibrium state of the molecule, the two atoms of mass m are in the same distance from the atom of mass M. For simplicity one should consider only vibrations along the molecule axis connecting the three atoms, where the complicated interatomic potential is approximated by two strings (with spring constant k).

(a) Establish the equation of motion.

(b) Calculate the eigenfrequencies and discuss the eigenvibrations of the system.

Solution (a) Let x_1, x_2, x_3 be the displacements of the atoms from the equilibrium positions at time t. From Newton's equations and Hooke's law then it follows that

$$m\ddot{x}_1 = -k(x_1 - x_2),$$

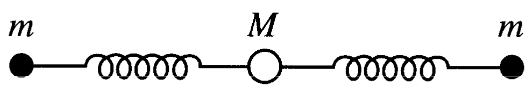

Figure 7.15.

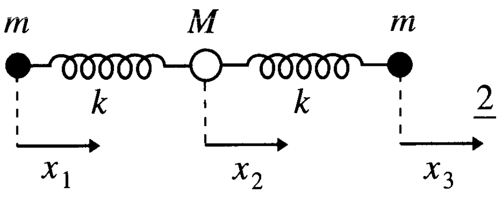

Figure 7.16.

$$M\ddot{x}_2 = -k(x_2 - x_3) - k(x_2 - x_1) = k(x_3 + x_1 - 2x_2), \qquad (7.46)$$
$$m\ddot{x}_3 = -k(x_3 - x_2).$$

(b) By inserting the *ansatz* $x_1 = a_1 \cos\omega t$, $x_2 = a_2 \cos\omega t$, and $x_3 = a_3 \cos\omega t$ into equation (7.46), one obtains

$$\begin{aligned} (m\omega^2 - k)a_1 + ka_2 &= 0, \\ ka_1 + (M\omega^2 - 2k)a_2 + ka_3 &= 0, \\ ka_2 + (m\omega^2 - k)a_3 &= 0. \end{aligned} \qquad (7.47)$$

The eigenfrequencies of this system are obtained by setting the determinant of coefficients equal to zero:

$$\begin{vmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{vmatrix} = 0. \qquad (7.48)$$

From this, it follows that

$$(m\omega^2 - k)[\omega^4 mM - \omega^2(kM + 2km)] = 0 \qquad (7.49)$$

or

$$\omega^2(m\omega^2 - k)[\omega^2 mM - k(M + 2m)] = 0.$$

By factorization of equation (7.49) with respect to ω, one obtains for the eigenvibrations of the system:

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{k}{m}}, \qquad \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}.$$

Discussion of the vibration modes:

(1) Insertion of $\omega = \omega_1 = 0$ into (7.47) yields $a_1 = a_2 = a_3$. The eigenfrequency $\omega_1 = 0$ does not correspond to a vibrational motion, but represents only a uniform translation of the entire molecule: •→ o→ •→.

(2) Inserting $\omega = \omega_2 = (k/m)^{1/2}$ into (7.47) yields $a_1 = -a_3$, $a_2 = 0$; i.e., the central atom is at rest, while the outer atoms vibrate against each other: ←• o •→.

(3) Inserting $\omega = \omega_3 = \{k/m(1 + 2m/M)\}^{1/2}$ into (7.47) yields $a_1 = a_3$, $a_2 = -(2m/M)a_1$, i.e., the two outer atoms vibrate in phase, while the central atom vibrates with opposite phase and with another amplitude: •→ ←o •→.

8 The Vibrating String

A string of length l is fixed at both ends. Thereby appear forces T that are constant in time and independent of the position. The string tension acts as a backdriving force when the string is displaced out of the rest position. A string element Δs at the position x experiences the force

$$F_y(x) = -T \sin \Theta(x)$$

in y-direction. At the position $x + \Delta x$ there acts in y-direction the force

$$F_y(x + \Delta x) = T \sin \Theta(x + \Delta x).$$

In y-direction, the string element Δs experiences the total force

$$F_y = T \sin \Theta(x + \Delta x) - T \sin \Theta(x). \tag{8.1}$$

Accordingly, along the x-direction the string element Δs is pulled by the force

$$F_x = T \cos \Theta(x + \Delta x) - T \cos \Theta(x).$$

In a first approximation we assume that the displacement in x-direction shall be zero. A displacement of the string in y-direction causes only a very small motion in the x-direction. This displacement is negligible compared to the displacement in y-direction, i.e.,

$$F_x = 0.$$

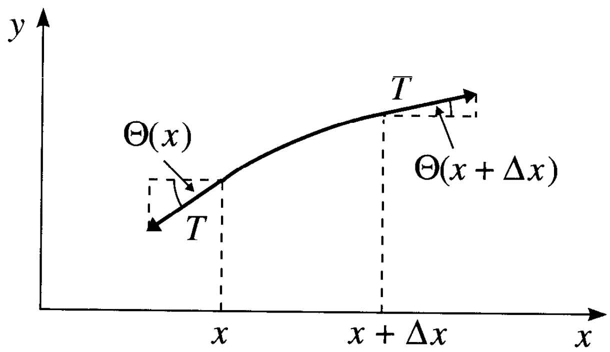

Figure 8.1. The string tension T.

Since we neglect the displacement in x-direction, the only acceleration component of the string element is given by $\partial^2 y/\partial t^2$. The mass of the element is $m = \sigma \Delta s$, where σ represents the line density. From that and by means of equation (8.1) we obtain the equation of motion:

$$F_y = \sigma \Delta s \frac{\partial^2 y}{\partial t^2} = T \sin \Theta(x + \Delta x) - T \sin \Theta(x). \tag{8.2}$$

Both sides are divided by Δx:

$$\frac{\sigma \Delta s d^2 y}{\Delta x dt^2} = \frac{T \sin \Theta(x + \Delta x) - T \sin \Theta(x)}{\Delta x}. \tag{8.3}$$

Inserting for Δs in the left-hand side of equation (8.3)

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2},$$

one has

$$\begin{aligned}\frac{\sigma \sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} \frac{\partial^2 y}{\partial t^2} &= \sigma \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \frac{\partial^2 y}{\partial t^2} \\ &= \frac{T \sin \Theta(x + \Delta x) - T \sin \Theta(x)}{\Delta x}.\end{aligned} \tag{8.4}$$

By forming the limit for Δx, $\Delta y \to 0$ on both sides of equation (8.4), we obtain

$$\sigma \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \frac{\partial^2 y}{\partial t^2} = T \frac{\partial}{\partial x} (\sin \Theta). \tag{8.5}$$

For $\sin \Theta$ we have $\sin \Theta = \tan \Theta / \sqrt{1 + \tan^2 \Theta}$. Since $\tan \Theta = \partial y/\partial x$ (inclination of the curve), we write

$$\sin \Theta = \frac{\partial y/\partial x}{\sqrt{1 + (\partial y/\partial x)^2}}. \tag{8.6}$$

By means of relation (8.6) the equation (8.5) can be transformed as follows:

$$\sigma \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \frac{\partial^2 y}{\partial t^2} = T \frac{\partial}{\partial x} \left(\frac{\partial y/\partial x}{\sqrt{1 + (\partial y/\partial x)^2}} \right). \tag{8.7}$$

In order to simplify the equation, we again consider only small displacements of the string in y-direction. Then $\partial y/\partial x \ll 1$, and $(\partial y/\partial x)^2$ can be neglected too.

Thus, we obtain

$$\sigma \frac{\partial^2 y}{\partial t^2} = T \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) \tag{8.8}$$

or

$$\sigma \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}. \tag{8.9}$$

We set $c^2 = T/\sigma$ (c has the dimension of a velocity). The desired differential equation (also called the *wave equation*) then reads

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) y(x,t) = 0. \tag{8.10}$$

Solution of the wave equation

The wave equation (8.10) is solved with given definite boundary conditions and initial conditions. The *boundary conditions* state that the string is tightly clamped at both ends $x = 0$ and $x = l$, i.e.,

$$y(0,t) = 0, \qquad y(l,t) = 0 \qquad \text{(boundary conditions).}$$

The *initial conditions* specify the state of the string at the time $t = 0$ (initial excitation).

The excitation is performed by a displacement of the form $f(x)$,

$$y(x,0) = f(x) \qquad \text{(first initial condition),}$$

and the velocity of the string is zero,

$$\left.\frac{\partial}{\partial t} y(x,t)\right|_{t=0} = 0 \qquad \text{(second initial condition).}$$

For solving the partial differential equation (PDE), we use the product *ansatz* $y(x,t) = X(x)\cdot T(t)$. Such an approach is obvious, since we are looking for eigenvibrations. These are defined so that all mass points (i.e., any string element at any position x) vibrate with the same frequency. By the *ansatz* $y(x,t) = X(x)\cdot T(t)$, the time behavior is decoupled from the spatial one. Thus we try to split the partial differential equation into a function of the position $X(x)$ and a function of the time $T(t)$. Inserting $y(x,t) = X(x)\cdot T(t)$ into the differential equation (8.10) yields

$$X(x)\ddot{T}(t) = c^2 X''(x) T(t),$$

where $\partial^2 T/\partial t^2 = \ddot{T}$ and $\partial^2 X/\partial x^2 = X''$. The above equation can be rewritten as

$$\frac{\ddot{T}(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}.$$

Since one side depends only on x and the other side depends on t, while x and t are independent of each other, there is only one possible solution: Both sides are constant. The constant will be denoted by $-\omega^2$,

$$\frac{\ddot{T}}{T} = -\omega^2 \quad \text{or} \quad \ddot{T} + \omega^2 T = 0, \tag{8.11}$$

or

$$\frac{X''}{X} = -\frac{\omega^2}{c^2} \quad \text{or} \quad X'' + \frac{\omega^2}{c^2} X = 0. \tag{8.12}$$

The solutions of the differential equations (continuous harmonic vibrations) have the form

$$T(t) = A\sin\omega t + B\cos\omega t,$$
$$X(x) = C\sin\frac{\omega}{c}x + D\cos\frac{\omega}{c}x.$$

The general solution then reads

$$y(x,t) = (A\sin\omega t + B\cos\omega t)\cdot\left(C\sin\frac{\omega}{c}x + D\cos\frac{\omega}{c}x\right). \tag{8.13}$$

The constants A, B, C, and D are determined from the boundary and initial conditions.

From the boundary conditions, it follows for (8.11) that

$$y(0,t) = 0 = D(A\sin\omega t + B\cos\omega t).$$

Since the expression in brackets differs from zero, we must have $D = 0$. Then (8.13) simplifies to

$$y(x,t) = C\sin\frac{\omega}{c}x(A\sin\omega t + B\cos\omega t).$$

With the second boundary condition, we get

$$y(l,t) = 0 = C\sin\frac{\omega}{c}l(A\sin\omega t + B\cos\omega t)$$
$$\Rightarrow \quad 0 = C\sin\frac{\omega}{c}l.$$

This equation will be satisfied if either of the following holds:

(a) $C = 0$, which means that the entire string is not displaced,

or

(b) $\sin(\omega l/c) = 0$. The sine equals zero if $(\omega/c)l = n\pi$, i.e., if $\omega = \omega_n = n\pi c/l$, where $n = 1, 2, 3, \ldots$ ($n = 0$ would lead to case (a)).

From the boundary conditions, we thus obtain the *eigenfrequencies* $\omega_n = n\pi c/l$ *of the string*. Since the string is a continuous system, there are *infinitely many eigenfrequencies.* The solution for an eigenfrequency, the normal vibration, was marked by the index n. The equation (8.11) becomes

$$y_n(x,t) = C\cdot\sin\frac{n\pi}{l}x\left(A_n\sin\frac{n\pi c}{l}t + B_n\cos\frac{n\pi c}{l}t\right),$$
$$y_n(x,t) = \sin\frac{n\pi}{l}x\left(a_n\sin\frac{n\pi c}{l}t + b_n\cos\frac{n\pi c}{l}t\right),$$

where we set $C\cdot A_n = a_n$ and $C\cdot B_n = b_n$.

From the initial conditions, we have

$$\left.\frac{\partial}{\partial t}y_n(x,t)\right|_{t=0} = 0 = \frac{n\pi c}{l}\sin\frac{n\pi}{l}x\left.\left(a_n\cos\frac{n\pi c}{l}t - b_n\sin\frac{n\pi c}{l}t\right)\right|_{t=0}.$$

Then

$$a_n \cdot \frac{n\pi c}{l} \cdot \sin \frac{n\pi}{l} x = 0$$

is satisfied for all x only if $a_n = 0$. Thus, the solution of the differential equation is

$$y_n(x,t) = b_n \cdot \sin \frac{n\pi}{l} x \cos \frac{n\pi c}{l} t. \qquad \textbf{(8.14)}$$

The parameter n describes the excitation states of a system, in this case those of the vibrating string. *In quantum physics such a discrete parameter n is called a quantum number.*

Interjection: If we had selected a negative separation constant in equation (8.11), i.e., $+\omega^2$ instead of $-\omega^2$, we would have arrived at the solution

$$y(x,t) = (Ae^{\omega t} + Be^{-\omega t})\left(Ce^{\frac{\omega}{c}x} + De^{-\frac{\omega}{c}x}\right).$$

The boundary conditions $y(0,t) = y(l,t) = 0$ would have led to the conditions

$$C + D = 0; \qquad Ce^{\frac{\omega}{c}l} + De^{-\frac{\omega}{c}l} = 0$$

with the solutions $C = D = 0$. The string would have remained at rest. But this is not the desired solution.

Since the one-dimensional wave equation is a linear differential equation, one can obtain the most general solution, according to the superposition principle, by the superposition (addition) of the particular solutions:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t = \sum_{n=1}^{\infty} b_n \sin k_n x \cos \omega_n t.$$

The coefficients b_n can be calculated from the given initial curve by using the considerations on the Fourier series (see the next chapter):

$$y(x,0) \equiv f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

The calculation of the Fourier coefficients b_n will be shown in the next chapter. One then gets the following general solution of the differential equation:

$$y(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x') \sin \frac{n\pi x'}{l} dx' \right) \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \qquad \textbf{(8.15)}$$

Normal vibrations

Normal vibrations are described by the following equation:

$$y_n(x,t) = C_n \sin(k_n x) \cos(\omega_n t). \qquad \textbf{(8.16)}$$

For a fixed time t, the spatial variation (positional dependence) of the normal vibration depends on the expression $\sin(n\pi x/l)$ (for $n > 1$, $\sin(n\pi x/l)$ has exactly $n-1$ nodes). All mass points (position x) vibrate with the same frequency ω_n.

At a definite position x, the time dependence of the normal vibration is represented by the expression $\cos(n\pi c/l)t$. The *wave number* k_n is defined as

$$k_n \equiv \frac{\omega_n}{c} = \frac{n\pi}{l} = \frac{2\pi}{\lambda_n}, \tag{8.17}$$

where $\lambda_n = 2l/n$ is the *wavelength.*

The *angular frequency* is defined as follows:

$$\omega_n \equiv \frac{n\pi c}{l} = 2\pi\nu_n. \tag{8.18}$$

Solving the equation (8.18) for ν_n, we obtain for the *frequency*

$$\nu_n = \frac{nc}{2l}, \tag{8.19}$$

i.e., the frequencies increase with increasing index n. By definition,

$$c = \sqrt{\frac{T}{\sigma}}; \tag{8.20}$$

c can be interpreted as *"sound" velocity* in the string, as we shall see below. T is the tension in the string, σ is the mass density. From the equations (8.19) and (8.20) we find

$$\nu_n = \frac{n}{2l}\sqrt{\frac{T}{\sigma}}, \tag{8.21}$$

i.e., the longer and thicker a string is, the smaller the frequency. The frequency increases with the string tension T. This agrees with our experience that long, thick strings sound deeper than short, thin ones. With increasing string tension the frequency increases. This property is utilized when tuning up a violin.

Multiplication of the wavelength by the frequency yields a constant c which has the dimension of a velocity:

$$\lambda_n\nu_n = \frac{2l}{n}\frac{nc}{2l} = c \qquad \text{(dispersion law)}. \tag{8.22}$$

c is the velocity (phase velocity) by which the wave propagates in a medium. This can be seen as follows: If an initial perturbation $y(x,0) = f(x)$ is given as in Figure 8.2, $f(x-ct)$ is also a solution of the wave equation (8.10), because with $z = x - ct$ we have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = -c\frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial t^2} = -c\frac{\partial^2 f}{\partial z^2}\frac{\partial z}{\partial t} = c^2\frac{\partial^2 f}{\partial z^2},$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}.$$

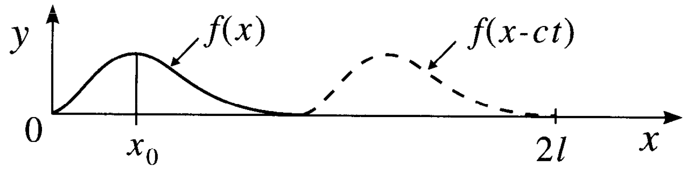

Figure 8.2. Propagation of a perturbation $f(x)$ along a long string: After the time t, the perturbation has moved away by ct; it is then described by $f(x - ct)$.

Hence,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}f(x-ct) = \frac{c^2}{c^2}\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f(x-ct)}{\partial x^2}.$$

$f(x - ct)$ thus satisfies the wave equation (8.10).

Let the maximum of the perturbation $f(x)$ be at x_0. After the time t, it lies at

$$x - ct = x_0.$$

It thus propagates with the velocity

$$\frac{dx}{dt} = c$$

along the string, namely to the right (positive x-direction). One can say that the perturbation $f(x)$ moves along the string with the velocity

$$\frac{dx}{dt} = c. \tag{8.23}$$

The propagation velocity of small perturbations is called the *sound velocity*. One easily realizes as above that $f(x + ct)$ is also a solution of the wave equation and represents a perturbation that moves to the left (negative x-direction). We are dealing here with *running* waves, while for the tightly clamped string we have *standing* waves.

If a string is excited with an arbitrary normal frequency, there are points on the string that remain at rest at any time (*nodes*).

The wavelength, the number of nodes, and the shape of normal vibrations can be represented as a function of the index n (see Figure 8.3).

n	Wavelength	Number of nodes	Figure
1	$2l$	0	(a)
2	l	1	(b)
3	$\frac{2}{3}l$	2	(c)
⋮	⋮	⋮	
n	$\frac{2}{n}l$	$n-1$	

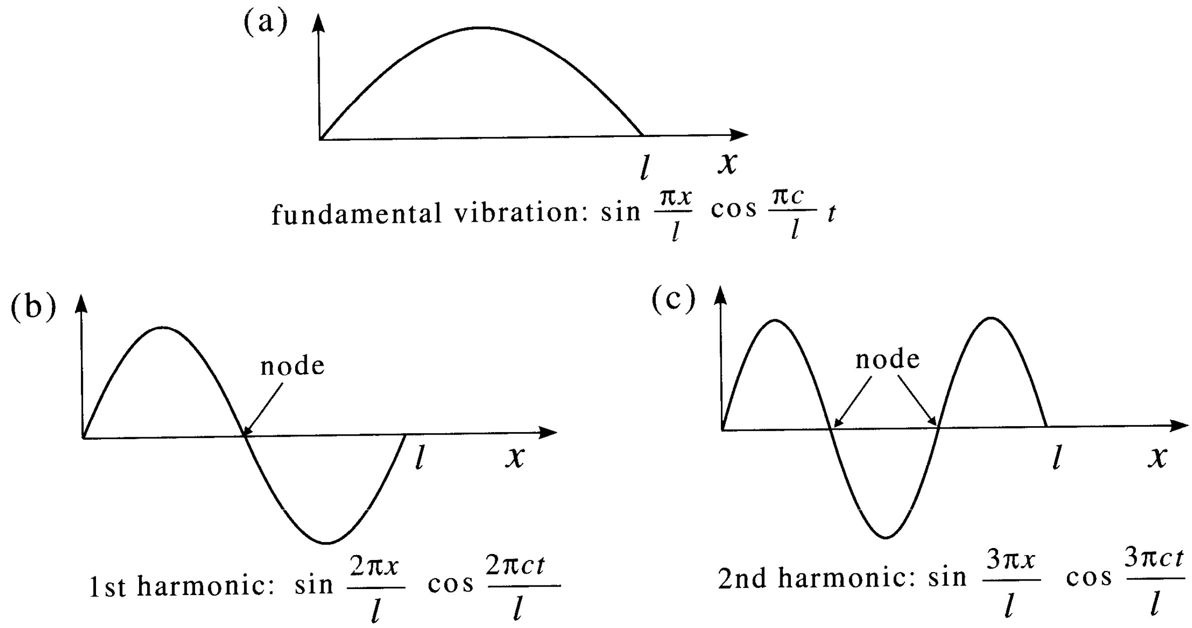

Figure 8.3. The lowest normal vibrations of a string.

Example 8.1: Exercise: Kinetic and potential energy of a vibrating string

Consider a string of density σ that is stretched between two points and is excited with small amplitudes.

(a) Calculate in general the kinetic and potential energy of the string.

(b) Calculate the kinetic and potential energy for waves of the form

$$y = C\cos\left(\frac{\omega(x-ct)}{c}\right)$$

with $T_0 = 500$ N, $C = 0.01$ m, and $\lambda = 0.1$ m.

Solution (a) The part $\overline{PQ}$ of the string has the mass $\sigma\,\Delta x$ and the velocity $\partial y/\partial t$. Its kinetic energy is then

$$\Delta T = \frac{1}{2}\sigma\,\Delta x\left(\frac{\partial y}{\partial t}\right)^2 . \tag{8.24}$$

The total kinetic energy of the string between $x = a$ and b is

$$T = \frac{1}{2}\sigma\int_a^b\left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{8.25}$$

The work which is needed to elongate the string from Δx to Δl is

$$dP = T_0(\Delta l - \Delta x), \qquad \frac{\Delta l}{\Delta x} \sim 1. \tag{8.26}$$

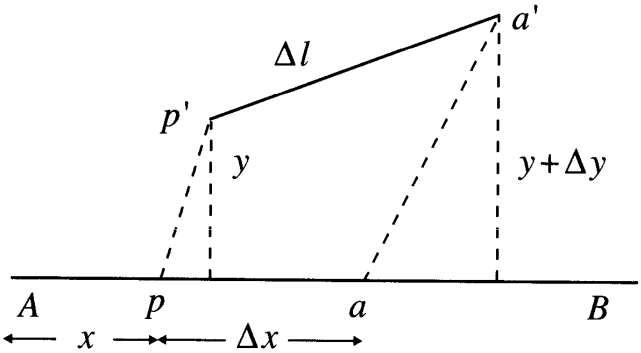

Figure 8.4. Displacement and deformation (elongation compression) of the string element Δx.

For small displacements, we have

$$\Delta l = (\Delta x^2 + \Delta y^2)^{1/2} = \Delta x \left[1 + \left(\frac{\partial y}{\partial x} \right)^2 \right]^{1/2} \simeq \Delta x \left[1 + \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right]. \quad \textbf{(8.27)}$$

The potential energy for the region $x = a$ to $x = b$ is then

$$P = \frac{1}{2} T_0 \int_a^b \left(\frac{\partial y}{\partial x} \right)^2 dx. \quad \textbf{(8.28)}$$

For a wave $y = F(x - ct)$ propagating in a direction, we have

$$T = P = \frac{1}{2} T_0 \int_a^b \Big[F'(x - ct) \Big]^2 dx, \qquad c^2 = \frac{T_0}{\sigma}. \quad \textbf{(8.29)}$$

Hence, the kinetic and potential energy are equal. If a, b are fixed points, then T and P vary with time. But if we admit that a and b can propagate with the sound velocity c, so that

$$a = A + ct \quad \text{and} \quad b = B + ct, \quad \textbf{(8.30)}$$

then P and T are constant:

$$T = P = \frac{1}{2} T_0 \int_A^B (F'(x))^2 \, dx. \quad \textbf{(8.31)}$$

(b)

$$\frac{\partial y}{\partial t} = C \sin \left(\frac{\omega}{c} x - \omega t \right) \omega$$

$$\Rightarrow \quad \left(\frac{\partial y}{\partial t} \right)^2 = C^2 \sin^2 \left(\frac{\omega}{c} x - \omega t \right) \omega^2 \quad \textbf{(8.32)}$$

Insertion into equation (8.25) yields ($a = 0, b = \lambda$)

$$T = \frac{1}{2}\frac{T_0}{c^2}C^2\omega^2 \int_0^\lambda \sin^2\left(\frac{\omega}{c}x - \omega t\right) dx = \frac{1}{2}\frac{T_0}{c^2}C^2\omega^2 \cdot I. \tag{8.33}$$

With the substitution $z = (\omega/c)x - \omega t$ for the integral I, we find

$$I = \frac{c}{\omega} \int_{-\omega t}^{(\omega/c)\lambda-\omega t} \sin^2 z\, dz = \frac{c}{\omega} \int_0^{(\omega/c)\lambda} \sin^2 z\, dz = \frac{c}{\omega} \int_0^{2\pi} \sin^2 z\, dz \tag{8.34}$$

$$= \frac{c}{\omega}\left[\frac{1}{2}z - \frac{1}{4}\sin(2z)\right]_0^{2\pi} = \frac{c}{\omega}\pi$$

$$\Rightarrow \quad T = \frac{1}{2}\frac{T_0}{c^2}C^2\omega^2\frac{c}{\omega}\pi = \frac{\pi^2 C^2 T_0}{\lambda}, \quad \lambda = 2\pi\frac{c}{\omega}. \tag{8.35}$$

One gets the same expression for the potential energy. Insertion of the numerical values yields

$$T = P = (0.01)^2 \cdot \pi^2 \frac{500\,\text{N}}{0.1\,\text{m}} \text{m}^2 \sim 5\,\text{Nm}.$$

Example 8.2: Exercise: Three different masses equidistantly fixed on a string

Calculate the eigenfrequencies of the system of three different masses that are fixed equidistantly on a stretched string, as is shown in Figure 8.5. (*Hint*: For small amplitudes, the string tension T does not change!)

Solution From Figure 8.6, we extract for the equations of motion

$$2m\ddot{x}_1 = T\left[\frac{(x_2 - x_1)}{L}\right] - T\left[\frac{x_1}{L}\right],$$

$$m\ddot{x}_2 = -T\left[\frac{(x_2 - x_1)}{L}\right] - T\left[\frac{(x_2 - x_3)}{L}\right],$$

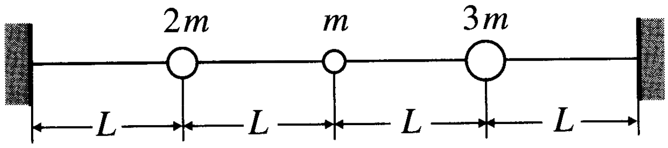

Figure 8.5.

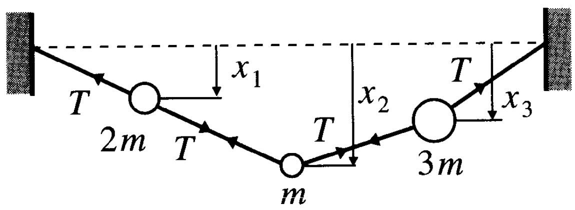

Figure 8.6.

$$3m\ddot{x}_3 = T\left[\frac{(x_2 - x_3)}{L}\right] - T\left[\frac{x_3}{L}\right]. \tag{8.36}$$

We look for the eigenvibrations. All mass points must then vibrate with the same frequency. We therefore start with

$$\begin{aligned} x_1 &= A\sin(\omega t + \psi), & \ddot{x}_1 &= -\omega^2 A\sin(\omega t + \psi), \\ x_2 &= B\sin(\omega t + \psi), & \ddot{x}_2 &= -\omega^2 B\sin(\omega t + \psi), \\ x_3 &= C\sin(\omega t + \psi), & \ddot{x}_3 &= -\omega^2 C\sin(\omega t + \psi). \end{aligned}$$

Hence, after insertion into equation (8.36) one gets

$$\begin{aligned} \left(\frac{2T}{L} - 2m\omega^2\right)A - \left(\frac{T}{L}\right)B &= 0, \\ -\left(\frac{T}{L}\right)A + \left(\frac{2T}{L} - m\omega^2\right)B - \left(\frac{T}{L}\right)C &= 0, \\ -\left(\frac{T}{L}\right)B + \left(\frac{2T}{L} - 3m\omega^2\right)C &= 0. \end{aligned} \tag{8.37}$$

For evaluating the eigenfrequencies of the system, i.e., for solving equation (8.37), the determinant of coefficients must vanish:

$$\begin{vmatrix} \left(2\frac{T}{L} - 2m\omega^2\right) & -\frac{T}{L} & 0 \\ -\frac{T}{L} & \left(2\frac{T}{L} - m\omega^2\right) & -\frac{T}{L} \\ 0 & -\frac{T}{L} & \left(2\frac{T}{L} - 3m\omega^2\right) \end{vmatrix} = 0.$$

Expansion of the determinant leads to

$$0 = 6m^3\omega^6 - \left(\frac{22Tm^2}{L}\right)\omega^4 + \left(\frac{19T^2m}{L^2}\right)\omega^2 - \left(\frac{4T^3}{L^3}\right)$$

or

$$0 = 6m^3\Omega^3 + \left(\frac{-22Tm^2}{L}\right)\Omega^2 + \left(\frac{19T^2m}{L^2}\right)\Omega + \left(\frac{-4T^3}{L^3}\right), \tag{8.38}$$

where we substituted $\Omega = \omega^2$. This leads to the cubic equation

$$a\Omega^3 + b\Omega^2 + c\Omega + d = 0,$$

where

$$a = 6m^3, \qquad b = \frac{-22Tm^2}{L}, \qquad c = \frac{19T^2m}{L^2}, \qquad d = \frac{-4T^3}{L^3}.$$

It can be transformed to the representation (reduction of the cubic equation)

$$y^3 + 3py + 2q = 0, \tag{8.39}$$

where

$$y = \Omega + \frac{b}{3a} = \Omega - \frac{11}{9}\frac{T}{Lm}$$

and

$$3p = -\frac{1}{3}\frac{b^2}{a^2} + \frac{c}{a} \quad \text{and} \quad 2q = \frac{2}{27}\frac{b^3}{a^3} - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a}.$$

Insertion leads to

$$3p = -\frac{71}{54}\frac{T^2}{L^2m^2}, \qquad 2q = -\frac{653}{1458}\frac{T^3}{L^3m^3}.$$

From this, it follows that

$$q^2 + p^3 < 0,$$

i.e., there exist 3 real solutions of the cubic equation (8.39).

For the case $q^2 + p^3 \le 0$, the solutions y_1, y_2, y_3 can be calculated using tabulated auxiliary quantities (see mathematical supplement 8.4). Direct application of Cardano's formula would lead to complex expressions for the real roots, hence the above method is convenient.

After insertion one obtains for the auxiliary quantities

$$\cos\varphi = \frac{-q}{\sqrt{-p^3}}, \qquad y_1 = 2\sqrt{-p}\cos\frac{\varphi}{3},$$
$$y_2 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3} + \frac{\pi}{3}\right),$$
$$y_3 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3} - \frac{\pi}{3}\right),$$

and finally, for the eigenfrequencies of the system

$$\omega_1 = 0.563\sqrt{\frac{T}{Lm}}, \qquad \omega_2 = 0.916\sqrt{\frac{T}{Lm}}, \qquad \omega_3 = 1.585\sqrt{\frac{T}{Lm}}.$$

Example 8.3: Exercise: Complicated coupled vibrational system

Determine the eigenfrequencies of the system of three equal masses suspended between springs with the spring constant k, as is shown in Figure 8.7. *Hint*: Consider the solution method of the preceding problem 8.2 and the mathematical supplement 8.4.

Solution From Figure 8.7, we extract for the equations of motion

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) - k(x_1 - x_3), \\ m\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) - k(x_2 - x_3), \\ m\ddot{x}_3 &= -kx_3 - k(x_3 - x_1) - k(x_3 - x_2), \end{aligned} \tag{8.40}$$

or

$$\begin{aligned} m\ddot{x}_1 + 3kx_1 - kx_2 - kx_3 &= 0, \\ m\ddot{x}_2 + 3kx_2 - kx_3 - kx_1 &= 0, \\ m\ddot{x}_3 + 3kx_3 - kx_1 - kx_2 &= 0. \end{aligned} \tag{8.41}$$

We look for the eigenvibrations. All mass points must vibrate with the same frequency. Thus, we adopt the *ansatz*

$$x_1 = A\cos(\omega t + \psi), \qquad \ddot{x}_1 = -\omega^2 A\cos(\omega t + \psi),$$

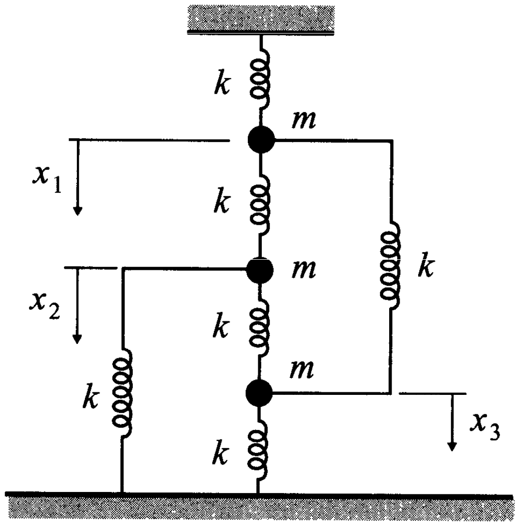

Figure 8.7. Vibrating coupled masses.

$$x_2 = B\cos(\omega t + \psi), \qquad \ddot{x}_2 = -\omega^2 B\cos(\omega t + \psi),$$
$$x_3 = C\cos(\omega t + \psi), \qquad \ddot{x}_3 = -\omega^2 C\cos(\omega t + \psi),$$

and after insertion into equation (8.41), we get

$$\begin{aligned} &(3k - m\omega^2)A - kB - kC = 0, \\ &-kA + (3k - m\omega^2)B - kC = 0, \\ &-kA - kB + (3k - m\omega^2)C = 0. \end{aligned} \tag{8.42}$$

To get a nontrivial solution of equation (8.42), the determinant of coefficients must vanish:

$$\begin{vmatrix} (3k - m\omega^2) & -k & -k \\ -k & (3k - m\omega^2) & -k \\ -k & -k & (3k - m\omega^2) \end{vmatrix} = 0.$$

Expansion of the determinant leads to

$$0 = \omega^6 - \frac{9k}{m}\omega^4 + \frac{24k^2}{m^2}\omega^2 - \frac{16k^3}{m^3}$$

or

$$0 = \Omega^3 - \frac{9k}{m}\Omega^2 + \frac{24k^2}{m^2}\Omega - \frac{16k^3}{m^3},$$

where we substituted $\Omega = \omega^2$ (see Example 8.2). The general cubic equation $a\Omega^3 + b\Omega^2 + c\Omega + d = 0$ (in our case $a = 1, b = -9k/m, c = 24k^2/m^2, d = -16k^3/m^3$) can according to mathematical supplement 8.4 be reduced to

$$y^3 + 3py + 2q = 0,$$

where

$$y = \Omega + \frac{b}{3a}, \qquad 3p = -\frac{1}{3}\frac{b^2}{a^2} + \frac{c}{a}, \qquad 2q = \frac{2}{27}\frac{b^3}{a^3} - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a}.$$

Insertion leads to

$$3p = -3\frac{k^2}{m^2}, \qquad 2q = 2\frac{k^3}{m^3} \qquad \Rightarrow \quad q^2 + p^3 = 0,$$

i.e., there exist 3 solutions (the real roots); 2 of them coincide. Hence, the vibrating system being treated here is *degenerate*. As in problem 8.2, the solutions can be calculated using tabulated auxiliary quantities. For these, we obtain

$$\cos\varphi = \frac{-q}{\sqrt{-p^3}}, \qquad y_1 = 2\sqrt{-p}\cos\frac{\varphi}{3},$$
$$y_2 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3} + \frac{\pi}{3}\right),$$
$$y_3 = -2\sqrt{-p}\cos\left(\frac{\varphi}{3} - \frac{\pi}{3}\right),$$

and, after insertion, for the eigenfrequencies of the system

$$\omega_3 = \sqrt{\frac{k}{m}}, \qquad \omega_1 = \omega_2 = 2\sqrt{\frac{k}{m}}.$$

Example 8.4: Mathematical supplement: The Cardano formula[1]

In theoretical physics, one often meets the problem of solving a cubic equation, just as in the Examples 8.2 and 8.3. We now will clarify this problem.

Reduction of the general cubic equation: If the general cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{8.43}$$

with nonvanishing coefficients a, b, and c is to be solved, one must first eliminate the quadratic term of the equation, i.e., *reduce* the equation. If the unknown x is replaced by $y + \lambda$, where y and λ are new, unknown quantities, equation (8.43) turns into

$$(y^3 + 3y^2\lambda + 3y\lambda^2 + \lambda^3) + (ay^2 + 2ay\lambda + a\lambda^2) + (by + b\lambda) + c = 0,$$
$$y^3 + (3\lambda + a)y^2 + (3\lambda^2 + 2a\lambda + b)y + (\lambda^3 + a\lambda^2 + b\lambda + c) = 0. \tag{8.44}$$

Since we have replaced one unknown quantity x by two unknown ones, y and λ, we can freely dispose of one of the two unknown quantities. This freedom is exploited so as to let the quadratic term of the equation disappear. This is achieved by setting the coefficient of y^2, that is, $3\lambda + a$, equal to zero, i.e., $\lambda = -a/3$. By inserting this value the equation (8.44) changes to

$$y^3 + \left(-\frac{a^2}{3} + b\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) = 0. \tag{8.45}$$

[1] We follow the exposition of E. v. Hanxleben and R. Hentze, *Lehrbuch der Mathematik*, Friedrich Vieweg & Sohn 1952, Braunschweig-Berlin-Stuttgart.

If we set the expressions determined by the known coefficients a, b, and c of the cubic equation,

$$-\frac{a^2}{3}+b=p \quad \text{and} \quad \frac{2a^3}{27}-\frac{ab}{3}+c=q, \tag{8.46}$$

the cubic equation takes the form

$$y^3+py+q=0 \qquad \text{(reduced cubic equation)}. \tag{8.47}$$

Result: To reduce the cubic equation given in the normal form, one sets $x = y - a/3$. Then equation (8.47) follows from equation (8.43).

Example: $x^3 - 9x^2 + 33x - 65 = 0$.

(1) *Solution*: Set $x = y - (-3) = y + 3$.

$$(y+3)^3 - 9(y+3)^2 + 33(y+3) - 65 = 0,$$
$$(y^3 + 9y^2 + 27y + 27) - 9(y^2 + 6y + 9) + 33(y+3) - 65 = 0,$$
$$y^3 + 6y - 20 = 0.$$

(2) *Solution*: Insert the values calculated from equation (8.46) into equation (8.47).

Special case: If in the general cubic equation, the linear term is missing ($b = 0$), i.e., the cubic equation is given in the form

$$x^3 + ax^2 + c = 0, \tag{8.48}$$

the reduction can also be performed by inserting

$$x = \frac{c}{y}. \tag{8.49}$$

From equations (8.48) and (8.49), we obtain the reduced equation

$$\frac{c^3}{y^3} + a\frac{c^2}{y^2} + c = 0 \quad \text{or} \quad y^3 + acy + c^2 = 0. \tag{8.50}$$

Solution of the reduced cubic equation: If one sets in the reduced cubic equation

$$y^3 + py + q = 0,$$
$$y = u + v, \tag{8.51}$$

one obtains

$$u^3 + 3u^2v + 3uv^2 + v^3 + p(u+v) + q = 0,$$
$$(u^3 + v^3 + q) + 3uv(u+v) + p(u+v) = 0,$$
$$(u^3 + v^3 + q) + (3uv + p)(u+v) = 0. \tag{8.52}$$

Since one can freely dispose of one of the unknown quantities u or v (justification?), these are suitably chosen so that the coefficient of $(u + v)$ vanishes. We therefore set

$$3uv + p = 0, \quad \text{i.e.,} \quad uv = -\frac{p}{3}. \tag{8.53}$$

Then equation (8.52) simplifies to

$$u^3 + v^3 + q = 0 \quad \text{or} \quad u^3 + v^3 = -q. \tag{8.54}$$

u and v are determined by equations (8.53) and (8.54). The quantities u and v can no longer be arbitrarily chosen. By raising equation (8.54) to the second power and equation (8.53) to the third power, one obtains

$$u^6 + 2u^3v^3 + v^6 = q^2,$$
$$4u^3v^3 = -4\left(\frac{p}{3}\right)^3.$$

Subtraction of the two equations yields

$$(u^3 - v^3)^2 = q^2 + 4\left(\frac{p}{3}\right)^3,$$
$$u^3 - v^3 = \pm\sqrt{q^2 + 4\left(\frac{p}{3}\right)^3}. \tag{8.55}$$

By addition and subtraction of equations (8.54) and (8.55), one obtains

$$u^3 = \frac{1}{2}\left[-q \pm \sqrt{q^2 + 4\left(\frac{p}{3}\right)^3}\right] \quad \text{and} \quad v^3 = \frac{1}{2}\left[-q \mp \sqrt{q^2 + 4\left(\frac{p}{3}\right)^3}\right],$$
$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \quad \text{and} \quad v = \sqrt[3]{-\frac{q}{2} \mp \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}. \tag{8.56}$$

If one sets

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} = m \quad \text{and} \quad \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} = n,$$

one gets

$$u_1 = m, \qquad u_2 = m\epsilon_2, \qquad u_3 = m\epsilon_3,$$
$$v_1 = n, \qquad v_2 = n\epsilon_2, \qquad v_3 = n\epsilon_3.$$

Here, the ϵ_i are the unit roots of the cubic equation $x^3 = 1$ which, as is evident, read

$$\epsilon_1 = 1, \qquad \epsilon_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad \epsilon = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Since now $y = u + v$, one can actually form 9 values for y (why?). But since the quantities u and v must satisfy the determining equation (8.53), the number of possible connections between u and v is restricted to 3, namely,

$$y_1 = u_1 + v_1, \qquad y_2 = u_2 + v_3, \qquad y_3 = u_3 + v_2;$$

hence,

$$y_1 = m + n = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}},$$
$$y_2 = m\epsilon_2 + n\epsilon_3 = -\frac{m+n}{2} + \frac{m-n}{2}i\sqrt{3}, \tag{8.57}$$
$$y_3 = m\epsilon_3 + n\epsilon_2 = -\frac{m+n}{2} - \frac{m-n}{2}i\sqrt{3}.$$

The real root of the cubic equation, i.e., the root

$$y_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

is known as the *"Cardano formula."* It was named in honor of the Italian Hieronimo Cardano[2] to whom the discovery of the formula was falsely ascribed. Actually, the formula is due to the Bolognesian professor of mathematics Scipione del Ferro,[3] who found this ingenious algorithm.

Example: $y^3 - 15y - 126 = 0$. Here,

$$p = -15, \qquad q = -126,$$
$$\frac{p}{3} = -5, \qquad \frac{q}{2} = -63.$$

By inserting into the Cardano formula, one obtains

$$\begin{aligned} y_1 &= \sqrt[3]{63 + \sqrt{63^2 - 5^3}} && + \sqrt[3]{63 - \sqrt{63^2 - 5^3}} \\ &= \sqrt[3]{63 + \sqrt{3844}} && + \sqrt[3]{63 - \sqrt{3844}c} \\ &= \sqrt[3]{63 + 62} && + \sqrt[3]{63 - 62} \\ &= \sqrt[3]{125} && + \sqrt[3]{1} \qquad (= m + n) \\ &= 6, \\ y_2 &= -\frac{5+1}{2} + \frac{5-1}{2} i\sqrt{3} = -3 + 2i\sqrt{3}, \\ y_3 &= -\frac{5+1}{2} - \frac{5-1}{2} i\sqrt{3} = -3 - 2i\sqrt{3}. \end{aligned}$$

Check the validity of the roots by insertion!

Discussion of Cardano's formula: The square root appearing in the Cardano formula only yields a real value if the radicand $(q/2)^2 + (p/3)^3 \geq 0$. If the radicand is negative, the three values for y yield complex numbers. We consider the possible cases:

[2]*Hieronimo Cardano*, Italian physicist, mathematician, and astrologer, b. Sept. 24, 1501, Pavia–d. Sept. 20, 1576, Rome. Cardano was the illegitimate son of Fazio (Bonifacius) Cardano, a friend of Leonardo da Vinci. He studied at the universities of Pavia and Padua, and in 1526 he graduated in medicine. In 1532, he went to Milan, where he lived in deep poverty, until he got a position teaching in mathematics. In 1539, he worked at a high school of physics, where he soon became the director. In 1543, he accepted a professorship for medicine in Pavia.

As a mathematician, Cardano was the most prominent personality of his age. In 1539, he published two books on arithmetic methods. At this time, the discovery of a solution method for the cubic equation became known. Nicolo Tartaglia, a Venetian mathematician, was the owner. Cardano tried in vain to get permission to publish it. Tartaglia left the method to him under the condition that he keeps it secret. In 1545, Cardano's book *Artis magnae sive de regulis algebraicis*, one of the cornerstones of the history of algebra, was published. The book contained, besides many other new facts, the method of solving cubic equations. The publication caused a serious controversy with Tartaglia.

[3]*Scipione del Ferro*, b. 1465(?)–d. 1526 (?). About his life we know only that he lectured from 1496 to 1526 at the university of Bologna. By 1500, he discovered the method of solving the cubic equation but did not publish it. Tartaglia rediscovered the method in 1535.

		$\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$	Form of the roots
(1)	$p > 0$	Real	A real value, two complex conjugate values
(2)	$p < 0$, namely,		
(a)	$\left\|\left(\frac{p}{3}\right)^3\right\| < \left(\frac{q}{2}\right)^2$	Real	As in 1.
(b)	$\left\|\left(\frac{p}{3}\right)^3\right\| = \left(\frac{q}{2}\right)^2$	$=0$	Three real values, among them a double root
(c)	$\left\|\left(\frac{p}{3}\right)^3\right\| > \left(\frac{q}{2}\right)^2$	Imaginary	All three roots by the form imaginary

The case (2c) was of particular interest to the mathematicians of the Middle Ages. Since any cubic equation has at least one real root, but they could not find it by means of Cardano's formula, the case was called the *casus irreducibilis*.[4] The first to solve this case was the French politician and mathematician Vieta.[5] He proved by using trigonometry that this case was solvable too, and that in this case the equation has three real roots.

Trigonometric solution of the irreducible case: Since p is negative in this case, one starts from the reduced cubic equation

$$y^3 - py + q = 0, \tag{8.58}$$

where p must now be kept fixed as absolute numerical value. According to the trigonometric formulae we have

$$\begin{aligned}
\cos 3\alpha &= \cos(2\alpha + \alpha) = \cos 2\alpha \cos\alpha - \sin 2\alpha \sin\alpha \\
&= (\cos^2\alpha - \sin^2\alpha)\cos\alpha - 2\sin^2\alpha\cos\alpha \\
&= \cos^3\alpha - \sin^2\alpha\cos\alpha - 2\sin^2\alpha\cos\alpha \\
&= \cos^3\alpha - (1-\cos^2\alpha)\cos\alpha - 2(1-\cos^2\alpha)\cos\alpha \\
&= \cos^3\alpha - \cos\alpha + \cos^3\alpha - 2\cos\alpha + 2\cos^3\alpha \\
&= 4\cos^3\alpha - 3\cos\alpha,
\end{aligned}$$

[4] *Casus irreducibilis* (Lat.) = "the nonreducible case."

[5] *Francois Vieta*, French mathematician, b. 1540, Fontenay-le-Comte–d. Dec. 13, 1603, Paris. Advocate and advisor of Parliament in the Bretagne. His greatest achievements were in the theory of equations and algebra, where he introduced and systematically used letter notations. He established the rules for the rectangular spherical triangle which are often ascribed to Neper. In his *Canon mathematicus*, a table of angular functions (1571), he emphasized the advantages of decimal notation. [BR]

thus,

$$\cos^3\alpha - \frac{3}{4}\cos\alpha - \frac{1}{4}\cos 3\alpha = 0. \tag{8.59}$$

If one considers $\cos\alpha$ to be unknown, equation (8.59) coincides with the form of equation (8.58). But since the value of the cosine varies only between the limits -1 and $+1$, while y, according to the values of p and q, can take any values, one cannot simply set $\cos\alpha = y$. By multiplying equation (8.59) by a still uncertain positive factor ϱ^3, one obtains

$$\varrho^3\cos^3\alpha - \frac{3}{4}\varrho^2\cdot\varrho\cos\alpha - \frac{1}{4}\varrho^3\cos 3\alpha = 0. \tag{8.60}$$

By setting $\varrho\cdot\cos\alpha = y$, $p = (3/4)\varrho^2$, and $q = -(1/4)\varrho^3\cos 3\alpha$, equation (8.60) turns into (8.58). From this, we find

$$\varrho = 2\cdot\sqrt{\frac{p}{3}} \tag{8.61}$$

and

$$\cos 3\alpha = -\frac{4q}{\varrho^3} = \frac{-4q}{8\cdot(p/3)\sqrt{p/3}} = -\frac{q/2}{\sqrt{(p/3)^3}}. \tag{8.62}$$

Equation (8.62) is ambiguous, since the cosine is a periodic function. One has

$$3\alpha = \varphi + k\cdot 360^\circ, \qquad \text{where} \quad k = 0, 1, 2, 3, \ldots. \tag{8.63}$$

From this, we find for α

$$\alpha_1 = \frac{\varphi}{3}, \qquad \alpha_2 = \frac{\varphi}{3} + 120^\circ, \qquad \alpha_3 = \frac{\varphi}{3} + 240^\circ.$$

Compare this consideration with the problem of cyclotomy! Which values are obtained for α if $k = 3, 4, \ldots$?

For y, one obtains

$$y_1 = 2\sqrt{\frac{p}{3}}\cos\frac{\varphi}{3}, \qquad y_2 = 2\sqrt{\frac{p}{3}}\cos\left(\frac{\varphi}{3} + 120^\circ\right), \qquad y_3 = 2\sqrt{\frac{p}{3}}\cos\left(\frac{\varphi}{3} + 240^\circ\right).$$

Now

$$\cos\left(\frac{\varphi}{3} + 120^\circ\right) = -\cos\left(60^\circ - \frac{\varphi}{3}\right)$$

and

$$\cos\left(\frac{\varphi}{3} + 240^\circ\right) = -\cos\left(60^\circ + \frac{\varphi}{3}\right),$$

so that the roots of the cubic equations are

$$\begin{aligned} y_1 &= 2\sqrt{\frac{p}{3}}\cos\frac{\varphi}{3}, \\ y_2 &= -2\sqrt{\frac{p}{3}}\cos\left(60^\circ - \frac{\varphi}{3}\right), \\ y_3 &= -2\sqrt{\frac{p}{3}}\cos\left(60^\circ + \frac{\varphi}{3}\right). \end{aligned} \tag{8.64}$$

Comment: The formulas of the *casus irreducibilis* can also be derived by means of the Moivre's theorem.

Example: Calculate the roots of the equation

$$y^3 - 981y - 11340 = 0.$$

Solution: Since $p < 0$ and

$$\left|\left(\frac{p}{3}\right)^3\right| = 327^3, \qquad \log\left|\left(\frac{p}{3}\right)^3\right| = 3 \cdot \log 327 \ = 7.5436,$$

$$\left(\frac{q}{2}\right)^2 = 5670^2, \qquad \log\left(\frac{q}{2}\right)^2 = 2 \cdot \log 5670 = 7.5072,$$

by comparing the logarithms it follows that $|(p/3)^3| > (q/2)^2$. Thus, the condition of the *casus irreducibilis* is fulfilled. According to equation (8.62)

$$\cos 3\alpha = +\frac{5670}{\sqrt{327^3}},$$

$$\log \cos 3\alpha = 3.7536 - 3.7718 = 9.9818 - 10,$$

$$\varphi = 3\alpha \approx 16°30', \qquad \text{hence,} \quad \frac{\varphi}{3} = \alpha = 5°30'.$$

From equation (8.64), we obtain $y_1 = 36,\ y_2 = -21,\ y_3 = -15$. Check the root values by insertion!

9 Fourier Series

When setting the initial conditions for the problem of the vibrating string, a trigonometric series was set equal to a given function $f(x)$. The expansion coefficients of the series had to be determined. To solve the problem, the function $f(x)$ should also be represented by a trigonometric series. These trigonometric series are called Fourier series.[1] The conditions that allow an expansion of a function into a Fourier series are summarized as follows:

(1) $f(x)$ is defined in the interval $a \leq x < a + 2l$,

(2) $f(x)$ and $f'(x)$ are piecewise continuous on $a \leq x < a + 2l$,

(3) $f(x)$ has a finite number of discontinuities which are finite jump discontinuities, and

(4) $f(x)$ has the period $2l$, i.e., $f(x + 2l) = f(x)$.

[1] *Jean Baptiste Joseph Fourier,* b. March 21, 1768, Auxerre, son of a tailor–d. May 16, 1830, Paris. Fourier attended the home École Militaire. Because of his origin he was excluded from an officer's career. Fourier decided to join the clergy, but did not take a vow because of the outbreak of the revolution of 1789. Fourier first took a teaching position in Auxerre. Soon he turned to politics and was arrested several times. In 1795, he was sent to Paris to study at the École Normale. He soon became member of the teaching staff of the newly founded École Polytechnique. In 1798, he became director of the Institut d'Egypte in Cairo. Only in 1801 did he return to Paris, where he was appointed by Napoleon as a prefect of the departement Isère. During his term of office from 1802 to 1815, he arranged the drainage of the malaria-infested marshes of Bourgoin. After the downfall of Napoleon, Fourier was dismissed from all posts by the Bourbons. However, in 1817 the king had to agree to Fourier's election to the Academy of Sciences, where he became permanent secretary in 1822. Fourier's most important mathematical achievement was his treatment of the notion of the function. The problem of the vibrating string that had been treated already by D'Alembert, Euler, and Lagrange, and had been solved in 1755 by D. Bernoulli by a trigonometric series. The subsequent question of whether an "arbitrary" function can be represented by such a series was answered 1807/12 by Fourier in the affirmative. The question about the conditions for such a representation could be answered only by his friend Dirichlet. Fourier became known mainly by his *Théorie analytique de la chaleur* (1822) which deals mainly with the discussion of the equation of heat propagation in terms of Fourier-series. This work represents the starting point for treating partial differential equations with boundary conditions by means of trigonometric series. Fourier also made import contributions to the theory of solving equations and to the probability calculus.

These conditions (Dirichlet conditions) are sufficient to represent $f(x)$ by a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right). \tag{9.1}$$

The Fourier coefficients a_n, b_n, and a_0 are determined as follows:

$$\begin{aligned} a_n &= \frac{1}{l} \int_a^{a+2l} f(x) \cos \frac{n\pi x}{l} \, dx, \\ b_n &= \frac{1}{l} \int_a^{a+2l} f(x) \sin \frac{n\pi x}{l} \, dx, \\ a_0 &= \frac{1}{l} \int_a^{a+2l} f(x) \, dx. \end{aligned} \tag{9.2}$$

To prove these formulas, one needs the so-called orthogonality relations of the trigonometric functions:

$$\begin{aligned} &\int_0^{2l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \, dx = l \, \delta_{nm}, \\ &\int_0^{2l} \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, dx = l \, \delta_{nm}, \\ &\int_0^{2l} \sin \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \, dx = 0. \end{aligned} \tag{9.3}$$

The first relation can be proven by means of the theorem

$$\cos A \cos B = \frac{1}{2} \Big(\cos(A + B) + \cos(A - B) \Big),$$

$$\int_0^{2l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \, dx = \frac{1}{2} \int_0^{2l} \left(\cos \frac{(n + m)\pi x}{l} + \cos \frac{(n - m)\pi x}{l} \right) dx = 0,$$

if $n \neq m$. The integral of the cosine function over a full period vanishes. For $n = m$ we have

$$\int_0^{2l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \, dx = \frac{1}{2} \int_0^{2l} \left(1 + \cos \frac{2n\pi x}{l} \right) dx = l.$$

The other relations can be proved in an analogous way.

The formula (9.2) for calculating the Fourier coefficients can be proved by means of the orthogonality relations.

To determine the a_n, one multiplies the equation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

by $\cos(m\pi x/l)$ and then integrates over the interval 0 to $2l$:

$$\begin{aligned}\int_0^{2l} f(x)\cos\frac{m\pi x}{l}\,dx &= \frac{a_0}{2}\int_0^{2l}\cos\frac{m\pi x}{l}\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2l}\sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx \\ &= \sum_{n=1}^{\infty} a_n l\delta_{nm} = l a_m,\end{aligned}$$

and therefore,

$$a_m = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{m\pi x}{l}\,dx, \tag{9.4}$$

as is given by the equations (9.2).

The analogous relation for the b_m can be confirmed by multiplication of equation (9.1) by $\sin(m\pi x/l)$ and integration from 0 to $2l$; the same holds for the calculation of a_0.

Functions that satisfy

$$f(x) = f(-x)$$

are called *even functions*; functions with the property

$$f(x) = -f(-x)$$

are called *odd functions*. For instance, $f(x) = \cos x$ evidently is an even function and $f(x) = \sin x$ an odd function. The part of (9.1)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

is obviously even, while

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

represents the odd part of the series expansion (9.1). Therefore, for even functions all $b_n = 0$, for odd functions a_0 and all a_n are equal to zero.

Any function $f(x)$ can be decomposed into an even and an odd part. Thus, $(f(x)+f(-x))/2$ is the even part and $(f(x)-f(-x))/2$ the odd part of $f(x)=[(f(x)+f(-x))/2+(f(x)-f(-x))/2]$.

Example 9.1: Inclusion of the initial conditions for the vibrating string by means of the Fourier expansion

A string is fixed at both ends. The center is displaced from the equilibrium position by the distance H and then released. From Figure 9.1 we see that the initial displacement is given by

$$y(x,0)=f(x)=\begin{cases} 2\dfrac{Hx}{l}, & 0\le x\le \dfrac{l}{2}, \\ \dfrac{2H(l-x)}{l}, & \dfrac{l}{2}\le x\le l. \end{cases}$$

If we assume $f(x)$ is an odd function (dashed line), we then obtain

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \\ &= \frac{2}{l}\left(\int_0^{l/2}\frac{2Hx}{l}\sin\frac{n\pi x}{l}\,dx+\int_{l/2}^{l}\frac{2H}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right), \end{aligned}$$

$$\begin{aligned} \int_0^{l/2}\frac{2Hx}{l}\sin\frac{n\pi x}{l}\,dx &= \frac{2H}{l}\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2} \\ &= \frac{2lH}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{Hl}{n\pi}\cos\frac{n\pi}{2}, \end{aligned}$$

$$\int_{l/2}^{l}\frac{2H}{l}(l-x)\sin\frac{n\pi x}{l}\,dx=\frac{2H}{l}\left(\int_{l/2}^{l} l\sin\frac{n\pi x}{l}\,dx-\int_{l/2}^{l} x\sin\frac{n\pi x}{l}\,dx\right)$$

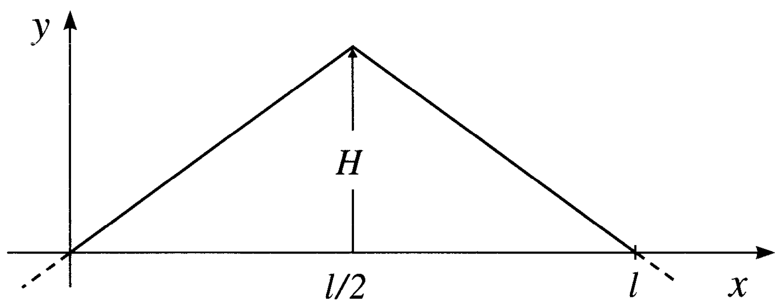

Figure 9.1.

$$= \frac{2H}{l}\left[-\frac{l^2}{n\pi}\cos\frac{n\pi x}{l} + \frac{xl}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}$$
$$= \frac{2lH}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{lH}{n\pi}\cos\frac{n\pi}{2},$$

$$b_n = \frac{2}{l}\left(\frac{2lH}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2lH}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$
$$= \frac{8H}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

By inserting the solution for the Fourier coefficient b_n into the general solution of the differential equation (8.15), we get the equation that describes the vibrations of a string:

$$y(x,t) = \sum_{n=1}^{\infty}\left(\frac{8H}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$
$$= \frac{8H}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right.$$
$$\left. + \frac{1}{5^2}\sin\frac{5\pi x}{l}\cos\frac{5\pi ct}{l} - \cdots\right).$$

Thus, by plucking the string in the center one essentially excites the fundamental mode (lowest eigenvibration) $\sin(\pi x/l)\cos(\pi ct/l)$. Several overtones are admixed with small amplitude. The initial displacement obviously corresponds to the fundamental vibration. If one wants to excite pure overtones, the initial displacement must be selected according to the desired higher harmonic vibration (compare the figures after equation (8.23)).

Example 9.2: Exercise: Fourier series of the sawtooth function

Find the Fourier series of the function

$$f(x) = 4x, \qquad 0 \le x \le 10, \qquad \text{with period} \quad 2l = 10, \qquad l = 5.$$

Solution The Fourier coefficients are

$$a_0 = \frac{1}{5}\int_0^{10} 4x\,dx = \frac{2}{5}x^2\Big|_0^{10} = 40,$$

$$a_n = \frac{1}{5}\int_0^{10} 4x\cos\frac{n\pi x}{5}\,dx = \frac{4x}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^{10} - \frac{4}{n\pi}\int_0^{10}\sin\frac{n\pi x}{5}\,dx$$
$$= 0 + \frac{20}{n^2\pi^2}\cos\frac{n\pi x}{5}\Big|_0^{10} = 0,$$

$$b_n = \frac{4}{5}\int_0^{10} x\sin\frac{n\pi x}{5}\,dx = -\frac{4x}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^{10} + \frac{4}{n\pi}\int_0^{10}\cos\frac{n\pi x}{5}\,dx$$
$$= -\frac{40}{n\pi} + \frac{20}{n^2\pi^2}\sin\frac{n\pi x}{5}\Big|_0^{10} = -\frac{40}{n\pi}.$$

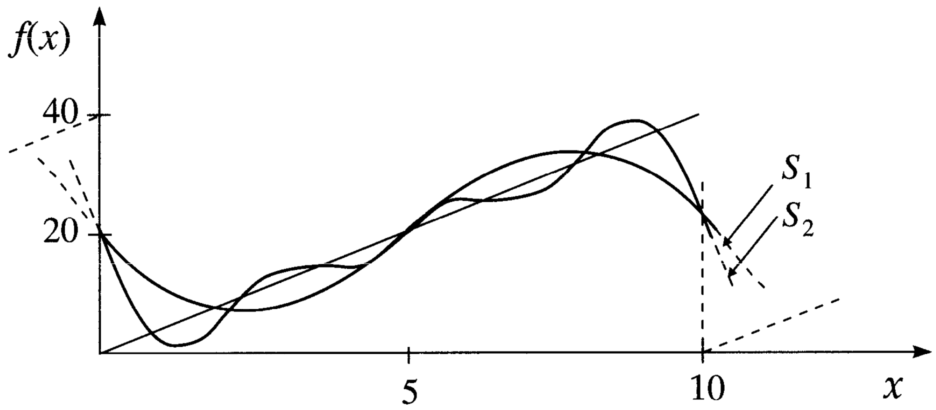

Figure 9.2.

Hence, the Fourier series reads

$$f(x) = 20 - \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{5}.$$

The first partial sums S_n of this series are drawn in Figure 9.2. A comparison of this series with the starting curve $f(x)$ illustrates the convergence of this Fourier series.

Example 9.3: Exercise: Vibrating string with a given velocity distribution

Find the transverse displacement of a vibrating string of length l with fixed endpoints if the string is initially in its rest position and has a velocity distribution $g(x)$.

Solution We look for the solution of the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{9.5}$$

where $y = y(x, t)$, with

$$\begin{aligned} &y(0, t) = 0, && y(l, t) = 0, \\ &y(x, 0) = 0, && \left.\frac{\partial}{\partial t} y(x, t)\right|_{t=0} = g(x). \end{aligned} \tag{9.6}$$

We use the separation *ansatz* $y = X(x) \cdot T(t)$. By inserting it into equation (9.5), one obtains

$$X \cdot \ddot{T} = c^2 X'' T \quad \text{or} \quad \frac{X''}{X}(x) = \frac{\ddot{T}}{c^2 T}(t). \tag{9.7}$$

Since the left-hand side of equation (9.7) depends only on x, the right side only on t, and x and t are independent of each other, the equation is satisfied only then if both sides are constant. The constant is denoted by $-\lambda^2$.

$$\frac{X''}{X} = -\lambda^2 \quad \text{and} \quad \frac{\ddot{T}}{c^2 T} = -\lambda^2,$$

or, transformed,

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad \ddot{T} + \lambda^2 c^2 T = 0. \tag{9.8}$$

The two equations have the solutions

$$X = A_1 \cos \lambda x + B_1 \sin \lambda x, \qquad T = A_2 \cos \lambda ct + B_2 \sin \lambda ct.$$

Since $y = X \cdot T$, we have

$$y(x,t) = (A_1 \cos \lambda x + B_1 \sin \lambda x)(A_2 \cos \lambda ct + B_2 \sin \lambda ct). \tag{9.9}$$

From the condition $y(0, t) = 0$, it follows that $A_1(A_2 \cos \lambda ct + B_2 \sin \lambda ct) = 0$. This condition is satisfied by $A_1 = 0$. Then

$$y(x,t) = B_1 \sin \lambda x (A_2 \cos \lambda ct + B_2 \sin \lambda ct).$$

We now set

$$B_1 A_2 = a, \qquad B_1 B_2 = b,$$

and it follows that

$$y(x,t) = \sin \lambda x (a \cos \lambda ct + b \sin \lambda ct). \tag{9.10}$$

From the condition $y(l, t) = 0$, it follows that $\sin \lambda l = 0$. This happens if

$$\lambda l = n\pi \quad \text{or} \quad \lambda = \frac{n\pi}{l}. \tag{9.11}$$

Here, $n = 1, 2, 3, \ldots$. The value $n = 0$ which seems possible at first sight leads to $y(x, t) \equiv 0$ and must be excluded. The relation (9.11) is inserted into (9.10). The normal vibration will be labeled by the index n:

$$y_n(x,t) = \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right). \tag{9.12}$$

Because $y(x, 0) = 0$, all $a_n = 0$, we have

$$y_n(x,t) = b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}. \tag{9.13}$$

By differentiation of (9.13), we get

$$\frac{\partial y_n}{\partial t} = b_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \tag{9.14}$$

For linear differential equations, the superposition principle holds, so that the entire solution looks as follows:

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c b_n}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \tag{9.15}$$

Because

$$\left. \frac{\partial}{\partial t} y(x,t) \right|_{t=0} = g(x),$$

it follows that

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c b_n}{l} \sin \frac{n\pi x}{l}. \tag{9.16}$$

The Fourier coefficients then follow by

$$\frac{n\pi c b_n}{l} = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx \tag{9.17}$$

or

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx. \tag{9.18}$$

By inserting (9.18) into (9.13), we obtain the final solution for $y(x,t)$:

$$y(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi c} \int_0^l g(x') \sin \frac{n\pi x'}{l}\, dx' \right) \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l}. \tag{9.19}$$

Example 9.4: Exercise: Fourier series for a step function

Given the function

$$f(x) = \begin{cases} 0, & \text{for } -5 \le x \le 0, \\ 3, & \text{for } 0 \le x \le 5 \end{cases} \qquad \text{period } 2l = 10.$$

(a) Sketch the function.

(b) Determine its Fourier series.

Solution (a)

$$f(x) = \begin{cases} 0, & \text{for } -5 \le x \le 0, \\ 3, & \text{for } 0 \le x \le 5 \end{cases} \qquad \text{period } 2l = 10.$$

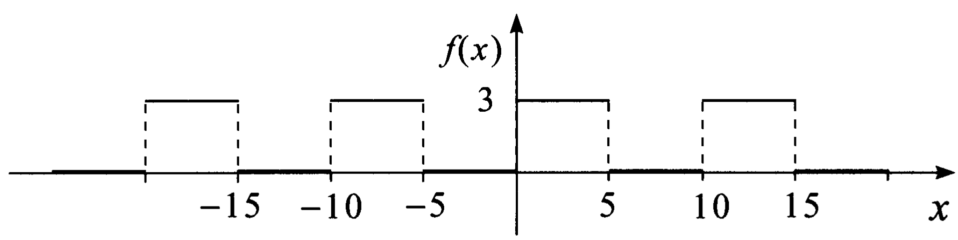

Figure 9.3.

(b) For period $2l = 10$ and $l = 5$, we choose the interval a to $a + 2l$ to be -5 to 5, i.e., $a = -5$:

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 3\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left\{\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right\}\Bigg|_0^5 = 0 \quad \text{for} \quad n \neq 0.$$

For $n = 0$, one has $a_n = a_0 = (3/5)\int_0^5 \cos(0\pi x/5)\,dx = (3/5)\int_0^5 dx = 3$. Furthermore,

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 3\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3}{n\pi}(1-\cos n\pi).$$

Thus,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3}{n\pi}(1-\cos n\pi)\sin\left(\frac{n\pi x}{5}\right),$$

i.e.,

$$f(x) = \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right).$$

Example 9.5: Exercise: On the unambiguousness of the tautochrone problem

Which trajectory of the mass of a mathematical pendulum yields a pendulum period that is independent of the amplitude?

Solution We consider the Figure 9.4. From energy conservation, we have

$$\frac{m}{2}\dot{s}^2(y) + gmy = mgh \tag{9.20}$$

or

$$\dot{s}(y) = \sqrt{2g(h-y)}. \tag{9.21}$$

From this, one can calculate the period by separation of the variables:

$$\frac{1}{4}T = \int_0^{T/4} dt = \int_0^{s(h)} \frac{ds}{\sqrt{2g(h-y)}} = \int_0^h \frac{(ds/dy)dy}{\sqrt{2g(h-y)}}. \tag{9.22}$$

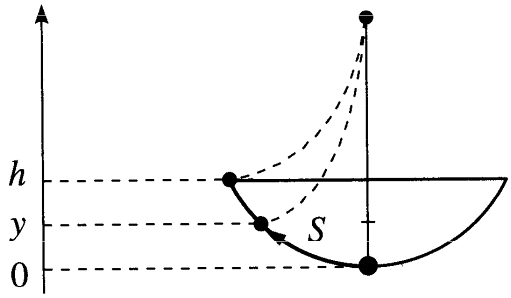

Figure 9.4.

Using the variable $u = y/h$, (9.22) changes to

$$\frac{T}{4} = \int_0^1 \frac{(ds/dy)\sqrt{h}\,du}{\sqrt{2g(1-u)}}. \tag{9.23}$$

We now require that T be independent of the maximum height h:

$$\frac{dT}{dh} = 0 \quad \text{for all } h. \tag{9.24}$$

Thus, we get from (9.23) ($s' \equiv ds/dy$)

$$\frac{d}{dh}\int_0^1 \frac{s'\sqrt{h}\,du}{\sqrt{2g(1-u)}} = \int_0^1 \frac{du}{\sqrt{2g(1-u)}}\left(\frac{1}{2}h^{-1/2}s' + \sqrt{h}\frac{ds'}{dh}\right) = 0 \quad \text{for all } h. \tag{9.25}$$

With the condition that we keep the dimensionless variable $u = y/h$ constant, we can rewrite the derivative with respect to h as a derivative with respect to y,

$$\frac{ds'}{dh} = \frac{u\,ds'}{d(uh)} = u\frac{ds'}{dy} = us'', \tag{9.26}$$

and thus, we can transform (9.25) into

$$\int_0^1 \frac{du}{\sqrt{8g(1-u)}}(s' + 2ys'')\frac{1}{\sqrt{h}} = 0 \quad \text{for all } h. \tag{9.27}$$

Any periodic function $f(u)$ satisfying $\int_0^1 f(u)\,du = 0$ can generally be expanded into a Fourier series:

$$f(u) = \sum_{m=1}^{\infty}\left[a_m \sin(2\pi m u) + b_m \cos(2\pi m u)\right]. \tag{9.28}$$

Therefore, from (9.27) it follows that

$$s'' + \frac{1}{2y}s' = \frac{\sqrt{8gh(1-u)}}{2y}\sum_{m=1}^{\infty}\left(a_m \sin(2\pi m u) + b_m \cos(2\pi m u)\right)$$

$$= \frac{\sqrt{8gh(h-y)}}{2y} \sum_{m=1}^{\infty} \left(a_m \sin\left(2\pi m \frac{y}{h}\right) + b_m \cos\left(2\pi m \frac{y}{h}\right)\right). \tag{9.29}$$

This holds for all values of h. The left-hand side of (9.29) does not contain h; therefore, the right-hand side must be independent of h too. This holds only for $a_m = b_m = 0$ (for all m), as we shall prove now.

To have the right-hand side of (9.29) independent of h, we must have

$$\sum_{m=1}^{\infty} \left[a_m \sin\left(2\pi m \frac{y}{h}\right) + b_m \cos\left(2\pi m \frac{y}{h}\right)\right] = \frac{\text{constant} \cdot (y/h)h^{1/2}}{\sqrt{8g(1-y/h)}} \tag{9.30}$$

or

$$\sum_{m=1}^{\infty} [a_m \sin(2\pi m u) + b_m \cos(2\pi m u)] = \frac{u}{\sqrt{1-u}} \frac{h^{1/2}}{\sqrt{8g}} C. \tag{9.31}$$

By integrating (9.31) from 0 to 1, we obtain

$$0 = \frac{h^{1/2}}{\sqrt{8g}} C \int_0^1 \frac{u}{\sqrt{1-u}}\, du = \frac{4}{3} \frac{h^{1/2}}{\sqrt{8g}} C, \tag{9.32}$$

thus, $C = 0$. (This reflects the fact that $u/\sqrt{1-u}$ cannot be expanded into a Fourier series *à la* (9.31).)

Inserting this result $C = 0$ again into (9.30), we have $a_m = b_m = 0\, \forall m$, and thus, from (9.29)

$$s'' + \frac{s'}{2y} = 0. \tag{9.33}$$

From this, one finds by integrating once

$$\frac{s''}{s'} = -\frac{1}{2y} \quad \Rightarrow \quad s' \equiv \frac{ds}{dy} = \tilde{C} e^{-(1/2)\ln y} = \frac{\tilde{C}}{\sqrt{y}}. \tag{9.34}$$

The constant is usually denoted by

$$\tilde{C} = \sqrt{\frac{l}{2}}, \tag{9.35}$$

so that we have to solve

$$\frac{ds}{dy} = \sqrt{\frac{l}{2}} \frac{1}{\sqrt{y}}. \tag{9.36}$$

This is the differential equation of a cycloid (see *Classical Mechanics: Point Particles and Relativity*, Exercise 24.4).

10 The Vibrating Membrane

We consider a two-dimensional system: the vibrating membrane. We shall see that the methods applied for the treatment of a vibrating string can be simply transferred in many respects.

The membrane is a skin without an elasticity of its own. The stretching of the membrane along the edge leads to a tension force that acts as a backdriving force on a deformed membrane.

Let the tangential tension in the membrane be spatially constant and time independent. We consider only vibrations with amplitudes so small that displacements within the membrane plane can be neglected.

Derivation of the differential equation

We introduce the following notations: σ is the surface density of the membrane, and the membrane tension is T (force per unit length). Let the coordinate system be oriented so that the membrane lies in the x,y-plane. The displacements perpendicular to this plane are denoted by $u = u(x, y, t)$.

To set up the equation of motion, we imagine a cut of length Δx through the membrane parallel to the x-axis, and a cut Δy parallel to the y-axis. The force acting on the membrane element $\Delta x \Delta y$ in the x-direction is the product of the tension and the length of the cut: $F_x = T\Delta y$. Analogously for the y-component we have $F_y = T\Delta x$.

The surface element $\Delta x \Delta y$ is pulled by the sum of the two forces. If the membrane is displaced, the u-component of this sum acts on it.

From Figure 10.1, we see

$$F_u = T\Delta x(\sin\varphi(y+\Delta y) - \sin\varphi(y)) + T\Delta y(\sin\vartheta(x+\Delta x) - \sin\vartheta(x)). \qquad \textbf{(10.1)}$$