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Cálculo Numérico
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Problemas Resueltos

Problemas

1. Sea una codificación binaria floating point tipo inicial, donde el número real máximo representable es $2^{63} - 2^{54}$.

   (a) Explique cuantos bits son necesarios para implementar esta codificación y como se distribuyen.

   (b) Sea $a = a_1 a_2 ... a_n$ donde n es el números de bits de la codificación usada y sea
   $a_i = \begin{cases} 1 & \text{si } i = 2j \vee i = 1 + 4j \text{ , } j \in \mathbb{N} \cup \{0\} \\ 0 & \text{en otro caso} \end{cases}$
   Obtenga el real representado por a

   (c) Sea $f(x) = e^{x^2}$. Para una aritmética finita de 5 cifras significativas de redondeo, se sabe que $f(0.5) = 1.2840$.

      i) Calcule el polinomio de Taylor de orden 4 de $f(x)$ en torno a $x_0 = 0$.

      ii) Calcule $p_4(0.5)$ con la misma aritmética.

      iii) Represente $f(0.5)$ y $p_4(0.5)$ según la codificación de punto flotante anterior. Calcule el error relativo y según esto concluya si $p_4(0.5)$ es o no una buena aproximación de $f(0.5)$

2. Considere la función $f(x) = \frac{x}{x^2+1}$. Para una aritmética finita de 4 cifras significativas con redondeo:

   (a) Determine el polinomio de Taylor de order 3 de $f(x)$, $p_3(x)$,en torno a $x_0 = 0$.

   (b) Obtenga una cota teórica del error

   (c) Utilice $p_3(x)$ para aproximar $f(x)$ en los puntos $x = 0.25$ y 0.5. Determine el error absoluto y relativo cometido por esta aproximación. Compare con la cota obtenida anteriormente.

   (d) Grafique $p_3(x)$ y $f(x)$

   (e) Aproxime $I = \int_0^1 f(x)dx$ mediante $I_3 = \int_0^1 p_3(x)dx$. Determine el error absoluto y relativo cometido por esta aproximación.

3. Realice un análisis de propagación del error en el siguiente algoritmo:
   $\varphi(x, y, z) = x + (y + z)$

4. Dado el SEL:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   (a) Determine la descomposición $A = LU$ a través del método de Gauss.

   (b) Resuelva el sistema de ecuaciones lineales $Ax = b$ utilizando la factorización $A = LU$.

   (c) Calcule el $\det(A)$. Si $\det(A) \neq 0$ calcule A^{-1} mediante el método de Gauss-Jordan.

   (d) Calcule $cond(A)$. La matriz A está bien o mal condicionada ? Qué efecto tiene el condicionamiento sobre la resolución de otro SEL similar al definido en este problema.

5. Resuelva el siguiente sistema de ecuaciones lineales, usando las estrategias de pivoteo completo y parcial, usando una aritmetica finita de 4 cifras significativas con redondeo. Calcule el error relativo. Calcule Cond(A).

$$\begin{bmatrix} 0 & 1 & 1 \\ -0.1 & 2 & 0.1 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6.125 \\ -0.375 \\ 1.375 \end{bmatrix}$$

6. Programe el algoritmo para obtener la descomposición LU.

7. Programe un algoritmo que calcule la determinante usando el método de Gauss

8. Cuente el número de operaciones del algoritmo anterior y el de Laplace para calcular la determinante, y compare.

9. Cuente el número de operaciones de Gauss y Gauss-Jordan

10. Sea la matriz de Hilbert definida como $H_n = [\frac{1}{i+j+1}]_{i,j=0,1\ldots n-1}$.

    (a) Resuelva el sistema $H_3 x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

    (b) Resuelva el sistema anterior, pero cambiando la primera componente del lado derecho por 1.01

    (c) Calcule $Cond(H_3)$

11. Dada la matriz tridiagonal A :

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

    (a) Determine su descomposición LU

    (b) Para $b = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^t$ resuelva el SEL (A, b) mediante el Método de Crout.

    (c) Es la matriz A definida positiva ? Si lo es determine su descomposición de Cholesky.

12. Sea el SEL definido por:

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

    (a) Resuelva el SEL mediante el método de iterativo de Jacobi (5 iteraciones)

    (b) Resuelva el SEL mediante el método iterativo de Gauss-Seidel (5 iteraciones)

    (c) Compare las soluciones entregadas por los método iterativos con respecto a la solución entregada por el método de Gauss: $x = \begin{bmatrix} \frac{3}{11} \\ \frac{2}{11} \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} \approx \begin{bmatrix} 0.272727272727 \\ 0.181818181818 \\ 0.181818181818 \\ 0.272727272727 \end{bmatrix}$

    Obs: ocupe $\vec{x}^0 = \vec{0}$

13. Dado el SEL:

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

    (a) Métodos Directos:

        i) Determine su descomposición $A = LU$ y resuelva el SEL utilizando esta factorización.

        iii) Es A definida positiva ? Si lo es determine su factorización de Cholesky y resuelva el SEL utilizando esta factorización.

    (b) Métodos Iterativos:

        i) Resuelva el SEL mediante el método Gauss-Seidel.

        ii) Para el método SOR aplicado a matrices definidas positivas y tridiagonales, el valor óptimo de ω está dado por:

$$\overline{\omega} = \frac{2}{1 + \sqrt{1 - \rho(T_G)}}$$

        Donde $\rho(T_G)$ es el radio espectral de la matriz del método de Gauss-Seidel T_G.
        Calcule $\overline{\omega}$ y resuelva el SEL mediante el método SOR para $\overline{\omega}$.

14. Analice el algoritmo de Cholesky.

15. Sea la matriz de Pascal:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

    (a) Determine si es definida positiva

(b) Si lo es, determine su descomposición de Cholesky.

16. Dado el SEL:

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Resuelva el SEL mediante el método Gauss-Seidel, y calcule el error relativo.

Obs: $\vec{x} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}$

17. Interpole la función $sen(\pi x)$ dentro del intervalo $[-1, 1]$ utilizando Lagrange en los puntos $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Acote el error de lagrange. Grafique $sen(\pi x), L_4(x)$ y $T_5(x)$ (polinomio de Taylor)

18. Encuentre el polinomio de newton con los siguientes datos, y vea a que función se parece

i	x_i	$f(x_i)$
0	-1	$\frac{1}{2}$
1	0	1
2	1	2
3	2	4
4	4	16

19. Dada la función $f(x) = \frac{1}{x+1}$, encuentre el polinomio de newton que interpole a f y a su derivada f' en los puntos $0, 1, 3$.

20. Por las fuertes lluvias que se pronostican para el mes de Mayo, se desean hacer arreglos en la carretera Autopista Central para evitar inundaciones, pero el Ingeniera que diseño la carretera era de la Universidad de las Américas, y perdió los planos. El gerente, desesperado, llama a un Ingeniero amigo suyo de Beauchef, y le plantea el problema. El Beauchefiano, que pasó Calculo Numérico, sabe muy bien que hacer. Le pide al gerente una tabla de datos tomados en los Portales de Peaje que indiquen tiempo y posición de un vehículo. La tabla que el gerente le entrega es la siguiente:

t	0	3	5	8	11
x	0	225	383	623	1001
y	0	9	25	64	121
y'	0				22

Haga lo que un Ingeniero de Beauchef haría.

Hint: Use Spline Cúbica y Polinomio de Newton

21. Una fábrica de mermeladas de membrillo desea expandir su mercado, por lo que decide envasar la mermelada en frascos que contengan 1 kilo de mermelada a temperatura ambiente (15°). Al preparar la mermelada, esta puede alcanzar temperaturas cerca de las $80°C$, por lo que aumenta su volumen, y luego al enfriar la mermelada, esta diminuye su volumen, y el proceso de sellado obliga a la fábrica a envasar la mermelada calientita a 50°. La fábrica produce una mermelada normal y una dietética, con 40% y 15% de concentración de azucar respectivamente, las cuales la empresa ya conoce sus propiedades para el envasado. El cocinero jefe ha creado una mermelada semi dietética con un 25% de concentración de azucar. La fábrica desea conocer el volumen que tendrá la mermelada al momento de envasado, para diseñar el tamaño y la forma de los frascos, y que volumen tandrá a temperatura ambiente, para ver el material del frasco, para que soporte la diferencia de presión. Para esto, se tienen los datos de las mermeladas normal y dietética, con 1 kilo cada una:

$\%\diagdown T°$	5°	10°	20°	40°	80°
15	880 cc	950 cc	1030 cc	1080 cc	1250 cc
40	900 cc	1010 cc	1090 cc	1150 cc	1280 cc

22. La empresa Ferrari realiza pruebas de su nuevo modelo "Relampago McQueen", por lo que se han tomado los siguientes datos:

$Tiempo\ [s]$	0	3	5
$Velocidad\ [\frac{km}{h}]$	0	63	108

(a) Calcule la velocidad en función del tiempo.

(b) Aproxime la distancia en función del tiempo integrando el polinomio anterior.

(c) Se midió la distancia de "Relampago McQueen" en $t = 7$, y se agregó el dato que ha recorrido $\frac{10\,567}{72}\ m$. Determine el polinomio de Hermite que interpola distancia y velocidad del automovil,

(d) Calcule la velocidad en $t = 7$.

23. Dibuje con Spline cúbica el contorno del lindo escarabajo rojo de la fotografía.

24. Determine el sistema para aproximar una función $f(x)$ en un intervalo $[\alpha, \beta]$ por un exponencial de la forma $y(x) = be^{ax}$, donde a y b son las constantes a determinar.

25. Use los ceros de $\tilde{T}_3$ y las transformaciones del intervalo dado y construya un polinomio interpolante de segundo grado para $f(x) = x \ln x$,$[1, 3]$

26. Obtenga el polinomio trigonométrico general de mínimos cuadrados continuos para $f(x) = \begin{matrix} 0 & si & -\pi < x \leq 0 \\ 1 & si & 0 < x < \pi \end{matrix}$

27. Detemine el polinomio trigonométrico $S_2(x)$ en $[-\pi, \pi]$ para $f(x) = x(\pi - x)$

Soluciones

1. Codificacion Binaria

   (a) Como el número real máximo representable en una codificacion binaria siempre es $\frac{2^m-1}{2^m} \times 2^{(2^e-1)}$, donde m es la cantidad de bits de la mantisa, y e la cantidad de bits del exponente, esto equivale a $2^{2^{e-1}} - 2^{(2^e-1)-m}$, queda un sistema:
   $2^{63} - 2^{54} = 2^{2^{e-1}} - 2^{(2^e-1)-m} \implies \begin{matrix} 2^e - 1 = 63 \\ (2^e-1) - m = 54 \end{matrix} \implies \begin{matrix} e = 6 \\ m = 9 \end{matrix}$
   Sumando el bit para el signo de la mantisa y el bit del signo del exponente, la codificación necesita $2 + 6 + 9 = 17$ bits.

   (b) **Como son 17 bits, entonces**

   i. Numero en codificacion binaria: $a = a_1 a_2 a_3 ... a_{17} = 11011101110111011$

   ii. Número real:

   A. Signo Mantisa: $-$

   B. Signo Exponente: $-$

   C. Exponente: $0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 29$

   D. Mantisa: $1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} + 1 \times 2^{-5} + 1 \times 2^{-6} + 0 \times 2^{-7} + 1 \times 2^{-8} + 1 \times 2^{-9} = \frac{443}{512} = 0.865\,234\,375$

   E. Numero: $a = -\frac{443}{512} \times 2^{-29} = -443 \times 2^{-38} = -1.\,611\,624\,611\,541\,628\,837\,6 \times 10^{-9}$

   (c) **Polinomio de Taylor**

   i. $p_4(x) = 1 + x^2 + \frac{1}{2}x^4$

   ii. Se debe calcular con la aritmetica finita de 5 cifras con redondeo, entonces $\frac{0.062\,5}{2} = 0.031\,25$, cumple la aritmética. Luego $p_4(0.5) = 1 + (0.5)^2 + \frac{1}{2}(0.5)^4 = 1 + 0.25 + \frac{0.062\,5}{2} = 1.25 + 0.031\,25 = 1.28125$, reduciendo a 5 cifras con redondeo, queda $p_4(0.5) = 1.2813$

   iii. En representacion de punto flotante quedan:

   A. $\boxed{f(0.5)}$ $1.\,284\,0 = 0.642 \times 2^1$

	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
	1	0	1	0	0	1	0	0	0
0.642	0.142		0.017			0.001375			

   $f(0.5) = 00000001101001000$

   B. $\boxed{p_4(0.5)}$ $1.2813 = 0.64065 \times 2^1$

	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
	1	0	1	0	0	1	0	0	0
0.64065	0.14065		0.015 65			0.000 025			

   $p_4(0.5) = 00000001101001000$

   C. $E_{rel} = \frac{|f(0.5) - p_4(0.5)|}{|f(0.5)|} = \frac{0.0027}{1.\,284\,0} = 2.\,102\,803\,738\,317\,757\,009\,3 \times 10^{-3}$

   La representación de $f(0.5)$ y de $p_4(0.5)$ en esta decodificación es exactamente la misma, por lo que quiere decir que la aproximación, en terminos del punto flotante, es muy buena.

2. $f(x) = \frac{x}{x^2+1}$

   (a) Taylor

   i. $f(x) = \frac{x}{x^2+1}$; $f^{(1)}(x) = \frac{1-x^2}{(x^2+1)^2}$; $f^{^{(2)}}(x) = \frac{2x^3-6x}{(x^2+1)^3}$; $f^{^{(3)}}(x) = -\frac{6\left(x^4-6x^2+1\right)}{(x^2+1)^4}$

   ii. $f(0) = 0$; $f'(0) = 1$; $f^{''}(0) = 0$; $f^{'''}(0) = -6$

   iii. $p_3(x) = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-6}{3!}x^3 = x - x^3$

   i. $f^{^{(4)}}(x) = \frac{24(x^5-10x^3+5x)}{(x^2+1)^5}$

(b) Cota del error

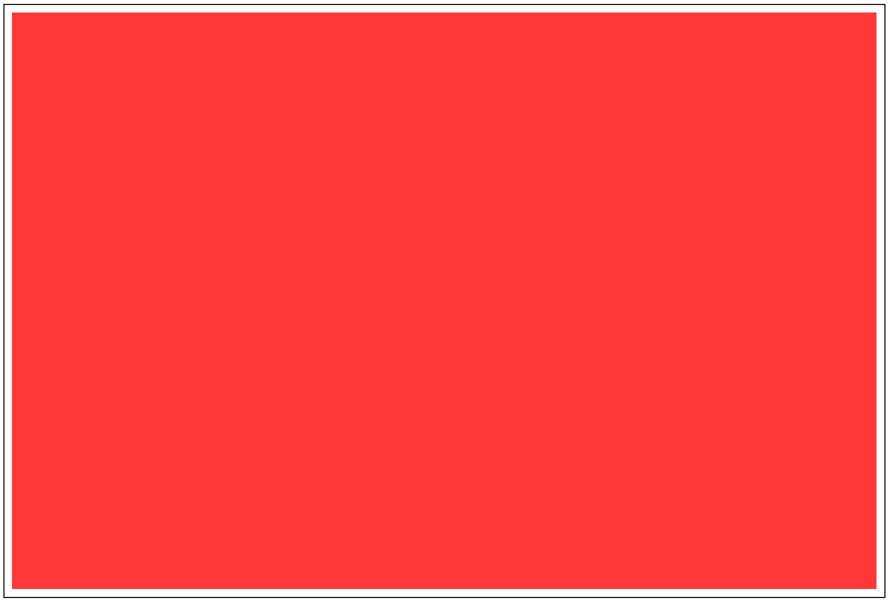
i. $E_3(x) = \frac{f^{(4)}(\xi)}{(4)!}x^4 \leq \frac{\max f^{(4)}(x)}{24}x^4 = \frac{f^{(4)}(2-\sqrt{3})}{24}x^4 = 0.812\,2x^4$

(c) Aproximar

i. $p_3(0.25) = 0.234\,4$; $f(0.25) = 0.235\,3$; $E_{abs} = |\ f(0.25) - p_3(0.25)| = 9 \times 10^{-4}$; $E_{rel} = \frac{|\ f(0.25) - p_3(0.25)|}{|f(0.25)|} = 3.825 \times 10^{-3}$; $E_3(0.25) = 3.\,173 \times 10^{-3}$

ii. $p_3(0.5) = 0.375\,$; $f(0.5) = 0.4$; $E_{abs} = |\ f(0.5) - p_3(0.5)| = 2.5\ \times 10^{-2}$; $E_{rel} = \frac{|\ f(0.5) - p_3(0.5)|}{|f(0.5)|} = 6.2\,5 \times 10^{-2}$; $E_3(0.5) = 5.\,076\ \times 10^{-2}$

(d) Grafico. $f(x)$ rojo, $p_3(x)$ negro



(e) Integral

$\int_0^1 p_3(x)dx = 0.25$;

$\int_0^1 f(x)dx = \frac{1}{2}\ln 2 = 0.346\,6$

$E_{abs} = |\ I - I_3| = 9.6\,6 \times 10^{-2}$

$E_{rel} = \frac{|\ I - I_3|}{|I|} = 2.78\,7 \times 10^{-1}$

3. Propagación de errores

(a) $x^{(0)} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $x^{(1)} = \begin{pmatrix} x \\ y+z \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$; $x^{(2)} = u + v = x + y + z$

(b) $D\varphi^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; $D\varphi^{(1)} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

(c) $fl(\varphi^{(0)}(x^{*(0)})) = \begin{pmatrix} x^* \\ y^* \oplus z^* \end{pmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \right) \cdot \begin{pmatrix} x^* \\ y^* + z^* \end{pmatrix} \Longrightarrow e_1 = 0 \wedge e_2 = \varepsilon_{syz} \Longrightarrow E^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_{syz} \end{bmatrix}$

(d) $fl(\varphi^{(1)}(x^{*(1)})) = u^* \oplus v^* = (1 + e_3)\,(u^* + v^*) \Longrightarrow e_3 = \varepsilon_{suv} \Longrightarrow E^{(2)} = \varepsilon_{suv}$

(e) $\triangle\varphi = D\varphi^{(1)} D\varphi^{(0)} \triangle x + D\varphi^{(1)} E^{(1)} \varphi^{(0)}(x^{*(0)}) + E^{(2)} \varphi^{(1)}(x^{*(1)})$

(f) $\triangle\varphi = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} \triangle x \\ \triangle y \\ \triangle z \end{pmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_{syz} \end{bmatrix} \begin{pmatrix} x^* \\ y^* + z^* \end{pmatrix} + \varepsilon_{suv}(x^* + y^* + z^*)$

(g) $\triangle\varphi = \triangle x + \triangle y + \triangle z + \varepsilon_{syz}\,(y^* + z^*) + \varepsilon_{suv}\,(x^* + y^* + z^*)$

(h) Pero $\varepsilon_{syz} = \frac{y}{y+z}\varepsilon_y + \frac{z}{y+z}\varepsilon_z$ y $\varepsilon_{suv} = \frac{u}{u+v}\varepsilon_u + \frac{v}{u+v}\varepsilon_v = \frac{x}{x+(y+z)}\varepsilon_x + \frac{(y+z)}{x+(y+z)}\varepsilon_{(y+z)}$, reemplazando

(i) $\triangle\varphi = \triangle x + \triangle y + \triangle z + \left(\frac{y}{y+z}\varepsilon_y + \frac{z}{y+z}\varepsilon_z \right)(y^* + z^*) + \left(\frac{x}{x+(y+z)}\varepsilon_x + \frac{(y+z)}{x+(y+z)}\varepsilon_{(y+z)} \right)(x^* + y^* + z^*)$

4. SEL

(a) Descomposicion LU

i. $A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & -2 & 0 & -2 \end{bmatrix};$

ii. $A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & -2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & -4 & 0 \end{bmatrix};$

iii. $A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & \frac{8}{7} \end{bmatrix}$

iv. $m_{21} = -1 \; m_{31} = 1 \; m_{41} = -1; \; m_{32=} - 2 \; m_{42=} 2; \; m_{43=} \frac{4}{7}$

v. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -2 & -\frac{4}{7} & 1 \end{bmatrix}; U = A^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & \frac{8}{7} \end{bmatrix}$

(b) Resolver $Ax = b$, con $A = LU$

i. $Ly = b \iff \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -2 & -\frac{4}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

ii. $y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{8}{7} \end{bmatrix}$

iii. $Ux = y \iff \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & \frac{8}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ \frac{8}{7} \end{bmatrix}$

iv. $x_n = \frac{u_{n(n+1)}}{u_{nn}} \iff x_4 = 1$

v. $x_k = \frac{1}{u_{kk}} \left(u_{k(n+1)} - \sum_{j=k+1}^{n} u_{kj} x_j \right) \quad k = (n-1), ..., 1$

A. $k = 3 \iff x_3 = \frac{1}{u_{33}} (y_3 - [u_{34} x_4]) = \frac{1}{7} (2 - [2 \times 1]) = 0$

B. $k = 2 \iff x_2 = \frac{1}{u_{22}} (y_2 - [u_{23} x_3 + u_{24} x_4]) = \frac{1}{1} (0 - [(-2) \times 0 + 1 \times 1]) = -1$

C. $k = 1 \iff x_1 = \frac{1}{u_{11}} (y_1 - [u_{12} x_2 + u_{13} x_3 + u_{14} x_4]) = \frac{1}{1} (1 - [(-1) \times 1 + 1 \times 0 + 1 \times 1]) = 1$

vi. $x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

(c) Determinante y A^{-1}

i. $\det(A) = u_{11} \times u_{22} \times u_{33} \times u_{44} = 1 \times 1 \times 7 \times \frac{8}{7} = 8 \neq 0$

ii. $\left[A \vdots I \right]^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

iii. $\left[A \vdots I \right]^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 0 & 1 \end{bmatrix}$

iv. $\left[A \vdots I \right]^2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 7 & 2 & 3 & -2 & 1 & 0 \\ 0 & 0 & -4 & 0 & -3 & 2 & 0 & 1 \end{bmatrix}$

v. $\left[A \vdots I\right]^3 = \begin{bmatrix}1 & 0 & -3 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{7} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3 & 0 & 2 & -1 & 0 & 0\\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 7 & 2 & 3 & -2 & 1 & 0\\ 0 & 0 & -4 & 0 & -3 & 2 & 0 & 1\end{bmatrix} =$

A. $\begin{bmatrix}1 & 0 & 0 & -\frac{6}{7} & \frac{5}{7} & -\frac{1}{7} & -\frac{3}{7} & 0\\ 0 & 1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & \frac{2}{7} & 0\\ 0 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} & 0\\ 0 & 0 & 0 & \frac{8}{7} & -\frac{9}{7} & \frac{6}{7} & \frac{4}{7} & 1\end{bmatrix}$

vi. $\left[A \vdots I\right]^4 = \begin{bmatrix}1 & 0 & 0 & \frac{6}{7}\\ 0 & 1 & 0 & -\frac{11}{7}\\ 0 & 0 & 1 & -\frac{2}{7}\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{7}{8}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & -\frac{6}{7} & \frac{5}{7} & -\frac{1}{7} & -\frac{3}{7} & 0\\ 0 & 1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & \frac{2}{7} & 0\\ 0 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} & 0\\ 0 & 0 & 0 & \frac{8}{7} & -\frac{9}{7} & \frac{6}{7} & \frac{4}{7} & 1\end{bmatrix} =$

A. $\begin{bmatrix}1 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{3}{4}\\ 0 & 1 & 0 & 0 & \frac{13}{8} & -\frac{3}{4} & -\frac{1}{2} & -\frac{11}{8}\\ 0 & 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{2} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 1 & -\frac{9}{8} & \frac{3}{4} & \frac{1}{2} & \frac{7}{8}\end{bmatrix}$

vii. $A^{-1} = I^4 = \begin{bmatrix}-\frac{1}{4} & \frac{1}{2} & 0 & \frac{3}{4}\\ \frac{13}{8} & -\frac{3}{4} & -\frac{1}{2} & -\frac{11}{8}\\ \frac{3}{4} & -\frac{1}{2} & 0 & -\frac{1}{4}\\ -\frac{9}{8} & \frac{3}{4} & \frac{1}{2} & \frac{7}{8}\end{bmatrix}$

(d) $cond(A) = \|A\| \cdot \left\|A^{-1}\right\| = \max\{4,6,7,4\} \cdot \max\left\{\frac{6}{4}, \frac{34}{8}, \frac{6}{4}, \frac{26}{8}\right\} = 7 \cdot \frac{34}{8} = \frac{119}{4} = 29.75 \gg 1$
En consecuencia, la matriz A está mal condicionada. Como la matriz está mal condicionada, al perturbar levemente sus coeficientes, se genera una gran alteración de los resultados al resolver sistemas de ecuaciones cuya matriz de coeficientes sea esta.

5. Gauss - Tecnicas de pivoteo

(a) Pivoteo Completo

$\begin{bmatrix}0 & 1 & 1\\ -0.1 & 2 & 0.1\\ 1 & 3 & \mathbf{6}\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 3 & \mathbf{6}\\ -0.1 & 2 & 0.1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Longrightarrow \begin{bmatrix}\mathbf{6} & 3 & 1\\ 0.1 & 2 & -0.1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}z\\ y\\ x\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

$\Longrightarrow \begin{bmatrix}\mathbf{6} & 3 & 1\\ 0.1 & 2 & -0.1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}z\\ y\\ x\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Longrightarrow \begin{bmatrix}6 & 3 & 1\\ 0 & 1.95 & -0.1167\\ 0 & 0.5 & -0.1667\end{bmatrix}\begin{bmatrix}z\\ y\\ x\end{bmatrix}$

$= \begin{bmatrix}1\\ -1.667 \times 10^{-2}\\ 0.8333\end{bmatrix}$

$\Longrightarrow \begin{bmatrix}6.0 & 3 & 1\\ 0 & \mathbf{1.95} & -0.1167\\ 0 & 0.5 & -0.1667\end{bmatrix}\begin{bmatrix}z\\ y\\ x\end{bmatrix} = \begin{bmatrix}1\\ -1.667 \times 10^{-2}\\ 0.8333\end{bmatrix} \Longrightarrow \begin{bmatrix}6.0 & 3 & 1\\ 0 & \mathbf{1.95} & -0.1167\\ 0 & 0 & -0.1368\end{bmatrix}\begin{bmatrix}z\\ y\\ x\end{bmatrix} = \begin{bmatrix}1\\ -0.01667\\ 0.8376\end{bmatrix}$

$\Longrightarrow x = \frac{0.8376}{-0.1368} = -6.123$

$\Longrightarrow y = \frac{-0.01667 - 0.1167 \times 6.123}{1.95} = -0.375$

$\Longrightarrow z = \frac{1 + 3 \times 0.375 + 6.123}{6} = 1.375$

$E_{rel} = \frac{0.002}{6.123} = 3.266 \times 10^{-4}$

(b) Pivoteo parcial

$\begin{bmatrix}0 & 1 & 1\\ -0.1 & 2 & 0.1\\ \mathbf{1} & 3 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Longrightarrow \begin{bmatrix}\mathbf{1} & 3 & 6\\ -0.1 & 2 & 0.1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

$\Longrightarrow \begin{bmatrix}1 & 3 & 6\\ -0.1 & 2 & 0.1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0.1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 3 & 6\\ 0 & \mathbf{2.3} & 0.7\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1.0\\ 0.1\\ 1.0\end{bmatrix}$

$\Longrightarrow \begin{bmatrix}1 & 3 & 6\\ 0 & \mathbf{2.3} & 0.7\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1.0\\ 0.1\\ 1.0\end{bmatrix} \Longrightarrow \begin{bmatrix}1.0 & 3.0 & 6.0\\ 0.0 & 2.3 & 0.7\\ 0.0 & 0 & 0.6957\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1.0\\ 0.1\\ 0.9565\end{bmatrix}$

$\Longrightarrow z = \frac{0.9565}{0.6957} = 1.375$

$\Longrightarrow y = \frac{0.1 - 1.375 \times 0.7}{2.3} = -0.375$

$\Longrightarrow x = 1 + 3 \times 0.375 - 6 \times 1.375 = -6.125$

$E_{rel} = 0$

(c) $\left\|\begin{matrix} 0 & 1 & 1 \\ -0.1 & 2 & 0.1 \\ \mathbf{1} & 3 & 6 \end{matrix}\right\| = \max\{2, 2.2, 10\} = 10$

$$\left\|\begin{bmatrix} 0 & 1 & 1 \\ -0.1 & 2 & 0.1 \\ \mathbf{1} & 3 & 6 \end{bmatrix}^{-1}\right\| = \left\|\begin{matrix} -7.3125 & 1.875 & 1.1875 \\ -0.4375 & 0.625 & 0.0625 \\ 1.4375 & -0.625 & -0.0625 \end{matrix}\right\| = 10.375$$

$Cond(A) = 10 \times 10.375 = 103.75 >> 1$

6. LU

```
For j=1 To (n-1) // los n-1 pivotes
    For i=(j+1) To n // restar a todas las filas debajo del pivote
        For k=j To n // desde el pivote hasta el final de la fila
            a[i,k]=a[i,k]-(a[i,j]/a[j,j])*a[j,k]
        End For k
      L[i,j]=a[i,j]/a[j,j] // copiar a la matriz L
    End For i
    For m=j To n // copiar a la matriz U
        U[j,m]=a[j,m]
    End For m
  End For j
```

7. Determinante

```
det=a[1,1]
For j=1 To (n-1) // los n-1 pivotes
  For i=(j+1) To n // restar a todas las filas debajo del pivote
      For k=j To n // desde el pivote hasta el final de la fila
          a[i,k]=a[i,k]-(a[i,j]/a[j,j])*a[j,k]
      End For k
  End For i
  det=det*a[j+1,j+1]
End For j
```

8. Operaciones

(a) Gauss: $\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\sum_{k=j}^{n} 3 = 3\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}(n-j+1) = 3\sum_{j=1}^{n-1}(n-j)(n-j+1)$

$= 3\sum_{j=1}^{n-1} n - j(1+2n) + j^2 + n^2 = 3\left[(n-1)\,n\ -(1+2n)\frac{(n-1)n}{2} + \frac{n(n-1)(2n-1)}{6} + (n-1)\,n^2\right] = n^3 - n$

$\Longrightarrow O(n^3)$

(b) Laplace: $D(n) = n(2 + D(n-1))$ con la condición inicial $D(1) = 0$

$D(n) = 2n + nD(n-1) \mid\ \times \frac{1}{n^n}$

$\frac{D(n)}{n^n} = \frac{2}{n^{n-1}} + \frac{D(n-1)}{n^{n-1}}\Big|$ se define $Y(n) = \frac{D(n)}{n^n}$

$Y(n) = \frac{2}{n^{n-1}} + Y(n-1) \quad \Longrightarrow \quad Y(n) = Y(1) + \sum_{k=2}^{n}\frac{2}{k^{k-1}}$

$\frac{D(n)}{n^n} = \frac{D(1)}{1^1} + \sum_{k=2}^{n}\frac{2}{k^{k-1}} = \sum_{k=2}^{n}\frac{2}{k^{k-1}} \Longrightarrow D(n) = n^n\sum_{k=2}^{n}\frac{2}{k^{k-1}}$

$\Longrightarrow O(n^n)$

(c) Claramente la diferencia es extremadamente mucha, ya que el algoritmo de Gauss es de orden $O(n^3)$, mientras que el algoritmo de Laplace es de orden $O(n^n)$. Claro que para el caso $n = 2$, el algoritmo de Laplace es más rápido que el algoritmo de Gauss, pero para $n \geq 3$, es más eficiente el algoritmo de Gauss.

9. Numero de Operaciones

(a) Gauss

$$\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\sum_{k=j}^{n+1}(D,M,R)\ +\ (D)\ +\sum_{k=1}^{n-1}\left(\sum_{j=k+1}^{n}(R,M)\ +(D)\ \right)$$
$$=(D,M,R)\ \sum_{j=1}^{n-1}((n-j-1+1)+(n+1-j+1))\ +\ (D)\ +\ \sum_{k=1}^{n-1}((R,M)\ (n-k-1+1)+(D))$$
$$=(D,M,R)2\sum_{j=1}^{n-1}(n-j+1)\ +\ (D)\ +\ (R,M)\sum_{k=1}^{n-1}(n-k)\ +\ (D)(n-1)$$
$$=(D,M,R)2(n(n-1)-\tfrac{n(n-1)}{2}+(n-1))\ +\ (D)\ +\ (R,M)(n(n-1)-\tfrac{n(n-1)}{2})\ +\ (D)(n-1)$$
$$=(D,M,R)(n(n-1)+2(n-1))+(D)n\ +\ (R,M)\tfrac{n(n-1)}{2}$$
$$D=n(n-1)+2(n-1)+n=n^2+2n-2$$
$$M=n(n-1)+2(n-1)+\tfrac{n(n-1)}{2}=\tfrac{3}{2}n^2+\tfrac{1}{2}n-2$$
$$R=n(n-1)+2(n-1)+\tfrac{n(n-1)}{2}=\tfrac{3}{2}n^2+\tfrac{1}{2}n-2$$

(b) Gauss-Jordan

$$\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\sum_{k=j}^{n+1}(D,M,R)\ +\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\sum_{k=j}^{n+1}(D,M,R)\ +\ \sum_{k=1}^{n}(D)$$
$$=2\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\sum_{k=j}^{n+1}(D,M,R)\ +\ (D)n$$
$$=(D,M,R)2(n(n-1)+2(n-1))\ +\ (D)n$$
$$D=2(n(n-1)+2(n-1))+n=2n^2+3n-4$$
$$M=2(n(n-1)+2(n-1))=2n^2+2n-4$$
$$R=2(n(n-1)+2(n-1))=2n^2+2n-4$$

(c) Contando el número total de operaciones:

i. Gauss: $D+M+R=n^2+2n-2+\frac{3}{2}n^2+\frac{1}{2}n-2+\frac{3}{2}n^2+\frac{1}{2}n-2=4n^2+3n-6$

ii. Gauss-Jordan: $D+M+R=2n^2+3n-4+2n^2+2n-4+2n^2+2n-4=6n^2+7n-12$

(d) Se ve claramente que el algoritmo de Gauss es más eficiente que el algoritmo de Gauss-Jordan, pero ambos algoritmos son de orden $O(n^2)$

10. $H_3=\begin{bmatrix}\frac{1}{0+0+1} & \frac{1}{0+1+1} & \frac{1}{0+2+1}\\ \frac{1}{1+0+1} & \frac{1}{1+1+1} & \frac{1}{1+2+1}\\ \frac{1}{2+0+1} & \frac{1}{2+1+1} & \frac{1}{2+2+1}\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{bmatrix}$

(a) $$\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \vdots & 1\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \vdots & 1\end{bmatrix}\Longrightarrow\begin{bmatrix}1 & 0 & 0\\ -\frac{1}{2} & 1 & 0\\ -\frac{1}{3} & 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \vdots & 1\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \vdots & 1\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ 0 & \frac{1}{12} & \frac{1}{12} & \vdots & \frac{1}{2}\\ 0 & \frac{1}{12} & \frac{4}{45} & \vdots & \frac{2}{3}\end{bmatrix}$$
$$\Longrightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -\frac{1}{12} & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ 0 & 1 & 1 & \vdots & 6\\ 0 & \frac{1}{12} & \frac{4}{45} & \vdots & \frac{2}{3}\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ 0 & 1 & 1 & \vdots & 6\\ 0 & 0 & \frac{1}{180} & \vdots & \frac{1}{6}\end{bmatrix}$$
$$\Longrightarrow\begin{bmatrix}1 & 0 & -\frac{1}{3}\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1\\ 0 & 1 & 1 & \vdots & 6\\ 0 & 0 & 1 & \vdots & 30\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & 0 & \vdots & -9\\ 0 & 1 & 0 & \vdots & -24\\ 0 & 0 & 1 & \vdots & 30\end{bmatrix}$$
$$\Longrightarrow\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & 0 & \vdots & -9\\ 0 & 1 & 0 & \vdots & -24\\ 0 & 0 & 1 & \vdots & 30\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & \vdots & 3\\ 0 & 1 & 0 & \vdots & -24\\ 0 & 0 & 1 & \vdots & 30\end{bmatrix}$$

(b) $$\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \vdots & 1\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \vdots & 1\end{bmatrix}\Longrightarrow\begin{bmatrix}1 & 0 & 0\\ -\frac{1}{2} & 1 & 0\\ -\frac{1}{3} & 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \vdots & 1\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \vdots & 1\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01\\ 0 & \frac{1}{12} & \frac{1}{12} & \vdots & 0.495\\ 0 & \frac{1}{12} & \frac{4}{45} & \vdots & 0.663\,333\,333\,333\,333\,333\,33\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01 \\ 0 & 1 & 1 & \vdots & 12 \times 0.495 \\ 0 & \frac{1}{12} & \frac{4}{45} & \vdots & 0.663\,333\,333\,333\,333\,333\,33 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01 \\ 0 & 1 & 1 & \vdots & 5.94 \\ 0 & 0 & \frac{1}{180} & \vdots & 0.168\,333\,333\,333\,333\,333\,33 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \vdots & 1.01 \\ 0 & 1 & 1 & \vdots & 5.94 \\ 0 & 0 & 1 & \vdots & 180 \times 0.168\,333\,333\,333\,333\,333\,33 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \vdots & -9.09 \\ 0 & 1 & 0 & \vdots & -24.36 \\ 0 & 0 & 1 & \vdots & 30.3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & \vdots & -9.09 \\ 0 & 1 & 0 & \vdots & -24.36 \\ 0 & 0 & 1 & \vdots & 30.3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \vdots & 3.09 \\ 0 & 1 & 0 & \vdots & -24.36 \\ 0 & 0 & 1 & \vdots & 30.3 \end{bmatrix}$$

(c) $Cond(H_3) = \left\| \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1} \right\| = \left\| \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \right\| \left\| \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \right\|$

$= \left|1 + \frac{1}{2} + \frac{1}{3}\right| |36 + 192 + 180| = 748$

11. matriz tridiagonal A :

(a) Descomposición LU

i. $i = 1$

A. $l_{11} = a_{11} = 1$

B. $u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{2}$

ii. $i = 2$

A. $l_{21} = a_{21} = \frac{1}{2}$

B. $l_{22} = a_{22} - l_{21}u_{12} = 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

C. $u_{23} = \frac{a_{23}}{l_{22}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$

iii. $i = 3$

A. $l_{32} = a_{32} = \frac{1}{2}$

B. $l_{33} = a_{33} - l_{32}u_{23} = 1 - \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}$

C. $u_{34} = \frac{a_{34}}{l_{33}} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$

iv. $i = 4$

A. $l_{43} = a_{43} = \frac{1}{2}$

B. $l_{44} = a_{44} - l_{43}u_{34} = 1 - \frac{1}{2} \times \frac{3}{4} = \frac{5}{8}$

v. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{5}{8} \end{bmatrix}; U = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) Método de Crout

i. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

A. $y_1 = \frac{b_1}{l_{11}} = 1$

B. $y_2 = \frac{b_2 - y_1 l_{21}}{l_{22}} = -\frac{2}{3}$

C. $y_3 = \frac{b_3 - y_2 l_{32}}{l_{33}} = \frac{1 + \frac{2}{3} \times \frac{1}{2}}{\frac{2}{3}} = 2$

D. $y_4 = \frac{b_4 - y_3 l_{43}}{l_{44}} = \frac{-2 \times \frac{1}{2}}{\frac{5}{8}} = -\frac{8}{5}$

ii. $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 2 \\ -\frac{8}{5} \end{bmatrix}$

A. $x_4 = y_4 = -\frac{8}{5}$

B. $x_3 = y_3 - x_4 u_{34} = 2 - \left(-\frac{8}{5}\right)\frac{3}{4} = \frac{16}{5}$

C. $x_2 = y_2 - x_3 u_{23} = -\frac{2}{3} - \frac{16}{5} \times \frac{2}{3} = -\frac{14}{5}$

D. $x_1 = y_1 - x_2 u_{12} = 1 - \left(-\frac{14}{5}\right)\frac{1}{2} = \frac{12}{5}$

(c) Descomposición de Cholesky

i. Definida positiva

A. $\det(A_{11}) = |1| = 1 > 0$

$$\det(A_{22}) = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} > 0$$

$$\det(A_{33}) = \begin{vmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} > 0$$

$$\det(A_{44}) = \begin{vmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{vmatrix} = \frac{5}{16} > 0$$

ii. $$L = \begin{bmatrix} \sqrt{a_{11}} & 0 & 0 & 0 \\ \frac{a_{21}}{l_{11}} & \sqrt{a_{22} - \sum_{k=1}^{2-1} (l_{2k})^2} & 0 & 0 \\ \frac{a_{31}}{l_{11}} & \frac{a_{32} - \sum_{k=1}^{2-1} l_{3k} l_{2k}}{l_{22}} & \sqrt{a_{33} - \sum_{k=1}^{3-1} (l_{3k})^2} & 0 \\ \frac{a_{41}}{l_{11}} & \frac{a_{42} - \sum_{k=1}^{2-1} l_{4k} l_{2k}}{l_{22}} & \frac{a_{43} - \sum_{k=1}^{3-1} l_{4k} l_{3k}}{l_{33}} & \sqrt{a_{44} - \sum_{k=1}^{4-1} (l_{4k})^2} \end{bmatrix}$$

iii. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \sqrt{1 - \left(\frac{1}{2}\right)^2} & 0 & 0 \\ 0 & \frac{\frac{1}{2} - 0}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} & \sqrt{1 - \left[0 + \left(\frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}\right)^2\right]} & 0 \\ 0 & \frac{0 - 0}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} & \frac{\frac{1}{2} - 0 - 0}{\sqrt{1 - \left[(0) + \left(\frac{1}{2}\right)^2\right]}} & \sqrt{1 - \left[0 + 0 + \left(\frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}\right)^2\right]} \end{bmatrix}$$

iv. $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{2}\sqrt{3} & 0 \\ 0 & 0 & \frac{1}{4}\sqrt{2}\sqrt{3} & \frac{1}{4}\sqrt{2}\sqrt{5} \end{bmatrix}$$

12. Métodos Iterativos

(a) Jacobi

$$M = -D^{-1}(L + U) = -\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix}$$

i. $$N = D^{-1} b = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

ii. $x^{k+1} = Mx^k + N \qquad \vee \qquad \begin{array}{ll} x_1^{k+1} = \frac{1}{3}\left(1 - x_2^k\right) & x_3^{k+1} = \frac{1}{3}\left(1 - \left[x_2^k + x_4^k\right]\right) \\ x_2^{k+1} = \frac{1}{3}\left(1 - \left[x_1^k + x_3^k\right]\right) & x_4^{k+1} = \frac{1}{3}\left(1 - x_3^k\right) \end{array}$

A. $$x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

B. $$x^1 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

C. $x^2 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{1}{9} \\ \frac{2}{9} \end{bmatrix}$

D. $x^3 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{1}{9} \\ \frac{2}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{8}{27} \\ \frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{27} \end{bmatrix}$

E. $x^4 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{8}{27} \\ \frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{27} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{27} \\ \frac{13}{81} \\ \frac{13}{81} \\ \frac{7}{27} \end{bmatrix}$

F. $x^5 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{7}{27} \\ \frac{13}{81} \\ \frac{13}{81} \\ \frac{7}{27} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{68}{243} \\ \frac{47}{243} \\ \frac{47}{243} \\ \frac{68}{243} \end{bmatrix} = \begin{bmatrix} 0.279\,84 \\ 0.193\,42 \\ 0.193\,42 \\ 0.279\,84 \end{bmatrix}$

(b) Método de Gauss-Seidel

i. $M = -(D+L)^{-1}U = -\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix}$

ii. $N = (D+L)^{-1}\,b = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix}$

iii. $x^{k+1} = Mx^k + N \qquad \Longleftrightarrow \qquad \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2^k + \frac{1}{3} \\ \frac{1}{9}x_2^k - \frac{1}{3}x_3^k + \frac{2}{9} \\ -\frac{1}{27}x_2^k + \frac{1}{9}x_3^k - \frac{1}{3}x_4^k + \frac{7}{27} \\ \frac{1}{81}x_2^k - \frac{1}{27}x_3^k + \frac{1}{9}x_4^k + \frac{20}{81} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{3}\left(1 - x_2^k\right) \\ \frac{1}{9}\left(x_2^k - 3x_3^k + 2\right) \\ \frac{1}{27}\left(7 - x_2^k + 3x_3^k - 9x_4^k\right) \\ \frac{1}{81}\left(x_2^k - 3x_3^k + 9x_4^k + 20\right) \end{bmatrix}$

A. $x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

B. $x^1 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix}$

C. $x^2 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} \frac{7}{27} \\ \frac{13}{81} \\ \frac{16}{81} \\ \frac{65}{243} \end{bmatrix}$

D. $x^3 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{7}{27} \\ \frac{13}{81} \\ \frac{16}{81} \\ \frac{65}{243} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} \frac{68}{243} \\ \frac{127}{729} \\ \frac{407}{2187} \\ \frac{1780}{6561} \end{bmatrix} =$

E. $x^4 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{68}{243} \\ \frac{127}{729} \\ \frac{407}{2187} \\ \frac{1780}{6561} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} \frac{602}{2187} \\ \frac{1178}{6561} \\ \frac{1201}{6561} \\ \frac{5360}{19\,683} \end{bmatrix}$

F. $x^5 = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{3} \\ 0 & \frac{1}{81} & -\frac{1}{27} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{602}{2187} \\ \frac{1178}{6561} \\ \frac{1201}{6561} \\ \frac{5360}{19\,683} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \\ \frac{7}{27} \\ \frac{20}{81} \end{bmatrix} = \begin{bmatrix} \frac{5383}{19\,683} \\ \frac{10\,697}{59\,049} \\ \frac{32\,272}{177\,147} \\ \frac{144\,875}{531\,441} \end{bmatrix} = \begin{bmatrix} 0.273\,48 \\ 0.181\,15 \\ 0.182\,18 \\ 0.272\,61 \end{bmatrix}$

(c) $E_{rel}(x_{GJ}^5) = \frac{\left\|x_J - x_{GJ}^5\right\|_\infty}{\left\|x_J\right\|_\infty} = \frac{\left\| \begin{bmatrix} \frac{3}{11} \\ \frac{2}{11} \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} - \begin{bmatrix} \frac{68}{243} \\ \frac{47}{243} \\ \frac{47}{243} \\ \frac{68}{243} \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} \frac{3}{11} \\ \frac{2}{11} \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} \right\|_\infty} = \frac{\frac{31}{2673}}{\frac{3}{11}} = \frac{31}{729} = 4.252\,4 \times 10^{-2}$

$E_{rel}(x_{GS}^5) = \frac{\left\|x_J - x_{GS}^5\right\|_\infty}{\left\|x_J\right\|_\infty} = \frac{\left\| \begin{bmatrix} \frac{3}{11} \\ \frac{2}{11} \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} - \begin{bmatrix} \frac{5383}{19\,683} \\ \frac{10\,697}{59\,049} \\ \frac{32\,272}{177\,147} \\ \frac{144\,875}{531\,441} \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} \frac{3}{11} \\ \frac{2}{11} \\ \frac{2}{11} \\ \frac{3}{11} \end{bmatrix} \right\|_\infty} = \frac{164}{59\,049} = 2.777\,4 \times 10^{-3}$

En consecuencia, puesto que el Error Relativo para la quinta iteración, al aplicar el Método de Gauss-Jacobi es casi un orden de magnitud mayor que si aplicamos Gauss-Seidel, podemos decir que la mejor solución es la entregada por este último método.

13. SEL

(a) **Métodos Directos**

i. $A^0 = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$A^1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{4} & 1 & 0 \\ -\frac{0}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$

$U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}$

Definiendo $U\vec{x} = \vec{y}$ el sistema queda

$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow \vec{y} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{2}{3} \end{bmatrix}$

Resolviendo $U\vec{x} = \vec{y}$

$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}$

ii. Es definida positiva si sus subdeterminantes son positivas

$|4| > 0$

$\begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix} = 12 > 0$

$\begin{vmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{vmatrix} = 32 > 0 \Longrightarrow$ es definida positiva

Entonces su factorizacion de Cholesky es

$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & \frac{2}{3}\sqrt{3} & \frac{2}{3}\sqrt{2}\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \sqrt{3} & \frac{2}{3}\sqrt{3} \\ 0 & 0 & \frac{2}{3}\sqrt{2}\sqrt{3} \end{bmatrix}$

(b) **Métodos Iterativos**

i. Gauss-Seidel

$M = -(D+L)^{-1}U = -\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$

$$\vec{N}=(D+L)^{-1}\vec{b}=\begin{bmatrix}4&0&0\\2&4&0\\0&2&4\end{bmatrix}^{-1}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}$$

$$\vec{x}^{k+1}=M\vec{x}^k+\vec{N}=\begin{bmatrix}0&-\frac{1}{2}&0\\0&\frac{1}{4}&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1^k\\x_2^k\\x_3^k\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}x_2^k+\frac{1}{4}\\\frac{1}{4}x_2^k-\frac{1}{2}x_3^k+\frac{1}{8}\\-\frac{1}{8}x_2^k+\frac{1}{4}x_3^k+\frac{3}{16}\end{bmatrix}$$

A. $\vec{x}^0=\begin{bmatrix}0\\0\\0\end{bmatrix}$

B. $\vec{x}^1=\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}$

C. $\vec{x}^2=\begin{bmatrix}\frac{3}{16}\\\frac{1}{16}\\\frac{7}{32}\end{bmatrix}$

D. $\vec{x}^3=\begin{bmatrix}\frac{7}{32}\\\frac{1}{32}\\\frac{15}{64}\end{bmatrix}$

E. $\vec{x}^4=\begin{bmatrix}\frac{15}{64}\\\frac{1}{64}\\\frac{31}{128}\end{bmatrix}=\begin{bmatrix}0.234\,375\\0.015\,625\\0.242\,187\,5\end{bmatrix}$

F. $\vec{x}^5=\begin{bmatrix}\frac{31}{128}\\\frac{1}{128}\\\frac{63}{256}\end{bmatrix}=\begin{bmatrix}0.242\,187\,5\\0.007\,812\,5\\0.246\,093\,75\end{bmatrix}$

Solo 4 iteraciones eran necesarias, es decir, hasta $\vec{x}^4$.

ii. $\rho(T_G)=\max\limits_{i=1\ldots n}|\lambda_i|$ donde λ_i es un valor propio de T_G

$\det(T_G-I\lambda)=0$

$$\begin{vmatrix}-\lambda&-\frac{1}{2}&0\\0&\frac{1}{4}-\lambda&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}-\lambda\end{vmatrix}=\frac{1}{2}\lambda^2-\lambda^3=0\Longrightarrow\lambda_1=\lambda_2=0\wedge\lambda_3=\frac{1}{2}\Longrightarrow\rho(T_G)=\frac{1}{2}$$

$$\overline{\omega}=\frac{2}{1+\sqrt{1-\rho(T_G)}}=\frac{2}{1+\sqrt{1-\frac{1}{2}}}=\frac{2}{\frac{1}{2}\sqrt{2}+1}=4-2\sqrt{2}=2\left(2-\sqrt{2}\right)=1.171\,572\,875\,253\,809\,902\,4$$

iii. Sor

$$\vec{x}^{k+1}=(1-\overline{\omega})\,\vec{x}^k+\overline{\omega}\left(M\vec{x}^k+\vec{N}\right)=(1-1.171\,5)\begin{bmatrix}x_1^k\\x_2^k\\x_3^k\end{bmatrix}+1.171\,5\left[\begin{bmatrix}0&-\frac{1}{2}&0\\0&\frac{1}{4}&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1^k\\x_2^k\\x_3^k\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}\right]=$$

$$-0.1715\begin{bmatrix}x_1^k\\x_2^k\\x_3^k\end{bmatrix}+1.171\,5\begin{bmatrix}-\frac{1}{2}x_2^k+\frac{1}{4}\\\frac{1}{4}x_2^k-\frac{1}{2}x_3^k+\frac{1}{8}\\-\frac{1}{8}x_2^k+\frac{1}{4}x_3^k+\frac{3}{16}\end{bmatrix}=\begin{bmatrix}-0.171\,5x_1^k-0.585\,75x_2^k+0.292\,875\\0.121\,375\,x_2^k-0.585\,75x_3^k+0.146\,437\,5\\-0.146\,437\,5x_2^k+0.121\,375\,x_3^k+0.219\,656\,25\end{bmatrix}$$

A. $\vec{x}^0=\begin{bmatrix}0\\0\\0\end{bmatrix}$

B. $\vec{x}^1=(1-1.171\,5)\begin{bmatrix}0\\0\\0\end{bmatrix}+1.171\,5\left[\begin{bmatrix}0&-\frac{1}{2}&0\\0&\frac{1}{4}&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}\right]=1.171\,5\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}=\begin{bmatrix}0.292\,875\\0.146\,437\,5\\0.219\,656\,25\end{bmatrix}$

C. $\vec{x}^2=(1-1.171\,5)\begin{bmatrix}0.292\,875\\0.146\,437\,5\\0.219\,656\,25\end{bmatrix}+1.171\,5\left[\begin{bmatrix}0&-\frac{1}{2}&0\\0&\frac{1}{4}&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}0.292\,875\\0.146\,437\,5\\0.219\,656\,25\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}\right]$

$=\begin{bmatrix}0.156\,871\,171\,875\\0.035\,547\,703\,125\\0.224\,873\,085\,937\,5\end{bmatrix}$

D. $\vec{x}^3=(1-1.171\,5)\begin{bmatrix}0.156\,871\,171\,875\\0.035\,547\,703\,125\\0.224\,873\,085\,937\,5\end{bmatrix}+1.171\,5\left[\begin{bmatrix}0&-\frac{1}{2}&0\\0&\frac{1}{4}&-\frac{1}{2}\\0&-\frac{1}{8}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}0.156\,871\,171\,875\\0.035\,547\,703\,125\\0.224\,873\,085\,937\,5\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\\\frac{1}{8}\\\frac{3}{16}\end{bmatrix}\right]=$

$\begin{bmatrix}0.245\,149\,526\,917\,968\,75\\0.019\,032\,692\,378\,906\,25\\0.241\,744\,704\,029\,296\,875\end{bmatrix}$

Hasta acá es necesario, es decir, 3 iteraciones

14. Algoritmos

(a) Cholesky

$$Ops = \sqrt{\ } + \sum_{j=2}^{n} D + \sum_{k=2}^{n-1}\left[\sum_{m=1}^{k-1}(M,S) + (\sqrt{\ },R) + \sum_{i=k+1}^{n}\left(\sum_{l=1}^{k-1}(S,M) + (R,D)\right)\right]$$

i. $= \sqrt{\ } + (n-1)D + \sum_{k=2}^{n-1}\left[(k-1)(M,S) + (\sqrt{\ },R) + \sum_{i=k+1}^{n}((k-1)(S,M) + (R,D))\right]$

$= \sqrt{\ } + (n-1)D + (M,S)\sum_{k=2}^{n-1}(k-1) + (\sqrt{\ },R)\,(n-2) + (S,M)\sum_{k=2}^{n-1}\sum_{i=k+1}^{n}(k-1) + (R,D)\sum_{k=2}^{n-1}(n-k)$

$= \sqrt{\ } + (n-1)D + (M,S)\,\frac{(n-1)(n-2)}{2} + (\sqrt{\ },R)\,(n-2) + (S,M)\sum_{k=2}^{n-1}(k-1)(n-k) + (R,D)\,\frac{(n-1)(n-2)}{2}$

$= \sqrt{\ }(n-1) + D(n-1 + \frac{(n-1)(n-2)}{2}) + R(n-2 + \frac{(n-1)(n-2)}{2}) + (S,M)(\frac{(n-1)(n-2)}{2} + \sum_{k=2}^{n-1}((n+1)k - n - k^2))$

$= \sqrt{\ }(n-1) + D(\frac{n(n-1)}{2}) + R(\frac{n(n-1)-2}{2}) + (S,M)(\frac{(n-1)(n-2)}{2} + (n+1)\frac{(n-2)(n+1)}{2} - n(n-2) - \frac{(2n-1)(n-1)n}{6} + 1)$

$= \sqrt{\ }(n-1) + D(\frac{n(n-1)}{2}) + R(\frac{n(n-1)-2}{2}) + (S,M)(\frac{1}{6}n^3 - \frac{7}{6}n + 1)$ / tomando todas las operaciones

$= (n-1) + (\frac{n(n-1)}{2}) + (\frac{n(n-1)-2}{2}) + (\frac{1}{6}n^3 - \frac{7}{6}n + 1) = \frac{n^3+6n^2-7n-6}{6}$

$\Longrightarrow O(n^3)$

15. $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

(a) Definida positiva

i. $\det(P_{11}) = |6| > 0$

ii. $\det(P_{22}) = \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3 > 0$

iii. $\det(P_{33}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 1 > 0$

(b) Descomposición de Cholesky

$$L = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ \frac{1}{\sqrt{1}} & \sqrt{2-\left(\frac{1}{\sqrt{1}}\right)^2} & 0 \\ \frac{1}{\sqrt{1}} & \frac{3-\frac{1}{\sqrt{1}}\times\frac{1}{\sqrt{1}}}{\sqrt{2-\left(\frac{1}{\sqrt{1}}\right)^2}} & \sqrt{6-\left(\frac{1}{\sqrt{1}}\right)^2-\left(\frac{3-\frac{1}{\sqrt{1}}\times\frac{1}{\sqrt{1}}}{\sqrt{2-\left(\frac{1}{\sqrt{1}}\right)^2}}\right)^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

16. Gauss-Seidel

(a) $M = -(D+L)^{-1}U = -\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$

(b) $\vec{N} = (D+L)^{-1}\vec{b} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{8} \\ \frac{3}{16} \end{bmatrix}$

(c) $\vec{x}^{k+1} = M\vec{x}^k + \vec{N} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{8} \\ \frac{3}{16} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2^k + \frac{1}{4} \\ \frac{1}{4}x_2^k - \frac{1}{2}x_3^k + \frac{1}{8} \\ -\frac{1}{8}x_2^k + \frac{1}{4}x_3^k + \frac{3}{16} \end{bmatrix}$

i. $\vec{x}^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ii. $\vec{x}^1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{8} \\ \frac{3}{16} \end{bmatrix}$

iii. $\vec{x}^2 = \begin{bmatrix} \frac{3}{16} \\ \frac{1}{16} \\ \frac{7}{32} \end{bmatrix}$

iv. $\vec{x}^3 = \begin{bmatrix} \frac{7}{32} \\ \frac{1}{32} \\ \frac{15}{64} \end{bmatrix}$

v. $\vec{x}^4 = \begin{bmatrix} \frac{15}{64} \\ \frac{1}{64} \\ \frac{31}{128} \end{bmatrix} = \begin{bmatrix} 0.234\,375 \\ 0.015\,625 \\ 0.242\,187\,5 \end{bmatrix}$

vi. $\vec{x}^5 = \begin{bmatrix} \frac{31}{128} \\ \frac{1}{128} \\ \frac{63}{256} \end{bmatrix} = \begin{bmatrix} 0.242\,187\,5 \\ 0.007\,812\,5 \\ 0.246\,093\,75 \end{bmatrix}$

(d) $E_{rel} = \dfrac{\left\| \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix} - \begin{bmatrix} 0.242\,187\,5 \\ 0.007\,812\,5 \\ 0.246\,093\,75 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix} \right\|} = \frac{0.007\,812\,5}{0.25} = 0.031\,25 \leq 5 \times 10^{-2}$

17. Lagrange

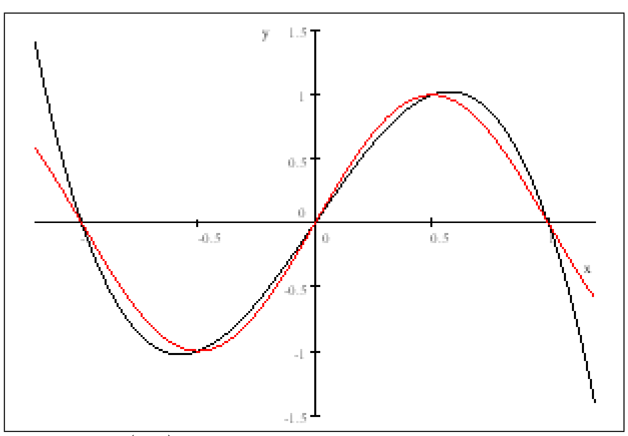
i	x_i	$f(x_i)$	$L_{4,i}(x)$		$L_{4,i}(x)$
0	-1	0	$\frac{(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)}{(-1+\frac{1}{2})(-1)(-1-\frac{1}{2})(-1-1)}$		$\frac{1}{6}x - \frac{1}{6}x^2 - \frac{2}{3}x^3 + \frac{2}{3}x^4$
1	$-\frac{1}{2}$	-1	$\frac{(x+1)x(x-\frac{1}{2})(x-1)}{(-\frac{1}{2}+1)(-\frac{1}{2})(-\frac{1}{2}-\frac{1}{2})(-\frac{1}{2}-1)}$		$-\frac{4}{3}x + \frac{8}{3}x^2 + \frac{4}{3}x^3 - \frac{8}{3}x^4$
2	0	0	$\frac{(x+1)(x+\frac{1}{2})(x-\frac{1}{2})(x-1)}{(1)(\frac{1}{2})(\frac{1}{2})(-1)}$	$=$	$5x^2 - 4x^4 - 1$
3	$\frac{1}{2}$	1	$\frac{(x+1)(x+\frac{1}{2})x(x-1)}{(\frac{1}{2}+1)(\frac{1}{2}+\frac{1}{2})\frac{1}{2}(\frac{1}{2}-1)}$		$\frac{4}{3}x + \frac{8}{3}x^2 - \frac{4}{3}x^3 - \frac{8}{3}x^4$
4	1	0	$\frac{(x+1)(x+\frac{1}{2})x(x-\frac{1}{2})}{(1+1)(1+\frac{1}{2})1(1-\frac{1}{2})}$		$-\frac{1}{6}x - \frac{1}{6}x^2 + \frac{2}{3}x^3 + \frac{2}{3}x^4$

$P_4(x) = 0 \times (L_{4,0}(x)) + (-1) \times \left(-\frac{4}{3}x + \frac{8}{3}x^2 + \frac{4}{3}x^3 - \frac{8}{3}x^4\right) + 0 \times (L_{4,2}(x)) + 1 \times \left(\frac{4}{3}x + \frac{8}{3}x^2 - \frac{4}{3}x^3 - \frac{8}{3}x^4\right) + 0 \times (L_{4,4}(x))$

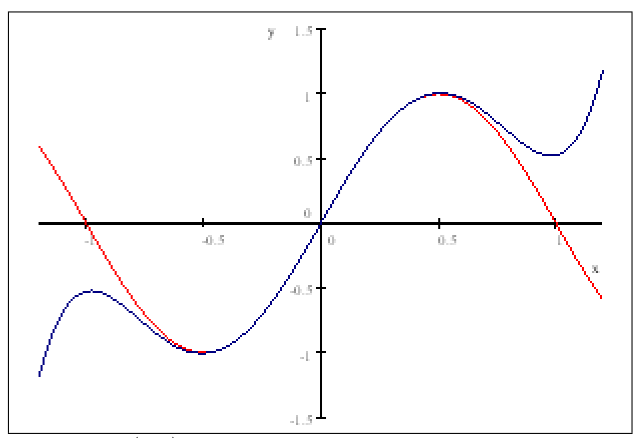
$P_4(x) = \frac{4}{3}x + \frac{8}{3}x^2 - \frac{4}{3}x^3 - \frac{8}{3}x^4 + \frac{4}{3}x - \frac{8}{3}x^2 - \frac{4}{3}x^3 + \frac{8}{3}x^4 = \frac{8}{3}x - \frac{8}{3}x^3$

b. $E_4(x) = \Pi_{i=0}^{4}(x - x_i)\frac{f^{(5)}(\xi)}{(5)!} = (x+1)\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1)\,\frac{\pi^5\cos(\pi\xi)}{120} \leq \pi^5\left(\frac{1}{2}\right)^5 = \frac{1}{32}\pi^5 \simeq 9.563$

c. $T_5(x) = \pi x - \frac{1}{6}\pi^3 x^3 + \frac{1}{120}\pi^5 x^5$



Rojo: $Sen(\pi x)$
Negro: Lagrange $\frac{8}{3}x - \frac{8}{3}x^3$



Rojo: $Sen(\pi x)$
Azul: Taylor $\pi x - \frac{1}{6}\pi^3 x^3 + \frac{1}{120}\pi^5 x^5$

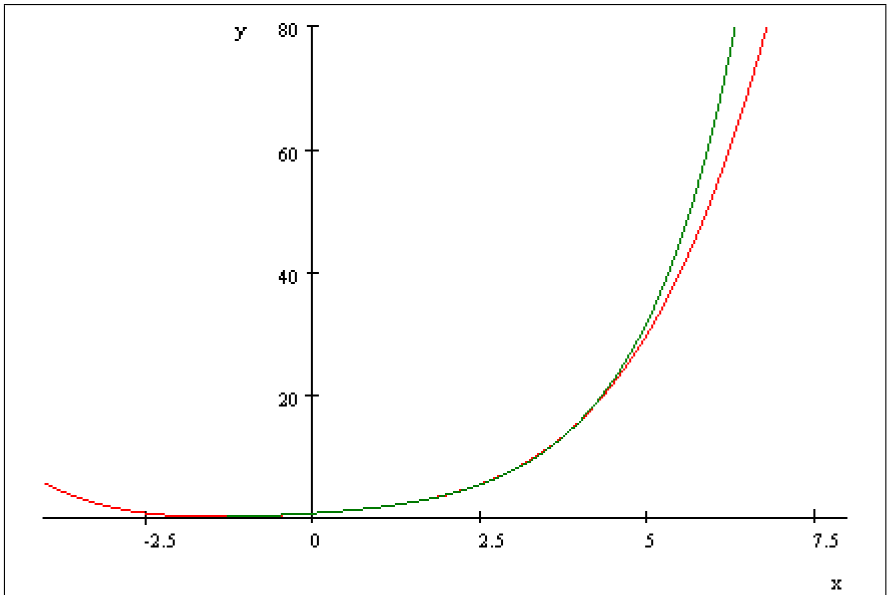
18. Newton

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}]$
-1	$\frac{1}{2}$	$\frac{1-\frac{1}{2}}{0+1} = 0.5$	$\frac{1-0.5}{1+1} = 0.25$	$\frac{0.5-0.25}{2+1} = \frac{1}{12}$	$\frac{\frac{5}{24}-\frac{1}{12}}{4+1} = \frac{1}{40}$
0	1	$\frac{2-1}{1-0} = 1$	$\frac{2-1}{2-0} = 0.5$	$\frac{\frac{4}{3}-0.5}{4-0} = \frac{5}{24}$	
1	2	$\frac{4-2}{2-1} = 2$	$\frac{6-2}{4-1} = \frac{4}{3}$		
2	4	$\frac{16-4}{4-2} = 6$			
4	16				

$P_4(x) = f[x_0] + (x - x_0)\, f[x_0, x_1] + (x - x_0)\,(x - x_1)\, f[x_0, x_1, x_2] + ... + \Pi_{j=0}^{n}(x - x_j) f[x_0 ... x_n]$

$P_4(x) = \frac{1}{2} + (x+1)\, 0.5 + (x+1)\, x 0.25 + (x+1)\, x(x-1)\frac{1}{12} + (x+1)\, x(x-1)(x-2)\frac{1}{40}$

$P_4(x) = 1 + \frac{43}{60}x + \frac{9}{40}\, x^2 + \frac{1}{30}x^3 + \frac{1}{40}x^4$



Rojo:Newton $1 + \frac{43}{60}x + \frac{9}{40}\, x^2 + \frac{1}{30}x^3 + \frac{1}{40}x^4$ Verde:2^x

19. Newton con derivadas

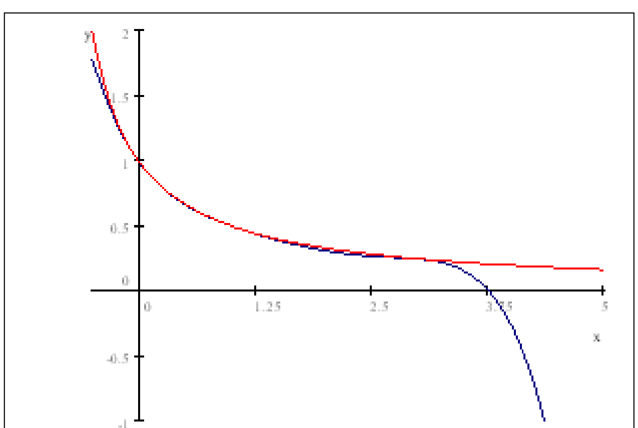
$f'(x) = -\frac{1}{(x+1)^2}$

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+}$
0	1	$f'(0) = -1$	$\frac{-\frac{1}{2}+1}{1-0} = \frac{1}{2}$	$\frac{\frac{1}{4}-\frac{1}{2}}{1-0} = -\frac{1}{4}$	$\frac{-\frac{1}{16}+\frac{1}{4}}{3-0} = \frac{1}{16}$	$\frac{\frac{1}{64}-\frac{1}{16}}{3-0} = -\frac{1}{64}$
0	1	$\frac{\frac{1}{2}-1}{1-0} = -\frac{1}{2}$	$\frac{-\frac{1}{4}+\frac{1}{2}}{1-0} = \frac{1}{4}$	$\frac{\frac{1}{16}-\frac{1}{4}}{3-0} = -\frac{1}{16}$	$\frac{-\frac{1}{64}+\frac{1}{16}}{3-0} = \frac{1}{64}$	
1	$\frac{1}{2}$	$f'(1) = -\frac{1}{4}$	$\frac{-\frac{1}{8}+\frac{1}{4}}{3-1} = \frac{1}{16}$	$\frac{\frac{1}{32}-\frac{1}{16}}{3-1} = -\frac{1}{64}$		
1	$\frac{1}{2}$	$\frac{\frac{1}{4}-\frac{1}{2}}{3-1} = -\frac{1}{8}$	$\frac{-\frac{1}{16}+\frac{1}{8}}{3-1} = \frac{1}{32}$			
3	$\frac{1}{4}$	$f'(3) = -\frac{1}{16}$				
3	$\frac{1}{4}$					

$P_5(x) = 1 + x\,(-1) + x^2\frac{1}{2} + x^2(x-1)\left(-\frac{1}{4}\right) + x^2(x-1)^2\frac{1}{16} + x^2(x-1)^2(x-3)\left(-\frac{1}{64}\right) : \; \frac{55}{64}x^2 - \frac{31}{64}x^3 + \frac{9}{64}x^4 - \frac{1}{64}x^5$

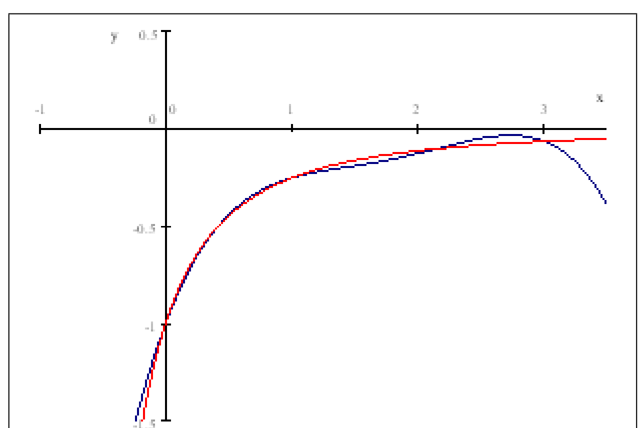
$P_5(x) = 1 - x + \frac{55}{64}x^2 - \frac{31}{64}x^3 + \frac{9}{64}x^4 - \frac{1}{64}x^5$

$\frac{\partial P_5(x)}{\partial x} = \frac{55}{32}x - \frac{93}{64}x^2 + \frac{9}{16}x^3 - \frac{5}{64}x^4 - 1$



Rojo: $\frac{1}{x+1}$

Azul: $1 - x + \frac{55}{64}x^2 - \frac{31}{64}x^3 + \frac{9}{64}x^4 - \frac{1}{64}x^5$



Rojo: $-\frac{1}{(x+1)^2}$

Azul: derivada $\frac{55}{32}x - \frac{93}{64}x^2 + \frac{9}{16}x^3 - \frac{5}{64}x^4 - 1$

20. Primero se construiye un Spline cúbico $S_x(t)$ el cual depende de $S_i(t), i = 0, 1, 2, 3,$ donde

$$S_x(t) = \begin{cases} S_0(t) = S_{0,0} + S_{0,1}(t-0) + S_{0,2}(t-0)^2 + S_{0,3}(t-0)^3 & t \in [0,3] \\ S_1(t) = S_{1,0} + S_{1,1}(t-3) + S_{1,2}(t-3)^2 + S_{1,3}(t-3)^3 & t \in [3,5] \\ S_2(t) = S_{2,0} + S_{2,1}(t-5) + S_{2,2}(t-5)^2 + S_{2,3}(t-5)^3 & t \in [5,8] \\ S_3(t) = S_{3,0} + S_{3,1}(t-8) + S_{3,2}(t-8)^2 + S_{3,3}(t-8)^3 & t \in [8,10] \end{cases}$$

La matriz es

$$\begin{bmatrix} 1^\circ & \text{Condicion} & \text{de} & \text{borde} & \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 \\ 2^\circ & \text{Condicion} & \text{de} & \text{borde} & \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 = 3(d_1 - d_0) \\ \mu_2 = 3(d_2 - d_1) \\ \mu_3 = 3(d_3 - d_2) \\ \mu_4 \end{bmatrix}$$

Los coeficientes son

$$\begin{array}{ll} h_0 = 3 - 0 = 3 & d_0 = \frac{225-0}{3} = 75 \\ h_1 = 5 - 3 = 2 & d_1 = \frac{383-225}{2} = 79 \\ h_2 = 8 - 5 = 3 & d_2 = \frac{623-383}{3} = 80 \\ h_3 = 11 - 8 = 3 & d_3 = \frac{1001-623}{3} = 126 \end{array}$$

Reemplazando en la matriz es

$$\begin{bmatrix} & & & & \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 2 & 10 & 3 & 0 \\ 0 & 0 & 3 & 12 & 3 \\ & & & & \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ 12 \\ 3 \\ 138 \\ \mu_4 \end{bmatrix}$$

C.B: Spline Natural

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 10 & 2 & 0 & 0 \\ 0 & 2 & 10 & 3 & 0 \\ 0 & 0 & 3 & 12 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 3 \\ 138 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{116}{59} \\ -\frac{226}{59} \\ \frac{735}{59} \\ 0 \end{bmatrix}$$

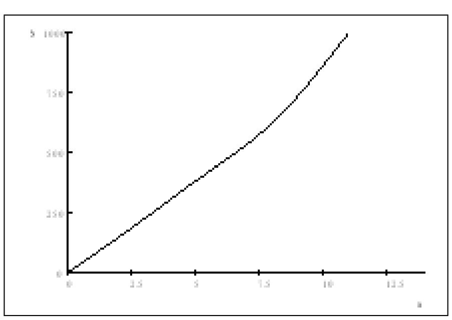
$$S_{k,0} = y_k \qquad S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{3} \qquad S_{k,2} = m_k \qquad S_{k,3} = \frac{m_{k+1} - m_k}{3h_k}$$

$$S_x(t) = \begin{cases} S_0(t) = 0 + \left(75 - \frac{3(2\times 0 + \frac{116}{59})}{3}\right)(t-0) + 0 \times (t-0)^2 + \frac{\frac{116}{59} - 0}{3\times 3}(t-0)^3 & t \in [0,3] \\ S_1(t) = 225 + \left(79 - \frac{2(2\times\frac{116}{59} - \frac{226}{59})}{3}\right)(t-3) + \frac{116}{59}(t-3)^2 + \frac{-\frac{226}{59} - \frac{116}{59}}{3\times 2}(t-3)^3 & t \in [3,5] \\ S_2(t) = 383 + \left(80 - \frac{3(2\times\left(-\frac{226}{59}\right) + \frac{735}{59})}{3}\right)(t-5) + \left(-\frac{226}{59}\right)(t-5)^2 + \frac{\frac{735}{59} + \frac{226}{59}}{3\times 3}(t-5)^3 & t \in [5,8] \\ S_3(t) = 623 + \left(126 - \frac{3(2\times\frac{735}{59} + 0)}{3}\right)(t-8) + \frac{735}{59} \times (t-8)^2 + \frac{0 - \frac{735}{59}}{3\times 3}(t-8)^3 & t \in [8,11] \end{cases}$$

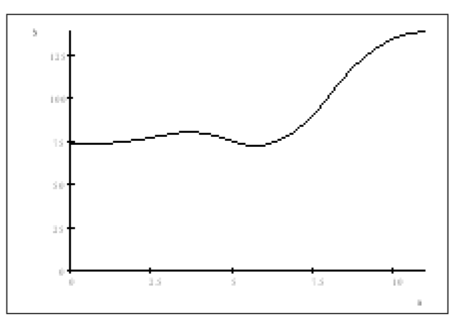
$$S_x(t) = \begin{cases} \frac{4309}{59}t + \frac{116}{531}t^3 & \text{if} \quad 0 \le t \wedge t \le 3 \\ \frac{2422}{59}t + \frac{629}{59}t^2 - \frac{57}{59}t^3 + \frac{1887}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ \frac{44\,116}{177}t - \frac{5483}{177}t^2 + \frac{961}{531}t^3 - \frac{167\,267}{531} & \text{if} \quad 5 \le t \wedge t \le 8 \\ \frac{2695}{59}t^2 - 364t - \frac{245}{177}t^3 + \frac{233\,695}{177} & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases} \qquad \begin{array}{c} S(0) = 0 \\ S(3) = 225 \\ S(5) = 383 \\ S(8) = 623 \\ S(11) = 1001 \end{array}$$

$$D_x(t) = \begin{cases} \frac{116}{177}t^2 + \frac{4309}{59} & \text{if} \quad 0 \le t \wedge t \le 3 \\ \frac{1258}{59}t - \frac{171}{59}t^2 + \frac{2422}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ -\frac{10\,966}{177}t + \frac{961}{177}t^2 + \frac{44\,116}{177} & \text{if} \quad 5 \le t \wedge t \le 8 \\ \frac{5390}{59}t - \frac{245}{59}t^2 - 364 & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases} \qquad \begin{array}{c} D(0) = \frac{17\,807}{241} = 73.887\,966\,804\,979\,253\,112 \\ D(3) = \frac{18\,611}{241} = 77.224\,066\,390\,041\,493\,776 \\ D(5) = \frac{19\,359}{241} = 80.327\,800\,829\,875\,518\,672 \\ D(8) = \frac{18\,804}{241} = 78.024\,896\,265\,560\,165\,975 \\ D(11) = \frac{87\,909}{1205} = 72.953\,526\,970\,954\,356\,846 \end{array}$$

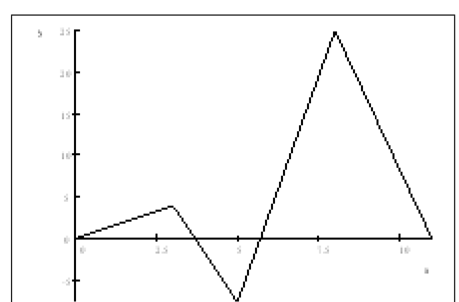
$$F_x(t) = \begin{cases} \frac{232}{177}t & \text{if} \quad 0 \le t \wedge t \le 3 \\ -\frac{342}{59}t + \frac{1258}{59} & \text{if} \quad 3 \le t \wedge t \le 5 \\ \frac{1922}{177}t - \frac{10\,966}{177} & \text{if} \quad 5 \le t \wedge t \le 8 \\ -\frac{490}{59}t + \frac{5390}{59} & \text{if} \quad 8 \le t \wedge t \le 11 \end{cases} \qquad \begin{array}{l} F(0) = 0 \\ F(3) = \frac{232}{59} = 3.932\,203\,389\,830\,508\,474\,6 \\ F(5) = -\frac{452}{59} = -7.661\,016\,949\,152\,542\,372\,9 \\ F(8) = \frac{1470}{59} = 24.915\,254\,237\,288\,135\,593 \\ F(11) = 0 \end{array}$$



$S_x(t)$



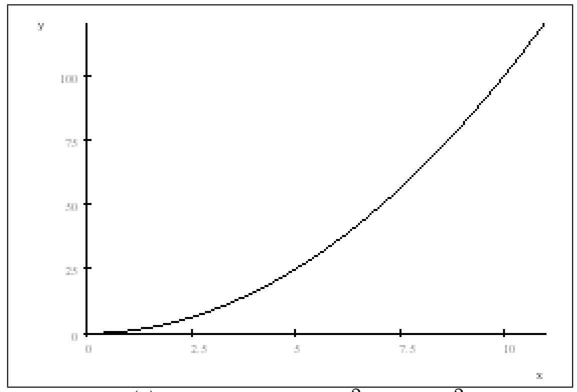
$D_x(t) = \frac{dS_x(t)}{dt}$



$F_x(t) = \frac{d^2 S_x(t)}{dt^2}$

Luego, para la coordenada y hacemos una interpolación de Newton

$$\begin{array}{cccccccc}
0 & 0 & y'(0)=0 & \frac{3-0}{3-0}=1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{9-0}{3-0}=3 & \frac{8-3}{5-0}=1 & 0 & 0 & 0 & \\
3 & 9 & \frac{25-9}{5-3}=8 & \frac{13-8}{8-3}=1 & 0 & 0 & & \\
5 & 25 & \frac{64-25}{8-5}=13 & \frac{19-13}{11-5}=1 & 0 & & & \\
8 & 64 & \frac{121-64}{11-8}=19 & \frac{22-19}{11-8}=1 & & & & \\
11 & 121 & y'(11)=22 & & & & & \\
11 & 121 & & & & & &
\end{array}$$



$y(t)=0+0\times t+1t^2+0=t^2$

despejando $t \Longrightarrow t=\sqrt{y}$

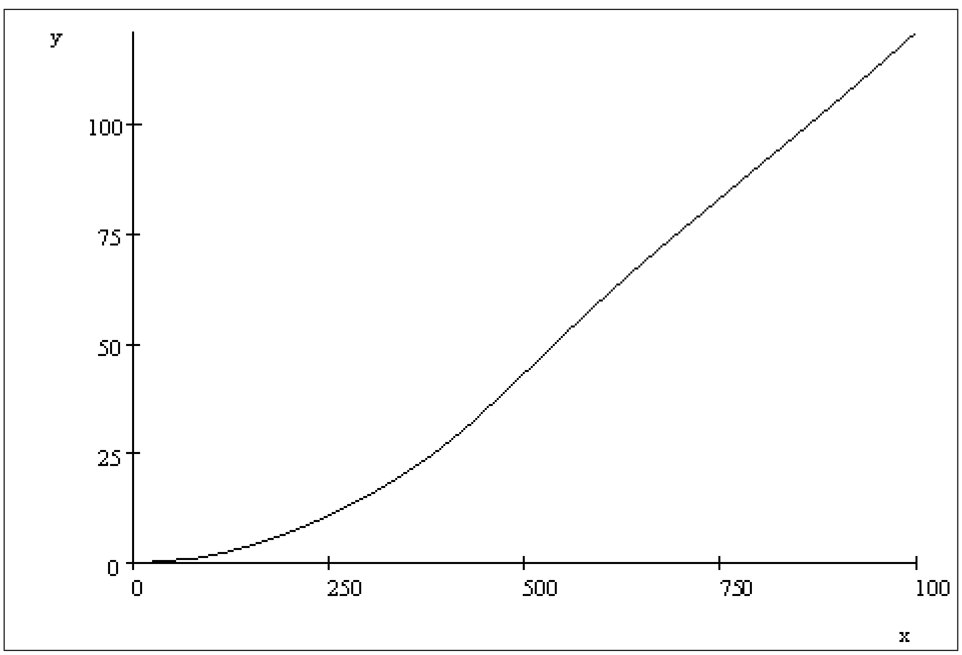
reemplazando en $S_x(t)$

$$S_x(t)=S_x(\sqrt{y})=\begin{cases} \frac{4309}{59}\sqrt{y}+\frac{116}{531}\sqrt{y}^3 & \text{if } \quad 0\le\sqrt{y}\wedge\sqrt{y}\le 3 \\ \frac{2422}{59}\sqrt{y}+\frac{629}{59}\sqrt{y}^2-\frac{57}{59}\sqrt{y}^3+\frac{1887}{59} & \text{if } \quad 3\le\sqrt{y}\wedge\sqrt{y}\le 5 \\ \frac{44\,116}{177}\sqrt{y}-\frac{5483}{177}\sqrt{y}^2+\frac{961}{531}\sqrt{y}^3-\frac{167\,267}{531} & \text{if } \quad 5\le\sqrt{y}\wedge\sqrt{y}\le 8 \\ \frac{2695}{59}\sqrt{y}^2-364\sqrt{y}-\frac{245}{177}\sqrt{y}^3+\frac{233\,695}{177} & \text{if } \quad 8\le\sqrt{y}\wedge\sqrt{y}\le 11 \end{cases}$$

transformando en una función que depende de y :

$$\Longrightarrow S_x(y)=\begin{cases} \frac{4309}{59}\sqrt{y}+\frac{116}{531}\sqrt{y}^3 & \text{if } \quad 0\le y\wedge y\le 9 \\ \frac{2422}{59}\sqrt{y}+\frac{629}{59}y-\frac{57}{59}\sqrt{y}^3+\frac{1887}{59} & \text{if } \quad 9\le y\wedge y\le 25 \\ \frac{44\,116}{177}\sqrt{y}-\frac{5483}{177}y+\frac{961}{531}\sqrt{y}^3-\frac{167\,267}{531} & \text{if } \quad 25\le y\wedge y\le 64 \\ \frac{2695}{59}y-364\sqrt{y}-\frac{245}{177}\sqrt{y}^3+\frac{233\,695}{177} & \text{if } \quad 64\le y\wedge y\le 121 \end{cases}$$

Graficando en un plano xy se obtiene el plano pedido.



21. Newton

(a) Volumen en función de la temperatura

i. 15% *Lagrange* : $\begin{bmatrix} i & t_i & V(t_i) & L_{4,i}(t) \\ 0 & 5 & 880 & \frac{(t-10)(t-20)(t-40)(t-80)}{(5-10)(5-20)(5-40)(5-80)} \\ 1 & 10 & 950 & \frac{(t-5)(t-20)(t-40)(t-80)}{(10-5)(10-20)(10-40)(10-80)} \\ 2 & 20 & 1030 & \frac{(t-5)(t-10)(t-40)(t-80)}{(20-5)(20-10)(20-40)(20-80)} \\ 3 & 40 & 1080 & \frac{(t-5)(t-10)(t-20)(t-80)}{(40-5)(40-10)(40-20)(40-80)} \\ 4 & 80 & 1250 & \frac{(t-5)(t-10)(t-20)(t-40)}{(80-5)(80-10)(80-20)(80-40)} \end{bmatrix}$

$P_{15}(t)=\sum_{i=0}^{4}V(t_i)L_{4,i}(t)$

$=880\frac{(t-10)(t-20)(t-40)(t-80)}{(5-10)(5-20)(5-40)(5-80)}+950\frac{(t-5)(t-20)(t-40)(t-80)}{(10-5)(10-20)(10-40)(10-80)}+1030\frac{(t-5)(t-10)(t-40)(t-80)}{(20-5)(20-10)(20-40)(20-80)}$

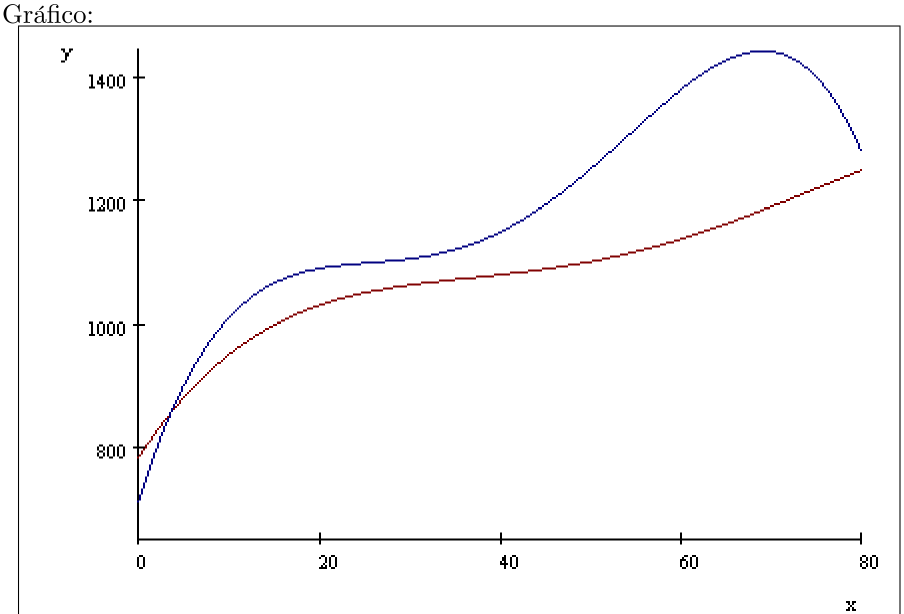
$+1080\frac{(t-5)(t-10)(t-20)(t-80)}{(40-5)(40-10)(40-20)(40-80)}+1250\frac{(t-5)(t-10)(t-20)(t-40)}{(80-5)(80-10)(80-20)(80-40)}$
$\Longrightarrow P_{15}(t)=\frac{49\,274}{63}+\frac{1915}{84}t-\frac{497}{720}t^2+\frac{157}{16\,800}t^3-\frac{53}{1260\,000}t^4$

ii. 40% $Newton:$

$$\begin{bmatrix} t_i & V(t_i) & V[t_i,t_{i+1}] & V[t_i...t_{i+2}] & V[t_i...t_{i+3}] & V[t_i...t_{i+4}] \\ 5 & 900 & \frac{1010-900}{10-5}=22 & \frac{8-22}{20-5}=-\frac{14}{15} & \frac{-\frac{1}{6}+\frac{14}{15}}{40-5}=\frac{23}{1050} & \frac{\frac{41}{16\,800}-\frac{23}{1050}}{80-5}=-\frac{109}{420\,000} \\ 10 & 1010 & \frac{1090-1010}{20-10}=8 & \frac{3-8}{40-10}=-\frac{1}{6} & \frac{\frac{1}{240}+\frac{1}{6}}{80-10}=\frac{41}{16\,800} & \\ 20 & 1090 & \frac{1150-1090}{40-20}=3 & \frac{\frac{13}{4}-3}{80-20}=\frac{1}{240} & & \\ 40 & 1150 & \frac{1280-1150}{80-40}=\frac{13}{4} & & & \\ 80 & 1280 & & & & \end{bmatrix}$$

$P_{40}(t)=900+22(t-5)-\frac{14}{15}(t-5)(t-10)$
$+\frac{23}{1050}(t-5)(t-10)(t-20)-\frac{109}{420\,000}(t-5)(t-10)(t-20)(t-40)$
$\Longrightarrow P_{40}(t)=\frac{14\,932}{21}+\frac{3995}{84}t-\frac{517}{240}t^2+\frac{139}{3360}t^3-\frac{109}{420\,000}t^4$

iii. Gráfico:



Rojo : $P_{15}(t)=\frac{49\,274}{63}+\frac{1915}{84}t-\frac{497}{720}t^2+\frac{157}{16\,800}t^3-\frac{53}{1260\,000}t^4$
Azul : $P_{40}(t)=\frac{14\,932}{21}+\frac{3995}{84}t-\frac{517}{240}t^2+\frac{139}{3360}t^3-\frac{109}{420\,000}t^4$

(b) Volumen en función de la concentración a 15°

i. $\begin{bmatrix} \% & V(\%) & L_{2,i} \\ 15 & P_{15}(15) = \frac{1006\,175}{1008} & \frac{c-40}{15-40} \\ 40 & P_{40}(15) = \frac{358\,255}{336} & \frac{c-15}{40-15} \end{bmatrix}$

ii. $V_{15}(c) = \frac{1006\,175}{1008}\frac{c-40}{15-40} + \frac{358\,255}{336}\frac{c-15}{40-15} = \frac{6859}{2520}c + \frac{965\,021}{1008}$
$\Longrightarrow V_{15}(c) = \frac{965\,021}{1008} + \frac{6859}{2520}c$

iii. $V_{15}(25) = \frac{344\,537}{336} = 1025.\,407\,738\,095\,238\,095\,2$

iv. La mermelada tendrá un volumen de 1025.4 cc a temperatura ambiente

(c) Volumen en función de la concentración a 50°

i. $\begin{array}{lll} \% & V(\%) & V[\%_i, \%_{i+1}] \\ 15 & P_{15}(50) = \frac{7711}{7} & \frac{\frac{8769}{7} - \frac{7711}{7}}{40-15} = \frac{1058}{175} \;:\; \frac{1058}{175} \\ 40 & P_{40}(50) = \frac{8769}{7} & \end{array}$

ii. $V_{50}(c) = \frac{7711}{7} + \frac{1058}{175}(c-15) = \frac{1058}{175}c + \frac{35\,381}{35}$
$\Longrightarrow V_{50}(c) = \frac{35\,381}{35} + \frac{1058}{175}c$

iii. $V_{50}(25) = \frac{40\,671}{35} = 1162.\,028\,571\,428\,571\,428\,6$

iv. La mermelada calientita tendrá un volumen de 1162 cc en el envasado.

22. Hermite por Newton

(a) $\begin{bmatrix} Tiempo\ [s] & 0 & 3 & 5 \\ Velocidad\ [\frac{km}{h}] & 0 & 63 & 108 \end{bmatrix} = \begin{bmatrix} Tiempo\ [s] & 0 & 3 & 5 \\ Velocidad\ [\frac{m}{s}] & 0 & \frac{63000}{3600} = \frac{35}{2} & \frac{108000}{3600} = 30 \end{bmatrix}$

$\begin{bmatrix} t_i & v(t_i) & v[t_i, t_{i+1}] & v[t_i, ..., t_{i+2}] \\ 0 & 0 & \frac{\frac{35}{2}-0}{3-0} = \frac{35}{6} & \frac{\frac{25}{4}-\frac{35}{6}}{5-0} = \frac{1}{12} \\ 3 & \frac{35}{2} & \frac{30-\frac{35}{2}}{5-3} = \frac{25}{4} & \\ 5 & 30 & & \end{bmatrix}$

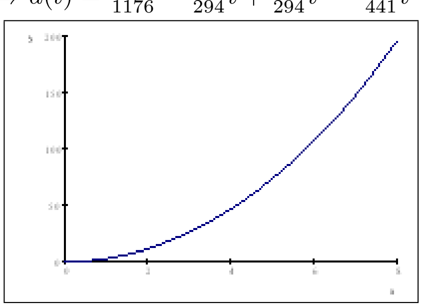
$\Longrightarrow v(t) = 0 + \frac{35}{6}t + \frac{1}{12}t(t-3) = \frac{67}{12}t + \frac{1}{12}t^2$

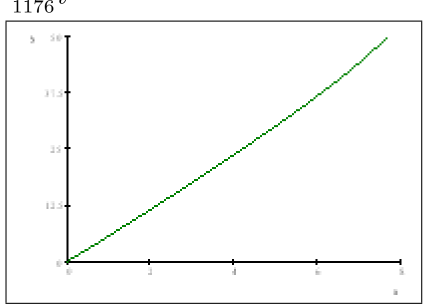
(b) $d(t) = \int_0^t v(t)dt = \int_0^t (\frac{67}{12}t + \frac{1}{12}t^2) = \frac{67}{24}t^2 + \frac{1}{36}t^3$
$d(0) = 0$
$d(3) = \frac{67}{24}3^2 + \frac{1}{36}3^3 = \frac{207}{8}$
$d(5) = \frac{67}{24}5^2 + \frac{1}{36}5^3 = \frac{5275}{72}$

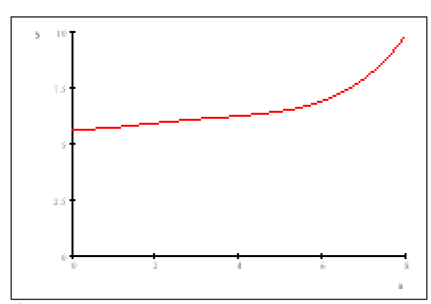
(c) $\begin{bmatrix} Tiempo\ [s] & 0 & 3 & 5 & 7 \\ Dis\tan cia[m] & 0 & \frac{207}{8} & \frac{5275}{72} & \frac{10\,567}{72} \\ Velocidad\ [\frac{m}{s}] & 0 & \frac{35}{2} & 30 & ? \end{bmatrix}$

$\begin{bmatrix} t_i & d(t_i) & d[t_i, t_{i+1}] & d[t_i, ..., t_{i+2}] & d[t_i, ..., t_{i+3}] & d[t_i, ..., t_{i+4}] & d[t_i, ..., t_{i+5}] & d[t_i, ..., t_{i+6}] \\ 0 & 0 & v(0) = 0 & \frac{\frac{69}{8}-0}{3-0} = \frac{23}{8} & \frac{\frac{71}{24}-\frac{23}{8}}{3-0} = \frac{1}{36} & \frac{\frac{1}{36}-\frac{1}{36}}{5-0} = 0 & \frac{0-0}{5-0} = 0 & \frac{\frac{1}{1008}-0}{7-0} = \frac{1}{7056} \\ 0 & 0 & \frac{\frac{207}{8}-0}{3-0} = \frac{69}{8} & \frac{\frac{35}{2}-\frac{69}{8}}{3-0} = \frac{71}{24} & \frac{\frac{223}{72}-\frac{71}{24}}{5-0} = \frac{1}{36} & \frac{\frac{1}{36}-\frac{1}{36}}{5-0} = 0 & \frac{\frac{1}{144}-0}{7-0} = \frac{1}{1008} & \\ 3 & \frac{207}{8} & v(3) = \frac{35}{2} & \frac{\frac{853}{36}-\frac{35}{2}}{5-3} = \frac{223}{72} & \frac{\frac{227}{72}-\frac{223}{72}}{5-3} = \frac{1}{36} & \frac{\frac{1}{18}-\frac{1}{36}}{7-3} = \frac{1}{144} & & \\ 3 & \frac{207}{8} & \frac{\frac{5275}{72}-\frac{207}{8}}{5-3} = \frac{853}{36} & \frac{30-\frac{853}{36}}{5-3} = \frac{227}{72} & \frac{\frac{27}{8}-\frac{227}{72}}{7-3} = \frac{1}{18} & & & \\ 5 & \frac{5275}{72} & v(5) = 30 & \frac{\frac{147}{4}-30}{7-5} = \frac{27}{8} & & & & \\ 5 & \frac{5275}{72} & \frac{\frac{10\,567}{72}-\frac{5275}{72}}{7-5} = \frac{147}{4} & & & & & \\ 7 & \frac{10\,567}{72} & & & & & & \end{bmatrix}$

$d(t) = (\frac{67}{24}t^2 + \frac{1}{36}t^3) + \frac{1}{7056}t^2(t-3)^2(t-5)^2$
$\Longrightarrow d(t) = \frac{6641}{2352}t^2 - \frac{11}{1764}t^3 + \frac{47}{3528}t^4 - \frac{1}{441}t^5 + \frac{1}{7056}t^6$
$\Longrightarrow v(t) = \frac{6641}{1176}t - \frac{11}{588}t^2 + \frac{47}{882}t^3 - \frac{5}{441}t^4 + \frac{1}{1176}t^5$
$\Longrightarrow a(t) = \frac{6641}{1176} - \frac{11}{294}t + \frac{47}{294}t^2 - \frac{20}{441}t^3 + \frac{5}{1176}t^4$

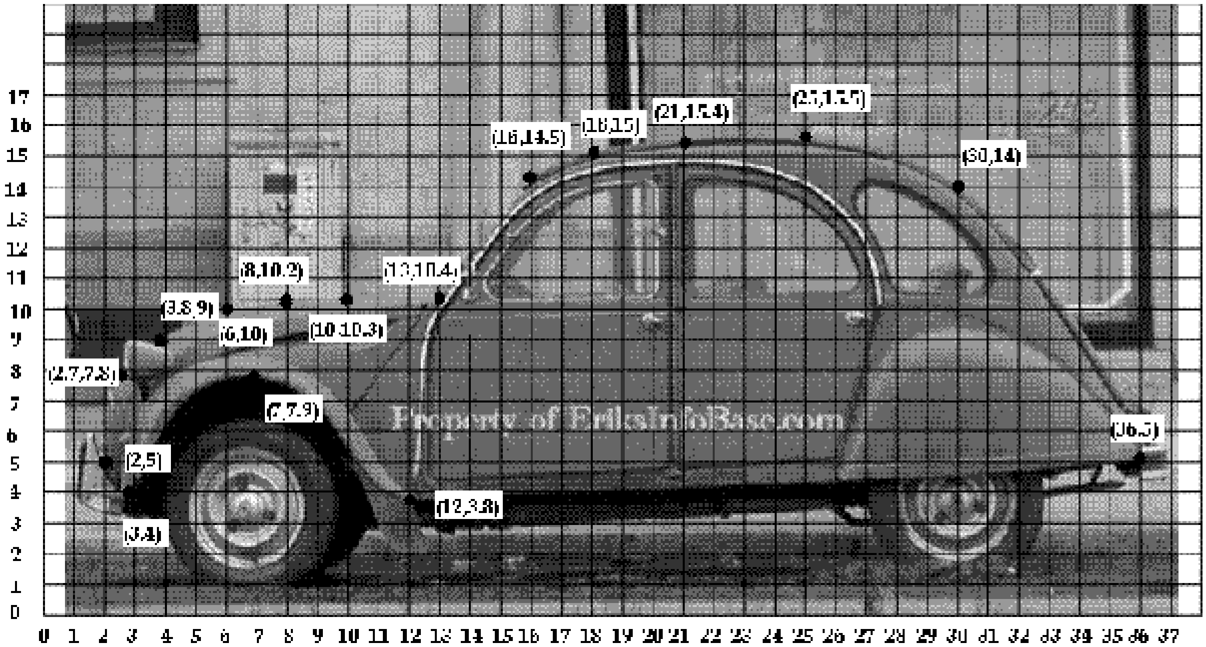

Distancia


Velocidad


Aceleración

23. Spline Cúbica

(a) Primero, la toma de datos se realiza usando una grilla cuadriculada sobre la fotografía:



(b) Luego, los datos se tabulan. Tabulemos los datos sel contorno superior

$$\begin{bmatrix} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ x & 2 & 2.7 & 3.8 & 6 & 8 & 10 & 13 & 16 & 18 & 21 & 25 & 30 & 36 \\ y & 5 & 7.8 & 9 & 10 & 10.2 & 10.3 & 10.4 & 14.5 & 15 & 15.4 & 15.5 & 14 & 5 \end{bmatrix}$$

(c) Ahora se calculan los elementos de la matriz:

$h_k = x_{k+1} - x_k$, $d_k = \frac{y_{k+1} - y_k}{h_k}$, $\mu_k = 3\left(d_k - d_{k-1}\right)$

$$\begin{bmatrix} k & 0 & 1 & 2 & 3 \\ h_k & 0.7 & 1.1 & 2.2 & 2 \\ d_k & \frac{2.8}{0.7} = 4 & \frac{1.2}{1.1} = \frac{12}{11} & \frac{1}{2.2} = \frac{5}{11} & \frac{0.2}{2} = \frac{1}{10} \\ \mu_k & & 3 \times (\frac{12}{11} - 4) = -\frac{96}{11} & 3 \times (\frac{5}{11} - \frac{12}{11}) = -\frac{21}{11} & 3 \times (\frac{1}{10} - \frac{5}{11}) = -\frac{117}{110} \end{bmatrix}$$

$$\begin{bmatrix} k & 4 & 5 & 6 & 7 \\ h_k & 2 & 3 & 3 & 2 \\ d_k & \frac{0.1}{2} = \frac{1}{20} & \frac{0.1}{3} = \frac{1}{30} & \frac{4.1}{3} = \frac{41}{30} & \frac{0.5}{2} = \frac{1}{4} \\ \mu_k & 3 \times (\frac{1}{20} - \frac{1}{10}) = -\frac{3}{20} & 3 \times (\frac{1}{30} - \frac{1}{20}) = -\frac{1}{20} & 3 \times (\frac{41}{30} - \frac{1}{30}) = 4 & 3 \times (\frac{1}{4} - \frac{41}{30}) = -\frac{67}{20} \end{bmatrix}$$

$$\begin{bmatrix} k & 8 & 9 & 10 & 11 \\ h_k & 3 & 4 & 5 & 6 \\ d_k & \frac{0.4}{3} = \frac{2}{15} & \frac{0.1}{4} = \frac{1}{40} & -\frac{1.5}{5} = -\frac{3}{10} & -\frac{9}{6} = -\frac{3}{2} \\ \mu_k & 3 \times (\frac{2}{15} - \frac{1}{4}) = -\frac{7}{20} & 3 \times (\frac{1}{40} - \frac{2}{15}) = -\frac{13}{40} & 3 \times (-\frac{3}{10} - \frac{1}{40}) = -\frac{39}{40} & 3 \times (-\frac{3}{2} + \frac{3}{10}) = -\frac{18}{5} \end{bmatrix}$$

(d) Usando las condiciones de borde de la Spline Natural, ie, $S^{''}(x_0) = 0$, $S^{''}(x_n) = 0$, y reemplazando valores, el sistema queda:

(e)
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{10} & \frac{18}{5} & \frac{11}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{10} & \frac{33}{5} & \frac{11}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{5} & \frac{42}{5} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 10 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 12 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 10 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 10 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 14 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 18 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 22 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{96}{11} \\ -\frac{21}{11} \\ -\frac{117}{110} \\ -\frac{3}{20} \\ -\frac{1}{20} \\ 4 \\ -\frac{67}{20} \\ -\frac{7}{20} \\ -\frac{13}{40} \\ -\frac{39}{40} \\ -\frac{18}{5} \\ 0 \end{bmatrix}$$

(f) Reesolviendo el sistema, y luego aproximando con 4 cifras significativas con redondeo

$$\begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{110\,884\,705\,363}{44\,676\,774\,692} \\ \frac{23\,196\,354\,987}{122\,861\,130\,403} \\ -\frac{189\,901\,004\,815}{982\,889\,043\,224} \\ \frac{70\,738\,032\,441}{982\,889\,043\,224} \\ -\frac{416\,919\,507\,977}{2457\,222\,608\,060} \\ \frac{167\,847\,444\,871}{335\,075\,810\,190} \\ -\frac{3690\,926\,192\,477}{7371\,667\,824\,180} \\ \frac{1136\,243\,352\,281}{14\,743\,335\,648\,360} \\ -\frac{146\,574\,685\,819}{3685\,833\,912\,090} \\ \frac{89\,412\,083}{670\,151\,620\,380} \\ -\frac{2412\,992\,893\,783}{14\,743\,335\,648\,360} \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ -2.482 \\ 0.1888 \\ -0.1932 \\ 0.07197 \\ -0.1697 \\ 0.5009 \\ -0.5007 \\ 0.07707 \\ -0.03977 \\ 0.0001334 \\ -0.1637 \\ 0 \end{bmatrix}$$

(g) Construyendo la Spline:

$S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3$

$S_{k,0} = y_k \qquad S_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{3} \qquad S_{k,2} = m_k \qquad S_{k,3} = \frac{m_{k+1} - m_k}{3 \times h_k}$

$$\begin{bmatrix} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ x & 2 & 2.7 & 3.8 & 6 & 8 & 10 & 13 \\ y & 5 & 7.8 & 9 & 10 & 10.2 & 10.3 & 10.4 \\ h_k & 0.7 & 1.1 & 2.2 & 2 & 2 & 3 & 3 \\ d_k & 4 & 1.091 & 0.4545 & 0.1 & 0.05 & 0.03333 & 1.367 \\ m_k & 0 & -2.482 & 0.1888 & -0.1932 & 0.07197 & -0.1697 & 0.5009 \end{bmatrix}$$

$$\begin{bmatrix} k & 7 & 8 & 9 & 10 & 11 & 12 \\ x & 16 & 18 & 21 & 25 & 30 & 36 \\ y & 14.5 & 15 & 15.4 & 15.5 & 14 & 5 \\ h_k & 2 & 3 & 4 & 5 & 6 & \\ d_k & 0.25 & 0.1333 & 0.025 & -0.3 & -1.5 & \\ m_k & -0.5007 & 0.07707 & -0.03977 & 0.0001334 & -0.1637 & 0 \end{bmatrix}$$

i. $k = 0$

$S_0(x) = 5 + \left(4 - \frac{0.7 \times (2 \times 0 - 2.482)}{3}\right) \times (x - 2) + 0 \times (x - 2)^2 + \frac{-2.482 - 0}{3 \times 0.7} \times (x - 2)^3$

$\Longrightarrow S_0(x) = 7.091\,x^2 - 9.604x - 1.182x^3 + 5.297$

ii. $k = 1$

$S_1(x) = 7.8 + \left(1.091 - \frac{1.1 \times (2 \times (-2.482) + 0.1888)}{3}\right) \times (x - 2.7) - 2.482 \times (x - 2.7)^2 + \frac{0.1888 + 2.482}{3 \times 1.1} \times (x - 2.7)^3$

$\Longrightarrow S_1(x) = 33.94x - 9.038x^2 + 0.8093x^3 - 33.9$

iii. $k = 2$

$S_2(x) = 9 + \left(0.4545 - \frac{2.2 \times (2 \times 0.1888 - 0.1932)}{3}\right) \times (x - 3.8) + 0.1888 \times (x - 3.8)^2 + \frac{-0.1932 - 0.1888}{3 \times 2.2} \times (x - 3.8)^3$

$\Longrightarrow S_2(x) = 0.8486x^2 - 3.623x - 0.05788x^3 + 13.69$

iv. $k = 3$

$S_3(x) = 10 + \left(0.1 - \frac{2 \times (2 \times (-0.1932) + 0.07197)}{3}\right) \times (x - 6) - 0.1932 \times (x - 6)^2 + \frac{0.07197 + 0.1932}{3 \times 2} \times (x - 6)^3$

$\Longrightarrow S_3(x) = 7.401x - 0.9887x^2 + 0.0442\,x^3 - 8.359$

v. $k = 4$

$S_4(x) = 10.2 + \left(0.05 - \frac{2 \times (2 \times 0.07197 - 0.1697)}{3}\right) \times (x - 8) + 0.07197 \times (x - 8)^2 + \frac{-0.1697 - 0.07197}{3 \times 2} \times (x - 8)^3$

$\Longrightarrow S_4(x) = 1.039x^2 - 8.818x - 0.04028x^3 + 34.89$

vi. $k = 5$

$S_5(x) = 10.3 + \left(0.03333 - \frac{3 \times (2 \times (-0.1697) + 0.5009)}{3}\right) \times (x - 10) - 0.1697 \times (x - 10)^2 + \frac{0.5009 + 0.1697}{3 \times 3} \times (x - 10)^3$

$\Longrightarrow S_5(x) = 25.62x - 2.405x^2 + 0.07451x^3 - 79.9$

vii. $k = 6$

$S_6(x) = 10.4 + \left(1.367 - \frac{3 \times (2 \times 0.5009 - 0.5007)}{3}\right) \times (x - 13) + 0.5009 \times (x - 13)^2 + \frac{-0.5007 - 0.5009}{3 \times 3} \times (x - 13)^3$

$\Longrightarrow S_6(x) = 4.841\,x^2 - 68.58x - 0.1113x^3 + 328.3$

viii. $k = 7$

$S_7(x) = 14.5 + \left(0.25 - \frac{2 \times (2 \times (-0.5007) + 0.07707)}{3}\right) \times (x - 16) - 0.5007 \times (x - 16)^2 + \frac{0.07707 + 0.5007}{3 \times 2} \times (x - 16)^3$

$\Longrightarrow S_7(x) = 90.84x - 5.123x^2 + 0.0963\,x^3 - 522$

ix. $k = 8$

$S_8(x) = 15 + \left(0.1333 - \frac{3 \times (2 \times 0.07707 - 0.03977)}{3}\right) \times (x - 18) + 0.07707 \times (x - 18)^2 + \frac{-0.03977 - 0.07707}{3 \times 3} \times (x - 18)^3$

$\Longrightarrow S_8(x) = 0.7781x^2 - 15.37x - 0.01298x^3 + 115.3$

x. $k = 9$

$S_9(x) = 15.4 + \left(0.025 - \frac{4 \times (2 \times (-0.03977) + 0.0001334)}{3}\right) \times (x - 21) - 0.03977 \times (x - 21)^2 + \frac{0.0001334 + 0.03977}{3 \times 4} \times (x - 21)^3$

$\Longrightarrow S_9(x) = 6.201x - 0.249\,3x^2 + 0.003325\,x^3 - 35.68$

xi. $k = 10$

$S_{10}(x) = 15.5 + \left(-0.3 - \frac{5\times(2\times(0.000\,133\,4)-0.163\,7)}{3}\right) \times (x-25) + 0.000\,133\,4 \times (x-25)^2 + \frac{-0.163\,7-0.000\,133\,4}{3\times 5} \times (x-25)^3$

$\Longrightarrow S_{10}(x) = 0.819\,3x^2 - 20.51x - 0.01092\,x^3 + 186.9$

xii. $k = 11$

$S_{11}(x) = 14 + \left(-1.5 - \frac{6\times(2\times(-0.163\,7)+0)}{3}\right) \times (x-30) - 0.163\,7 \times (x-30)^2 + \frac{0+0.163\,7}{3\times 6} \times (x-30)^3$

$\Longrightarrow S_{11}(x) = 33.53x - 0.982\,2x^2 + 0.009094x^3 - 353.5$

(h) Luego, la Spline Cúbica es

$$S(x) = \begin{cases} S_0(x) & \text{if} \quad 2 \leq x \wedge x \leq 2.7 \\ S_1(x) & \text{if} \quad 2.7 \leq x \wedge x \leq 3.8 \\ S_2(x) & \text{if} \quad 3.8 \leq x \wedge x \leq 6 \\ S_3(x) & \text{if} \quad 6 \leq x \wedge x \leq 8 \\ S_4(x) & \text{if} \quad 8 \leq x \wedge x \leq 10 \\ S_5(x) & \text{if} \quad 10 \leq x \wedge x \leq 13 \\ S_6(x) & \text{if} \quad 13 \leq x \wedge x \leq 16 \\ S_7(x) & \text{if} \quad 16 \leq x \wedge x \leq 18 \\ S_8(x) & \text{if} \quad 18 \leq x \wedge x \leq 21 \\ S_9(x) & \text{if} \quad 21 \leq x \wedge x \leq 25 \\ S_{10}(x) & \text{if} \quad 25 \leq x \wedge x \leq 30 \\ S_{11}(x) & \text{if} \quad 30 \leq x \wedge x \leq 36 \end{cases}$$

(i) Ahora, tabulando los datos de la parte inferior del escarabajo, interpolemos por trazos usando polinomios de 1° y 2° grado:

$$\begin{bmatrix} x & 2 & 3 & 7 & 12 & 36 \\ y & 5 & 4 & 7.9 & 3.8 & 5 \end{bmatrix}$$

(j) $f_0(x) = 5\frac{(x-3)}{2-3} + 4\frac{(x-2)}{3-2} = 7 - x$

$\Longrightarrow f_0(x) = 7 - x$

(k) $f_1(x) = 4\frac{(x-7)(x-12)}{(3-7)(3-12)} + 7.9\frac{(x-3)(x-12)}{(7-3)(7-12)} + 3.8\frac{(x-3)(x-7)}{(12-3)(12-7)} = 2.969\,x - 0.199\,4x^2 - 3.113$

$\Longrightarrow f_1(x) = 2.969\,x - 0.199\,4x^2 - 3.113$

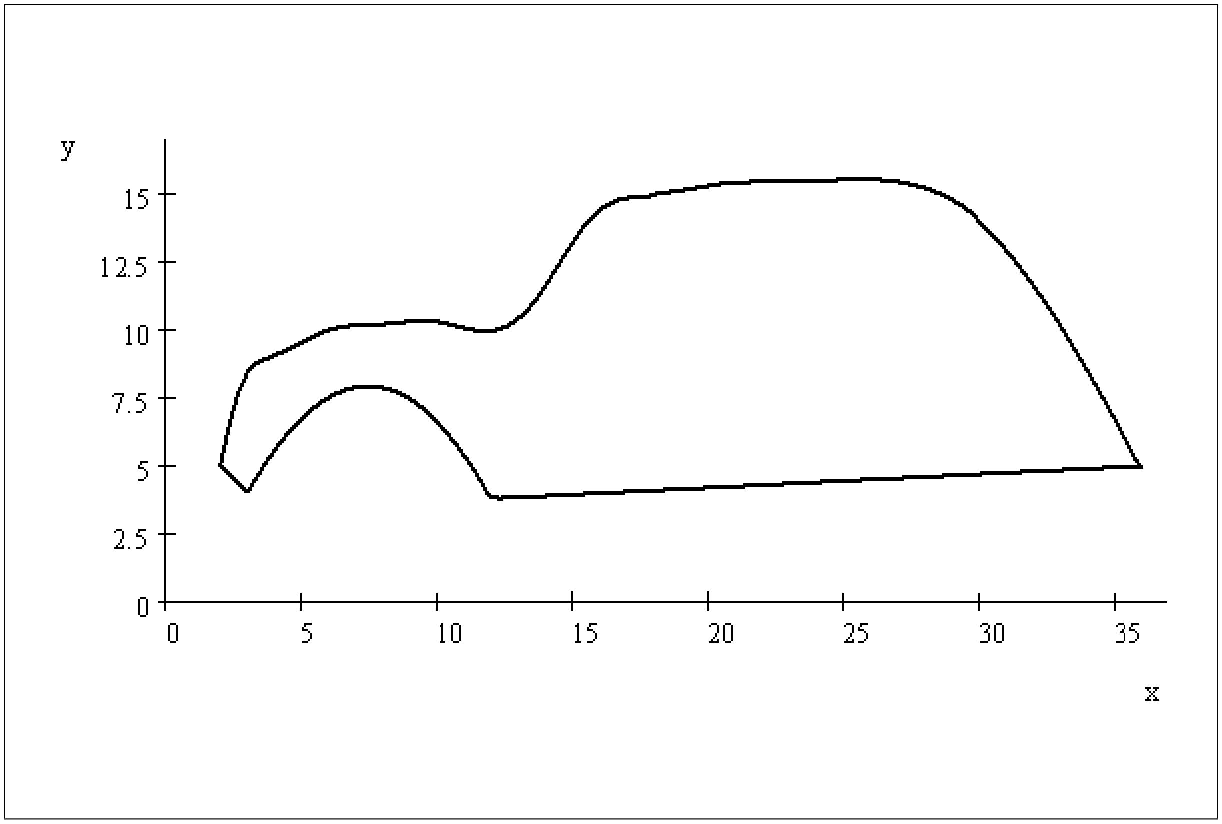
(l) $f_2(x) = 3.8\frac{(x-36)}{(12-36)} + 5\frac{(x-12)}{(36-12)} = 0.05x + 3.2$

$\Longrightarrow f_2(x) = 0.05x + 3.2$

(m) Luego, esta función por trozos es:

$$f(x) = \begin{cases} f_0(x) & \text{if} \quad 2 \leq x \wedge x \leq 3 \\ f_1(x) & \text{if} \quad 3 \leq x \wedge x \leq 12 \\ f_2(x) & \text{if} \quad 12 \leq x \wedge x \leq 36 \end{cases}$$

(n) Finalmente, el gráfico de $S(x)$ y $f(x)$ es:



24. Primero se hace una transformación para tener un sistema lineal. En este caso se usa el ln

  (a) $y = be^{ax} \Longrightarrow \ln(y) = \ln(b) + ax$
  Ahora buscamos el minimo del error cuadrático

  (b) $E = \int_{\alpha}^{\beta} (\ln(f(x)) - (\ln(b) + ax))^2 dx$

  (c) $\frac{\partial E}{\partial a} = \int_{\alpha}^{\beta} x\,(2\ln b + 2ax - 2\ln(f(x)))\;dx = 0 \Longrightarrow \int_{\alpha}^{\beta} (x\ln b + ax^2\,)dx = \int_{\alpha}^{\beta} x\ln(f(x))\,dx$

  (d) $\frac{\partial E}{\partial b} = \frac{1}{b}\int_{\alpha}^{\beta} (2\ln b + 2ax - 2\ln(f(x)))\;dx = 0 \Longrightarrow \int_{\alpha}^{\beta} (\ln b + ax\,)dx = \int_{\alpha}^{\beta} \ln(f(x))\,dx$
  El sistema de forma matricial queda de la forma

  (e) $\begin{bmatrix} \int_{\alpha}^{\beta} x\,dx & \int_{\alpha}^{\beta} x^2 dx \\ \int_{\alpha}^{\beta} dx & \int_{\alpha}^{\beta} x dx \end{bmatrix} \begin{bmatrix} \ln(b) \\ a \end{bmatrix} = \begin{bmatrix} \int_{\alpha}^{\beta} x\ln(f(x))\,dx \\ \int_{\alpha}^{\beta} \ln(f(x))\,dx \end{bmatrix}$

25. Primero, los ceros de $\tilde{T}_3$ se encuentran en $\bar{x}_k = \cos(\frac{2k-1}{2n}\pi), k = 1, 2, 3.$

  (a) $\bar{x}_1 = \cos(\frac{1}{6}\pi) = 0.866\,025\,403\,784\,438\,646\,76$

  (b) $\bar{x}_2 = \cos(\frac{3}{6}\pi) = 0.0$

  (c) $\bar{x}_3 = \cos(\frac{5}{6}\pi) = -0.866\,025\,403\,784\,438\,646\,76$
  Debemos usar una transformacion lineal para pasar de $[-1, 1]$ a $[1, 3]$.Esto es $\tilde{x}_k = 2 + \bar{x}_k$.

  (d) $\tilde{x}_1 = 2.\,866\,025\,403\,784\,438\,646\,8 \simeq 2.866$

  (e) $\tilde{x}_2 = 2$

  (f) $\tilde{x}_3 = 1.\,133\,974\,596\,215\,561\,353\,2 \simeq 1.134$
  Ahora debemos calcular los valores de $f(x)$ en $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$

  (g) $f(\tilde{x}_1) = f(2.866) = 3.\,017\,661\,066\,077\,538\,987\,9 \simeq 3.\,018$

  (h) $f(\tilde{x}_2) = f(2) = 1.\,386\,294\,361\,119\,890\,618\,8 \simeq 1.\,386$

  (i) $f(\tilde{x}_3) = f(1.134) = 0.142\,601\,866\,816\,505\,412\,01 \simeq 0.143$
  Las diferencias divididas son:

  (j) $\begin{array}{cccc} x & f(x) & f[x,x] & f[x,x,x] \\ 2.866 & 3.\,018 & \frac{1.\,386-3.\,018}{2-2.866} = 1.\,885 & \frac{1.\,435-1.\,885}{1.134-2.866} = 0.26 \\ 2 & 1.\,386 & \frac{0.143-1.\,386}{1.134-2} = 1.\,435 & \\ 1.134 & 0.143 & & \end{array}$
  El polinomio interpolante de segunda grado es:

(k) $\tilde{P}_3(x) = 3.018 + 1.885(x-2.866) + 0.26(x-2.866)(x-2) = 0.619\,84x + 0.26x^2 - 0.894\,09$
El error del polinomio está acotado de la forma

(l) $\max_{x\in[1,3]} \left|\tilde{P}_3(x) - f(x)\right| \leq \frac{1}{2^3(3+1)!} \max_{x\in[1,3]} \left|f^{(4)}(x)\right| = \frac{1}{192} \max_{x\in[1,3]} \left|\frac{1}{2x^3}\right| = \frac{1}{192}\frac{1}{2} = \frac{1}{384} < 0.002\,604\,2$

26. Calculemos a_k y b_k

(a) $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(kx)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(kx)dx = \frac{1}{\pi k}\; sen(kx)\big|_0^{\pi} = 0$

(b) $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sen(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} sen(kx)dx = -\frac{1}{\pi k}\cos(kx)\big|_0^{\pi} = \frac{1}{\pi k}(1-(-1)^k)$
Luego, el polinomio general $S_n(x)$ es

(c) $S_n(x) = \frac{a_0}{2} + a_n\cos(nx) + \sum_{k=1}^{n-1} a_k\cos(kx) + b_k sen(kx) = \sum_{k=1}^{n-1}\frac{1}{\pi k}(1-(-1)^k) sen(kx)$

27. El polinomio es $S_n(x) = \frac{a_0}{2} + a_2\cos(2x) + a_1\cos(x) + b_1 sen(x)$. Calculemos los coeficientes

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi-x)dx = \int_{-\pi}^{\pi} xdx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx = x^2\big|_{-\pi}^{\pi} - \frac{1}{3\pi}\; x^3\big|_{-\pi}^{\pi} = -\frac{2}{3}\pi^2$

(b) $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi-x)\cos(x)dx = \int_{-\pi}^{\pi} x\cos(x)dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(x)dx = -\frac{1}{\pi}\left[x^2 sen(x)\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x sen(x)dx\right] = -\frac{2}{\pi}\int_{-\pi}^{\pi} x sen(x)dx = -\frac{2}{\pi}\left[-x\cos(x)\big|_{-\pi}^{\pi} + \int_{\pi}^{-\pi}\cos(x)dx\right] = \frac{2}{\pi}2\pi = 4$

(c) $a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi-x)\cos(2x)dx = \frac{1}{\pi}\int_{-2\pi}^{2\pi}\frac{u}{2}(\pi-\frac{u}{2})\cos(u)\frac{du}{2} = \frac{1}{8\pi}\int_{-2\pi}^{2\pi} u(2\pi-u)\cos(u)du$
$= \frac{1}{4}\int_{-2\pi}^{2\pi} u\cos(u)du - \frac{1}{8\pi}\int_{-2\pi}^{2\pi} u^2\cos(u)du = -\frac{1}{8\pi}\int_{-2\pi}^{2\pi} u^2\cos(u)du = -\frac{1}{8\pi}\left[u^2 sen(u)\big|_{-2\pi}^{2\pi} - \int_{-2\pi}^{2\pi} 2u sen(u)du\right]$
$= \frac{1}{4\pi}\int_{-2\pi}^{2\pi} u\sin(u)du = \frac{1}{4\pi}\left[-x\cos(x)\big|_{-2\pi}^{2\pi} + \int_{2\pi}^{-2\pi}\cos(x)dx\right] = -\frac{1}{4\pi}4\pi = -1$