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Pauta Control 2

1. Interpolación polinomial

(a) Lagrange y Hermite

i. Velocidad: $\left[\frac{km}{h}\right] = \frac{1000}{3600} \left[\frac{m}{s}\right] = \frac{10}{36} \left[\frac{m}{s}\right]$

A. Usando Newton,

$t[s]$	$v\left[\frac{m}{s}\right]$					
0	$30 \times \frac{10}{36} = \frac{25}{3}$	$\frac{\frac{50}{3} - \frac{25}{3}}{\frac{5}{3} - 0} = \frac{5}{3}$	$\frac{-\frac{5}{3} - \frac{5}{3}}{\frac{10}{3} - 0} = -\frac{2}{9}$	$\frac{\frac{1}{6} + \frac{2}{9}}{\frac{15}{3} - 0} = \frac{7}{270}$	$\frac{-\frac{4}{135} - \frac{7}{270}}{20 - 0} = -\frac{1}{360}$	
5	$60 \times \frac{10}{36} = \frac{50}{3}$	$\frac{\frac{125}{9} - \frac{50}{3}}{\frac{10}{3} - 5} = -\frac{5}{9}$	$\frac{\frac{10}{3} + \frac{5}{9}}{15 - 5} = \frac{1}{6}$	$\frac{-\frac{5}{18} - \frac{1}{6}}{20 - 5} = -\frac{4}{135}$		
10	$50 \times \frac{10}{36} = \frac{125}{9}$	$\frac{\frac{175}{9} - \frac{125}{9}}{\frac{15}{3} - 10} = \frac{10}{9}$	$\frac{-\frac{5}{3} - \frac{10}{9}}{20 - 10} = -\frac{5}{18}$			
15	$70 \times \frac{10}{36} = \frac{175}{9}$	$\frac{\frac{100}{9} - \frac{175}{9}}{20 - 15} = -\frac{5}{3}$				
20	$40 \times \frac{10}{36} = \frac{100}{9}$					

B. $V(t) = \frac{25}{3} + \frac{5}{3}t - \frac{2}{9}t(t-5) + \frac{7}{270}t(t-5)(t-10) - \frac{1}{360}t(t-5)(t-10)(t-15)$

C. $V(t) = \frac{25}{3} + \frac{665}{108}t - \frac{11}{8}t^2 + \frac{59}{540}t^3 - \frac{1}{360}t^4$

ii. Distancia

A. $D(t) = \int_0^t V(x)dx = \frac{25}{3}t + \frac{1}{2} \frac{665}{108}t^2 - \frac{1}{3} \frac{11}{8}t^3 + \frac{1}{4} \frac{59}{540}t^4 - \frac{1}{5} \frac{1}{360}t^5$

$D(t) = \frac{25}{3}t + \frac{665}{216}t^2 - \frac{11}{24}t^3 + \frac{59}{2160}t^4 - \frac{1}{1800}t^5$

B. $D(0) = 0$, $D(5) = \frac{33125}{432} \simeq 76.7$, $D(10) = \frac{8125}{54} \simeq 150$, $D(15) = \frac{11125}{48} \simeq 232$, $D(20) = \frac{8750}{27} \simeq 324$

iii. Hermite

$t[s]$	$D[m]$					
0	0	$V(0) = \frac{25}{3}$	$\frac{15.3 - \frac{25}{3}}{5 - 0} = 1.4$	$\frac{0.273 - 1.4}{5 - 0} = -0.225$	$\frac{-0.0666 + 0.225}{10 - 0} = 0.0158$	
0	0	$\frac{76.7 - 0}{5 - 0} = 15.3$	$\frac{\frac{50}{3} - 15.3}{5 - 0} = 0.273$	$\frac{-0.393 - 0.273}{10 - 0} = -0.0666$	$\frac{0.0462 + 0.0666}{10 - 0} = 0.0113$	
5	76.7	$V(5) = \frac{50}{3}$	$\frac{14.7 - \frac{50}{3}}{10 - 5} = -0.393$	$\frac{-0.162 + 0.393}{10 - 5} = 0.0462$	$\frac{0.0664 - 0.0462}{15 - 5} = 0.00202$	
5	76.7	$\frac{150 - 76.7}{10 - 5} = 14.7$	$\frac{\frac{125}{9} - 14.7}{10 - 5} = -0.162$	$\frac{0.502 + 0.162}{15 - 5} = 0.0664$	$\frac{0.0214 - 0.0664}{15 - 5} = -0.0045$	
10	150	$V(10) = \frac{125}{9}$	$\frac{\frac{82}{3} - \frac{125}{9}}{15 - 10} = 0.502$	$\frac{0.609 - 0.502}{15 - 10} = 0.0214$	$\frac{-0.0818 - 0.0214}{20 - 10} = -0.0103$	
10	150	$\frac{232 - 150}{15 - 10} = \frac{82}{5}$	$\frac{\frac{175}{9} - \frac{82}{5}}{15 - 10} = 0.609$	$\frac{-0.209 - 0.609}{20 - 10} = -0.0818$	$\frac{-0.250 + 0.0818}{20 - 10} = -0.0168$	
15	232	$V(15) = \frac{175}{9}$	$\frac{\frac{92}{3} - \frac{175}{9}}{20 - 15} = -0.209$	$\frac{-1.46 + 0.209}{20 - 15} = -0.250$		
15	232	$\frac{324 - 232}{20 - 15} = \frac{92}{5}$	$\frac{\frac{100}{9} - \frac{92}{5}}{20 - 15} = -1.46$			
20	324	$V(20) = \frac{100}{9}$				
20	324					
...						
		$\frac{0.0113 - 0.0158}{10 - 0} = -0.00045$	$\frac{-0.000619 + 0.00045}{15 - 0} = -0.0000113$	$\frac{-0.0000022 + 0.0000113}{15 - 0} = 6.07 \times 10^{-7}$		
		$\frac{0.00202 - 0.0113}{15 - 0} = -0.000619$	$\frac{-0.000652 + 0.000619}{15 - 0} = -0.0000022$	$\frac{0.0000177 + 0.0000022}{20 - 0} = 0.000000995$		
...		$\frac{-0.0045 - 0.00202}{15 - 5} = -0.000652$	$\frac{-0.000387 + 0.000652}{20 - 5} = 0.0000177$	$\frac{-0.0000175 - 0.0000177}{20 - 5} = -2.35 \times 10^{-6}$...	
		$\frac{-0.0103 + 0.0045}{20 - 5} = -0.000387$	$\frac{-0.00065 + 0.000387}{20 - 5} = -0.0000175$			
		$\frac{-0.0168 + 0.0103}{20 - 10} = -0.00065$				
...		$\frac{0.000000995 - 6.07 \times 10^{-7}}{20 - 0} = 0.000000194$	$\frac{-0.00000016725 - 0.000000194}{20 - 0} = -0.000000093325$			
		$\frac{-2.35 \times 10^{-6} - 0.000000995}{20 - 0} = -0.00000016725$				

$H(t) = 0 + \frac{25}{3}t + 1.4t^2 - 0.225t^2(t-5) + 0.0158t^2(t-5)^2 - 0.00045t^2(t-5)^2(t-10) - 0.0000113t^2(t-5)^2(t-10)^2 + 6.07 \times 10^{-7}t^2(t-5)^2(t-10)^2(t-15) + 0.000000194t^2(t-5)^2(t-10)^2(t-15)^2 - 0.000000093325t^2(t-5)^2(t-10)^2(t-15)^2(t-20)$

(b) Simpson Compuesta

i. $f(x) = \sin\left(\frac{\pi x}{2}\right)$, $\frac{df(x)}{dx} = \frac{\pi}{2} \cos\left(\frac{\pi x}{2}\right)$, $g(x) = f(x) \sqrt{1 + \left[\frac{df(x)}{dx}\right]^2}$

ii. $n = 8$, $a = 0$, $b = 1$, $h = \frac{1}{8}$

	i	x_i	$f(x_i)$	$\frac{df(x_i)}{dx}$	$g(x)$
	0	0	$\sin(0) = 0$	$\frac{\pi}{2} \cos(0) = 1.57$	0
	1	$\frac{1}{8}$	$\sin(\frac{\pi}{16}) = 0.195$	$\frac{\pi}{2} \cos(\frac{\pi}{16}) = 1.54$	$0.195(1 + 1.54^2) = 0.657462$
	2	$\frac{2}{8}$	$\sin(\frac{\pi}{8}) = 0.383$	$\frac{\pi}{2} \cos(\frac{\pi}{8}) = 1.45$	$0.383(1 + 1.45^2) = 1.1882575$
iii.	3	$\frac{3}{8}$	$\sin(\frac{3\pi}{16}) = 0.556$	$\frac{\pi}{2} \cos(\frac{3\pi}{16}) = 1.31$	$0.556(1 + 1.31^2) = 1.5101516$
	4	$\frac{4}{8}$	$\sin(\frac{\pi}{4}) = 0.707$	$\frac{\pi}{2} \cos(\frac{\pi}{4}) = 1.11$	$0.707(1 + 1.11^2) = 1.5780947$
	5	$\frac{5}{8}$	$\sin(\frac{5\pi}{16}) = 0.831$	$\frac{\pi}{2} \cos(\frac{5\pi}{16}) = 0.873$	$0.831(1 + 0.873^2) = 1.464329199$
	6	$\frac{6}{8}$	$\sin(\frac{3\pi}{8}) = 0.924$	$\frac{\pi}{2} \cos(\frac{3\pi}{8}) = 0.601$	$0.924(1 + 0.601^2) = 1.257749724$
	7	$\frac{7}{8}$	$\sin(\frac{7\pi}{16}) = 0.981$	$\frac{\pi}{2} \cos(\frac{7\pi}{16}) = 0.306$	$0.981(1 + 0.306^2) = 1.072856916$
	8	$\frac{8}{8}$	$\sin(\frac{\pi}{2}) = 1$	$\frac{\pi}{2} \cos(\frac{\pi}{2}) = 0$	1

iv. $S = \int_0^1 f(x) \sqrt{1 + \left[\frac{df(x)}{dx}\right]^2} dx = \int_0^1 g(x) dx$
 $\simeq \frac{h}{3} [g(x_0) + 4g(x_1) + g(x_8) + 2g(x_2) + 4g(x_3) + 2g(x_4) + 4g(x_5) + 2g(x_6) + 4g(x_7)]$
 $= \frac{1}{24} [0 + 4 \times 0.657 + 1 + 2 \times 1.19 + 4 \times 1.51 + 2 \times 1.58 + 4 \times 1.46 + 2 \times 1.26 + 4 \times 1.07]$
 $= 1.160$

2. Spline Cúbica

(a) Spline Sujeta: $f(x) = x \ln(x)$, $f'(x) = \ln(x) + 1$

i. $x_0 = 1 \quad f(x_0) = 0 \quad f'(x_0) = 1 \quad h_0 = 1 \quad d_0 = 1,39 - 0 = 1,39$
 $x_1 = 2 \quad f(x_1) = 1,39 \quad f'(x_1) = 1,69 \quad h_1 = 1 \quad d_1 = 3,30 - 1,39 = 1,91$
 $x_2 = 3 \quad f(x_2) = 3,30 \quad f'(x_2) = 2,10 \quad h_2 = 1 \quad d_2 = 5,55 - 3,30 = 2,25$
 $x_3 = 4 \quad f(x_3) = 5,55 \quad f'(x_3) = 2,39$

ii.
$$\begin{bmatrix} 2h_0 & h_1 & 0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 \\ 0 & 0 & h_2 & 2h_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \mu_0 = 3(d_0 - f'(x_0)) \\ \mu_1 = 3(d_1 - d_0) \\ \mu_2 = 3(d_2 - d_1) \\ \mu_3 = 3(f'(x_3) - d_2) \end{bmatrix}$$

iii.
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3(1,39 - 1) = 1,17 \\ 3(1,91 - 1,39) = 1,56 \\ 3(2,25 - 1,91) = 1,02 \\ 3(2,39 - 2,25) = 0,42 \end{bmatrix}$$

iv.
$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0,469 \\ 0,231 \\ 0,165 \\ 0,127 \end{bmatrix}$$

Ahora, se construye la Spline de la siguiente forma:

v. $S_k(x) = S_{k,0} + S_{k,1}(x - x_k) + S_{k,2}(x - x_k)^2 + S_{k,3}(x - x_k)^3$ donde
 $S_{k,0} = y_k \quad S_{k,1} = d_k - \frac{h_k(2c_k + c_{k+1})}{3} \quad S_{k,2} = c_k \quad S_{k,3} = \frac{c_{k+1} - c_k}{3h_k}$

vi. $S_0(x) = 0 + (1,39 - \frac{1}{3}(2 \times 0,469 + 0,231))(x - 1) + 0,469(x - 1)^2 + \frac{0,231 - 0,469}{3}(x - 1)^3$
 $= (x - 1) + 0,469(x - 1)^2 - 0,0793(x - 1)^3$

vii. $S_1(x) = 1,39 + (1,91 - \frac{1}{3}(2 \times 0,231 + 0,165))(x - 2) + 0,231(x - 2)^2 + \frac{0,165 - 0,231}{3}(x - 2)^3$
 $= 1,39 + 1,7(x - 2) + 0,231(x - 2)^2 - 0,022(x - 2)^3$

viii. $S_2(x) = 3,3 + (2,25 - \frac{1}{3}(2 \times 0,165 + 0,127))(x - 3) + 0,165(x - 3)^2 + \frac{0,127 - 0,165}{3}(x - 3)^3$
 $= 3,3 + 2,1(x - 3) + 0,165(x - 3)^2 - 0,0127(x - 3)^3$

ix. $S(x) = \begin{cases} (x - 1) + 0,469(x - 1)^2 - 0,0793(x - 1)^3 & si \quad x \in [1, 2] \\ 1,39 + 1,7(x - 2) + 0,231(x - 2)^2 - 0,022(x - 2)^3 & si \quad x \in [2, 3] \\ 3,3 + 2,1(x - 3) + 0,165(x - 3)^2 - 0,0127(x - 3)^3 & si \quad x \in [3, 4] \end{cases}$

(b) Extrapolación

i. $S(1.5) = S_0(1.5) = 0.6073375 \implies E_r = \frac{|S_0(1.5) - f(1.5)|}{|f(1.5)|} = \frac{|0.6073375 - 0.6081976622|}{|0.6081976622|} = 1.41 \times 10^{-3}$

ii. $S(2.5) = S_1(2.5) = 2.295 \implies E_r = \frac{|S_1(2.5) - f(2.5)|}{|f(2.5)|} = \frac{|2.295 - 2.29072683|}{|2.29072683|} = 1.87 \times 10^{-3}$

iii. $S(3.5) = S_2(3.5) = 4.3896625 \implies E_r = \frac{|S_2(3.5) - f(3.5)|}{|f(3.5)|} = \frac{|4.3896625 - 4.38467039|}{|4.38467039|} = 1.14 \times 10^{-3}$

(c) Integral

i. $\int_1^4 S(x) dx = \int_1^2 S_0(x) dx + \int_2^3 S_1(x) dx + \int_3^4 S_2(x) dx$
 $= \left[\frac{1}{2}(x - 1)^2 + \frac{0,469}{3}(x - 1)^3 - \frac{0,0793}{4}(x - 1)^4 \right]_1^2$

$$\begin{aligned}
& + \left[1.39x + \frac{1.7}{2}(x-2)^2 + \frac{0.231}{3}(x-2)^3 - \frac{0.022}{4}(x-2)^4 \right]_2^3 \\
& + \left[3.3x + \frac{2.1}{2}(x-3)^2 + \frac{0.165}{3}(x-3)^3 - \frac{0.0127}{4}(x-3)^4 \right]_3^4 \\
& = 0.63650833 + 2.3115 + 4.401825 = 7.34983333 \\
\text{ii. } \int_1^4 f(x)dx &= \int_1^4 x \ln(x)dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \Big|_1^4 = 8 \ln 4 - 4 - 0 + \frac{1}{4} = 7.3403549 \\
\text{iii. } E_r &= \frac{|\int_1^4 f(x)dx - \int_1^4 S(x)dx|}{|\int_1^4 f(x)dx|} = \frac{|7.3403549 - 7.34983333|}{|7.3403549|} = 1.29 \times 10^{-3}
\end{aligned}$$

3. Mínimos cuadrados

(a) Legendre

$$\begin{aligned}
\text{i. } \phi_0(x) &= 1 \\
&\Rightarrow \int_{-1}^1 dx = 2, \quad \int_{-1}^1 x \sin(\pi x) dx = \frac{2}{\pi} \\
\text{ii. } \phi_1(x) &= x - B_1 \\
B_1 &= \frac{\int_{-1}^1 x dx}{\int_{-1}^1 dx} = 0 \Rightarrow \phi_1(x) = x \\
&\Rightarrow \int_{-1}^1 x^2 dx = \frac{2}{3}, \quad \int_{-1}^1 x(x \sin(\pi x)) dx = 0 \\
\text{iii. } \phi_2(x) &= (x - B_2) \phi_1(x) - C_2 \phi_0(x) \\
B_2 &= \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2 dx} = 0, \quad C_2 = \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 dx} = \frac{1}{3} \Rightarrow \phi_2(x) = x^2 - \frac{1}{3} \\
&\Rightarrow \int_{-1}^1 (x^2 - \frac{1}{3})^2 dx = \frac{8}{45}, \quad \int_{-1}^1 (x^2 - \frac{1}{3}) x \sin(\pi x) dx = \frac{4}{3\pi} - \frac{12}{\pi^3} \\
\text{iv. } \phi_3(x) &= (x - B_3) \phi_2(x) - C_3 \phi_1(x) \\
B_3 &= \frac{\int_{-1}^1 x(x^2 - \frac{1}{3})^2 dx}{\int_{-1}^1 (x^2 - \frac{1}{3})^2 dx} = 0, \quad C_3 = \frac{\int_{-1}^1 x^2(x^2 - \frac{1}{3}) dx}{\int_{-1}^1 x^2 dx} = \frac{4}{15} \Rightarrow \phi_3(x) = x(x^2 - \frac{1}{3}) - \frac{4}{15}x = \\
& x^3 - \frac{3}{5}x \\
&\Rightarrow \int_{-1}^1 (x^3 - \frac{3}{5}x)^2 dx = \frac{8}{175}, \quad \int_{-1}^1 (x^3 - \frac{3}{5}x) x \sin(\pi x) dx = 0 \\
\text{v. Los coeficientes son} \\
a_0 &= \frac{2}{\pi} = \frac{1}{\pi} \\
a_1 &= \frac{0}{\frac{2}{\pi}} = 0 \\
a_2 &= \frac{\frac{4}{3\pi} - \frac{12}{\pi^3}}{\frac{8}{45}} = \frac{15}{2\pi} - \frac{135}{2\pi^3} \\
a_3 &= \frac{0}{\frac{8}{175}} = 0 \\
\text{vi. Finalmente, el polinomio de Legendre es:} \\
L(x) &= a_0 \phi_0(x) + a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) \\
&= \frac{1}{\pi} + 0 + \left(\frac{15}{2\pi} - \frac{135}{2\pi^3} \right) \left(x^2 - \frac{1}{3} \right) + 0 \\
&= \frac{45}{2\pi^3} - \frac{3}{2\pi} + \frac{15}{2\pi} x^2 - \frac{135}{2\pi^3} x^2
\end{aligned}$$

(b) Trigonométricos

$$\begin{aligned}
\text{i. Como el intervalo es } [-1, 1], & \text{ para aplicar los polinomios trigonométricos debemos trans-} \\
& \text{formar el intervalo } [-\pi, \pi] \text{ a } [-1, 1]. \text{ Para esto se usa la transformación lineal } \frac{x}{\pi} = u. \\
\text{ii. La base trigonométrica queda de la forma:} \\
\phi_0(u) &= \frac{1}{2} \\
\phi_1(u) &= \cos(\pi u) \\
\phi_2(u) &= \cos(2\pi u) \\
\phi_3(u) &= \sin(\pi u) \\
\text{iii. Los coeficientes son:} \\
a_0 &= \int_{-1}^1 u \sin(\pi u) du = \frac{2}{\pi} \\
a_1 &= \int_{-1}^1 \cos(\pi u) [u \sin(\pi u)] du = -\frac{1}{2\pi} \\
a_2 &= \int_{-1}^1 \cos(2\pi u) [u \sin(\pi u)] du = -\frac{2}{3\pi} \\
b_1 &= \int_{-1}^1 \sin(\pi u) [u \sin(\pi u)] du = 0 \\
\text{iv. El polinomio resultante es:} \\
S(u) &= \frac{a_0}{2} + a_2 \cos(2\pi u) + a_1 \cos(\pi u) + b_1 \sin(\pi u) \\
&= \frac{2}{\pi} - \frac{2}{3\pi} \cos(2\pi u) - \frac{1}{2\pi} \cos(\pi u) + 0 \\
&= \frac{1}{\pi} - \frac{1}{2\pi} \cos(\pi u) - \frac{2}{3\pi} \cos(2\pi u)
\end{aligned}$$

(c) Veamos el error cuadrático de cada aproximación

i. Legendre

$$\begin{aligned}
E_L &= \int_{-1}^1 (f(x) - L(x))^2 dx = \int_{-1}^1 (x \sin(\pi x) - \frac{45}{2\pi^3} + \frac{3}{2\pi} - \frac{15}{2\pi} x^2 + \frac{135}{2\pi^3} x^2)^2 dx \\
&\text{Para simplificar cálculo, llamemos } c_1 = \frac{45}{2\pi^3} - \frac{3}{2\pi}, \text{ y } c_2 = \frac{15}{2\pi} - \frac{135}{2\pi^3}. \text{ Entonces} \\
E_L &= \int_{-1}^1 (x \sin(\pi x) - c_1 - c_2 x^2)^2 dx
\end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^1 (c_1^2 - 2xc_1 \sin \pi x + 2x^2 c_1 c_2 - 2x^3 c_2 \sin \pi x + x^4 c_2^2 + x^2 \sin^2 \pi x) dx \\
&= c_1^2 \int_{-1}^1 dx - 2c_1 \int_{-1}^1 x \sin \pi x + 2c_1 c_2 \int_{-1}^1 x^2 dx - 2c_2 \int_{-1}^1 x^3 \sin \pi x dx + c_2^2 \int_{-1}^1 x^4 dx + \int_{-1}^1 x^2 \sin^2 \pi x dx \\
&= 2c_1^2 - 2c_1 \frac{2}{\pi} + 2c_1 c_2 \frac{2}{3} - 2c_2 \left(\frac{2}{\pi} - \frac{12}{\pi^3} \right) + c_2^2 \frac{2}{5} + \left(\frac{1}{3} - \frac{1}{2\pi^2} \right) \\
&\text{Ahora, evaluando con } c_1 \simeq 0.248, c_2 \simeq 0.210 \\
&E_L \simeq 7.2164763566785362844 \times 10^{-2}
\end{aligned}$$

ii. Trigonométrico

$$\begin{aligned}
E_T &= \int_{-1}^1 (f(x) - S(x))^2 dx = \int_{-1}^1 \left(x \sin(\pi x) - \frac{1}{\pi} + \frac{1}{2\pi} \cos(\pi x) + \frac{2}{3\pi} \cos(2\pi x) \right)^2 dx \\
&= 2 \int_{-1}^1 \left(-\frac{1}{\pi} x \sin(\pi x) + \frac{1}{2\pi} \cos(\pi x) x \sin(\pi x) + \frac{2}{3\pi} \cos(2\pi x) x \sin(\pi x) - \frac{1}{2\pi^2} \cos(\pi x) - \frac{2}{3\pi^2} \cos(2\pi x) + \right. \\
&\quad \left. \frac{1}{3\pi^2} \cos(\pi x) \cos(2\pi x) \right) dx + \int_{-1}^1 x^2 \sin^2(\pi x) dx + \int_{-1}^1 \frac{1}{\pi^2} dx + \int_{-1}^1 \frac{1}{4\pi^2} \cos^2(\pi x) dx + \int_{-1}^1 \frac{4}{9\pi^2} \cos^2(2\pi x) dx \\
&= 2 \left(-\frac{2}{\pi^2} - \frac{1}{4\pi^2} - \frac{4}{9\pi^2} + 0 + 0 + 0 \right) + \left(\frac{1}{3} - \frac{1}{2\pi^2} \right) + \frac{2}{\pi^2} + \frac{1}{4\pi^2} + \frac{4}{9\pi^2} \\
&= \frac{1}{3} - \frac{115}{36\pi^2} \simeq 9.6684411425321189987 \times 10^{-3}
\end{aligned}$$

iii. Entonces, como $E_T < E_L$, eso implica que la aproximación por el polinomio trigonométrico tiene menor error, por lo que es mejor

(d) Observaciones:

- i. Muchas funciones las cuales se integraban eran impares, por lo que se podía poner que vale cero sin hacer cálculos
- ii. Muchas integrales salen muchas veces, por lo que no era necesario repetir las mismas integrales si se anotaban ya una vez
- iii. En la parte b), si se aplicaba la definición del método de los mínimos cuadrados se obtenía la forma de las constantes, y lo importante era conservar la ortogonalidad de la base trigonométrica en $[-1, 1]$.