Homework

Exercise 1.

clique

Given a graph $G = (V, E)$, the clique problem asks to find the maximum size of a subset of vertices $C \subseteq V$ such that all pairs of vertices in C are connected.

Question 1. *Propose an integer program for this problem.*

Question 2. *Show that the relaxation of this program has an integrality gap of* $\Omega(n)$.

Exercise 2.

Densest subgraph

Given a graph $G = (V, E)$, the *densest subgraph* problem is to find a subgraph $G' = (V', E')$ that maximizes the ratio $|E'|/|V'|$. Consider the following linear program with variables x_v for each $v \in V$ and y_e for each $e \in E$.

$$
\begin{array}{lll}
\text{Maximize :} & \displaystyle\sum_{e \in E} y_e & \\
\text{Subject to :} & \displaystyle\sum_{v \in V} x_v = 1 & \\
& y_e \leq \min(x_u, x_v), & e = (u, v) \in E \\
& y_e \geq 0, & e \in E \\
& x_v \geq 0, & v \in V
\end{array}
$$

Question 1. *Show that this linear program is a relaxation of the densest subgraph problem.*

We consider the following rounding scheme. Let $r \in [0, 1]$, $V_r = \{v \in V | x_v \geq r\}$ and $E_r = \{e \in E | y_e \geq r\}$.

Question 2. *Show that* $\displaystyle\int_0^1 |E_r| dr = \sum_{e \in E} y_e$.

Question 3. *Show that* $\displaystyle\int_0^1 |V_r| dr = \sum_{v \in V} x_v = 1$.

Question 4. *Show that there exists a value of r such that $|E_r|/|V_r| \geq \sum_{e \in E} y_e / \sum_{v \in V} x_v$. What does it imply for the complexity of the densest-subgraph problem ? How would you find the value of r that gives the best sets V_r and E_r ?*

Exercise 3.

Alternative LP-relaxation for multiway cut

Given a graph $G = (V, E)$ with positive costs on edges, and a set $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ of terminals, a multiway cut is a set of edges whose removal disconnects the terminals from each other. The multiway cut problem is to find a multiway cut of minimum cost.

From $G = (V, E)$ obtain the directed graph H by replacing each edge $\{u, v\}$ of G by two directed edges $(u \to v)$ and $(v \to u)$, each having the same cost as $\{u, v\}$.

Assign a $0/1$ indicator variable d_e to each arc e in H. Suppose the terminals are numbered $s_1, \ldots, s_k$ in some order. Let P be the collection of all simple paths from a lower-numbered terminal to a higher-numbered terminal.

Consider the following bidirected integer programming formulation for the multiway cut problem.

$$
\begin{array}{lll}
\text{Minimize :} & \displaystyle\sum_{e \in H} c(e) d_e & \\
\text{Subject to :} & \displaystyle\sum_{e \in P} d_e \geq 1, & p \in P \\
& d_e \in \{0, 1\}, & e \in H
\end{array}
$$

Question 1. *Show that an optimal solution to this integer program yields an optimal solution to the multiway cut problem.*

Question 2. *Obtain the LP-relaxation and dual program. Give a good physical interpretation of the dual.*

Question 3. *Give an instance where the cost of an optimal integer solution is different from the cost of an optimal fractional solution. What is the corresponding integrality gap ? How far can you go (until getting bored) ?*

Exercise 4.

Capacitated vertex cover

Given a graph $G = (V, E)$ with weighted vertices $w_v \in \mathbb{Q}^+$ and vertex capacities $k_v \in \mathbb{Z}^+$, the capacitated vertex cover problem is to find a vertex cover of minimum cost such that each vertex :

1. can be taken multiple times (i.e. x_v times)
2. covers at most $k_v x_v$ edges.

The minimum capacitated vertex cover is to find such a cover that minimizes the total weight of the chosen vertices, $\sum_{v \in V} w_v \cdot x_v$.
We consider the following linear program for this problem :

$$\begin{array}{lll} \text{Minimize :} & \displaystyle\sum_{v \in V} x_v w_v & \\ \text{Subject to :} & y_{e,v} + y_{e,u} \geq 1, & e \in E \\ & \displaystyle\sum_{e : v \in e} y_{e,v} \leq k_v x_v, & v \in V \\ & x_v \in \mathbb{N} & \\ & y_{e,v} \in \{0, 1\} & \end{array}$$

Question 1. *Show that this integer linear program encodes the minimum capacitated vertex cover problem. Propose a relaxation to this program.*

Question 2. *Show that the constraint $x_v \geq y_{e,v}$ for each $e : v \in e$ can be added to the LP relaxation. What does it corresponds to ? In the following, we consider the relaxed LP with this additional constraint.*

Consider the following rounding strategy :
Let $y^*_{e,v} = \begin{cases} 1 & \text{if } y_{e,v} \geq \frac{1}{2} \\ 0 & \text{if } y_{e,v} < \frac{1}{2} \end{cases}$, and $y^*_v = \sum_{e : v \in e} y^*_{e,v}$
Set $x^*_v = \left\lceil \dfrac{y^*_v}{k_v} \right\rceil$ and take x_v times node v.

Question 3. *Show that this gives a valid capacitated vertex cover*

Question 4. *Show that $y^*_v \leq 2 k_v x_v$*

Question 5. *Show that if $y^*_v \geq k_v$, $4 x_v \geq x^*_v$*

Question 6. *Suppose $y^*_v < k_v$. Remark that $x^*_v \in \{0, 1\}$. Show that if $x^*_v = 1$ then $x_v \geq \frac{1}{2}$. Conclude on the approximation ratio of the rounding strategy.*