1. Transformada de Laplace.

1.1. Sea $f : [0, \infty) \to \Re$ función continua a trozos y de orden exponencial. Demuestre que si $F(s)$ denota la transformada de Laplace de f, entonces:

$$\lim_{s \to \infty} F(s) = 0$$

Solución:

$f(t)$ es de orden exponencial $\Rightarrow$

$$|f(x)| \leq Me^{cx}$$

$$|e^{-sx} f(x)| \leq Me^{-(s-c)x}$$

Notar que la integral de la función de la derecha converge para $s > c$, entonces la transformada de Laplace de $f(x)$ converge absolutamente para $s > c$.

$$|F(s)| = |\int_o^\infty e^{-sx} f(x)dx| \leq \int_0^\infty |e^{-sx} f(x)dx| \leq M \int_0^\infty e^{-(s-c)x} dx = \frac{M}{s-c}; s > c$$

Luego

$$\lim_{s \to \infty} F(s) = 0$$

1.2. Sea $f : (0, \infty) \to \Re$ continua en $(0, \infty)$ y de orden exponencial y tal que

$$\lim_{t \to 0^+} f(t) = +\infty$$

(a) Si $\lim_{t \to 0^+} t^\alpha f(t) = 1$, $\alpha \in (0, 1)$, demuestre que la transformada de laplace de la función existe.

(b) La transformada de Laplace de la función $t^{-\frac{1}{2}} \cosh(t)$ tiene la forma

$$I = h(s) \sqrt{s + \sqrt{(s^2 - 1)}}$$

Encuentre $h(s)$ y de aquí la expresión final de la transformada.

Solución:

$$Ł\{f(t)\} = I = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}dt$$

sea:

$I_1 = \int_0^T f(t)e^{-st}dt$

$I_2 = \int_T^\infty f(t)e^{-st}dt$

I_2 existe puesto que $f(t)$ es de orden exponecial en $(0, \infty)$

$$I_1 = \int_0^T f(t)e^{-st}dt = \lim_{\varepsilon \to 0} \int_\varepsilon^T f(t)e^{-st}dt$$

Sea $t \in (0, \varepsilon_0)$. Ocupando el dato se tiene que $|t^\alpha f(t) - 1| \leq \mu$ entonces:

$$-\mu \leq t^\alpha f(t) - 1 \leq \mu$$

$$-\mu + 1 \leq t^\alpha f(t) \leq \mu + 1$$

$\Leftrightarrow$

$$|t^\alpha f(t)| \leq \mu + 1$$

$$|f(t)| \leq \frac{\mu + 1}{t^\alpha}$$

Ocupando estas cotas:

$$\int_\varepsilon^T f(t)e^{-st}dt = \int_\varepsilon^{\varepsilon_0} f(t)e^{-st}dt + \int_{\varepsilon_0}^T f(t)e^{-st}dt$$

de esta última expresión se debe analizar el primer término puesto que el segundo existe.

$$|\int_\varepsilon^{\varepsilon_0} f(t)e^{-st}dt| \leq \int_\varepsilon^{\varepsilon_0} \frac{1+\mu}{t^\alpha} e^{-st}dt \leq (1+\mu) \int_\varepsilon^{\varepsilon_0} t^{-\alpha}dt \leq \frac{1+\mu}{1-\alpha}(\varepsilon_0^{1-\alpha} - \varepsilon^{1-\alpha})$$

luego

$$\lim_{\varepsilon\to 0}\frac{(1+\mu)(\varepsilon_0^{1-\alpha}-\varepsilon^{1-\alpha})}{1-\alpha}=\frac{(1+\mu)\varepsilon_0^{1-\alpha}}{1-\alpha}$$

$\Rightarrow I_1 \leq \infty \Rightarrow I < \infty$ y luego $\quad Ł\{f(t)\} \quad$ existe.

Parte b:

$$Ł\left\{t^{-\frac{1}{2}}\coth(t)\right\}=\frac{1}{2}\left(Ł\left\{t^{-\frac{1}{2}e^{t}}\right\}+Ł\left\{t^{-\frac{1}{2}e^{-t}}\right\}\right)$$

Para obtener la transformada de la función $t^{-\frac{1}{2}}$ se aplica la definición de transformada de laplace y se hace el cambio de variable $st=y^2$, $sdt=2ydy$

$$Ł\left\{t^{-\frac{1}{2}}\right\}=\int_0^\infty e^{-st}t^{-\frac{1}{2}}dt=\int_0^\infty e^{-y^2}\left(\frac{y^2}{s}\right)^{-\frac{1}{2}}\left(\frac{2y}{s}\right)dy=2s^{-\frac{1}{2}}\int_0^\infty e^{-y^2}dy=2s^{-\frac{1}{2}}\frac{1}{2}\pi^{\frac{1}{2}}=\sqrt{\frac{\pi}{s}}$$

Ocupando esta transformada, se aplica la propiedad:

$$Ł\left\{f(t)e^{at}\right\}=F(s-a)$$

$$Ł\left\{t^{-\frac{1}{2}}\cosh(t)\right\}=\frac{1}{2}\left(\sqrt{\frac{\pi}{s+1}}+\sqrt{\frac{\pi}{s-1}}\right)=\frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{s-1}+\sqrt{s+1}}{\sqrt{s^2-1}}\right)$$

Sea

$$A=\sqrt{s-1}+\sqrt{s+1}$$

$$A^2=2s+2\sqrt{s^2-1}$$

$$A=\sqrt{2s+2\sqrt{s^2-1}}$$

$$Ł\left\{t^{-\frac{1}{2}}\cosh(t)\right\}=\frac{\sqrt{\pi}}{2\sqrt{s^2-1}}\sqrt{2s+2\sqrt{s^2-1}}=\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{s^2-1}}\sqrt{s+\sqrt{s^2-1}}$$

$$\Rightarrow h(s)=\sqrt{\frac{\pi}{2(s^2-1)}}$$

finalmente

$$I=\sqrt{\frac{\pi}{2(s^2-1)}}\sqrt{s+\sqrt{s^2-1}}$$

1.3. Encuentre: $f(t)$ si $F(s) = \frac{s^2+1}{s^3-2s^2-8s}$

$$F(s) = \frac{s^2+1}{s(s^2-2s-8)} = \frac{s^2+1}{s(s-4)(s+2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+2}$$

$$F(s) = \frac{A(s^2-2s-8) + B(s^2+2s) + C(s^2-4s)}{s(s-4)(s+2)}$$

$\Rightarrow$

$$s^2(A+B+C) + s(-2A+2B-4C) - 8A = s^2+1$$

$\Rightarrow$

(1) $-8A = 1$

(2) $A - B + 2C = 0$

(3) $A + B + C = 1$

$$A = -\frac{1}{8} \quad ; \quad B = \frac{17}{24} \quad ; \quad C = \frac{5}{12}$$

$$F(s) = -\frac{1}{8s} + \frac{17}{24(s-4)} + \frac{5}{12(s+2)}$$

$$f(t) = -\frac{1}{8}Ł^{-1}\left\{\frac{1}{s}\right\} + \frac{17}{24}Ł^{-1}\left\{\frac{1}{s-4}\right\} + \frac{5}{12}Ł^{-1}\left\{\frac{1}{s+2}\right\}$$
$$f(t) = \left(-\frac{1}{8} + \frac{17}{24}e^{4t} + \frac{5}{12}e^{-2t}\right)U(t)$$

1.4. Encuentre

(a) $f(t)$ si $F(s) = \frac{s}{(s^2+a^2)(s^2+b^2)}$, $a^2 \neq b^2$

(b) $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solución:

(a)

$$F(s) = \frac{As+B}{(s^2+a^2)} + \frac{Cs+D}{(s^2+b^2)} = \frac{(As+B)(s^2+b^2)+(Cs+D)(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}$$

(1) $A + C = 0$

(2) $Asb^2 + Csa^2 = s \quad \Leftrightarrow \quad Ab^2 + Ca^2 = 1$

(3) $Bs^2 + Ds^2 = 0$

(4) $Bb^2 + Da^2 = 0$

$$A = \frac{1}{(b^2-a^2)} \quad ; \quad B = 0 \quad ; \quad C = \frac{1}{(a^2-b^2)} \quad ; \quad D = 0$$

$$Ł^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = Ł^{-1}\left\{\frac{s}{(b^2-a^2)(s^2+a^2)}\right\} + Ł^{-1}\left\{\frac{s}{(a^2-b^2)(s^2+b^2)}\right\}$$

$$Ł^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{(b^2-a^2)}Ł^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} + \frac{1}{(a^2-b^2)}Ł^{-1}\left\{\frac{s}{(s^2+b^2)}\right\}$$

finalmente

$$Ł^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{(b^2-a^2)}\cos(at) + \frac{1}{(a^2-b^2)}\cos(bt)$$

(b)

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$$
$$\Rightarrow A(s^2+2s+5) + (Bs+C)(s-1)$$

(1) $(A+B)s^2 = 0 \qquad \Rightarrow A = -B$

(2) $(2A - B + C)s = 5s$

(3) $\quad 5A - C = 3 \qquad \Rightarrow C = 5A - 3$

$$(1)y(3)en(2) \Rightarrow 2A + A + 5A - 3 = 5 \Rightarrow 8A = 8$$

$$\rightarrow A = 1 \qquad \rightarrow B = -1 \qquad \rightarrow C = 2$$

$$L^{-1}\left\{\frac{1}{s-1} + \frac{2-s}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{s-1} + \frac{2}{s^2+2s+5} - \frac{s}{s^2+2s+5}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1} + \frac{2}{(s+1)^2+2^2} - \frac{s+1-1}{(s+1)^2+2^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1} + \frac{2}{(s+1)^2+2^2} - \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^t + e^{-t}\sin(2t) - e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) = e^t + \frac{3}{2}e^{-t}\sin(2t) - e^{-t}\cos(2t)$$

1.5. Sea $H(t)$ una función igual a t cuando $0 < t < 4$ e igual a 5 cuando $t > 4$. Obtenga Ł $\{H(t)\}$ por definición y usando escalón unitario

Solución 1:
Aplicando la definición de transformada de Laplace:

$$Ł\{H(t)\} = \int_0^\infty H(t)e^{-st}dt = \int_0^4 te^{-st}dt + \int_4^\infty 4e^{-st}dt$$

$\int te^{-st}dt = -\frac{te^{-st}}{s} + \int \frac{e^{-st}}{s}dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}$

$u = t \Rightarrow du = dt; \qquad dv = e^{-st}dt \Rightarrow v = -\frac{e^{-st}}{s}$

$$Ł\{H(t)\} = \left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)_0^4 + 5\left(-\frac{e^{-st}}{s^2}\right)_4^\infty$$

$$Ł\{H(t)\} = \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}$$

Solución 2:
(usando escalón unitario)

$$H(t) = tU(t) - [t-5]U(t-4) + U(t-4)$$

$\Rightarrow$

$$H(S) = Ł\left\{H(t)\right\} = Ł\left\{tU(t)\right\} - Ł\left\{[t-4]U(t-4)\right\} + Ł\left\{U(t-4)\right\}$$

$$Ł\left\{H(t)\right\} = \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}$$

1.6. Ocupando transformada de Laplace resuelva la siguiente ecuación diferencial

$$x'' + k^2 x = f(t) \quad ; \quad x(0) = x'(0) = 0$$

$$f(t) = \sum_{i=0}^{\infty} U(t-i)$$

Solución

Se asume que la transformada de Laplace de la Serie es la Serie de las transformadas de Laplace:

$$Ł\{f(t)\} = Ł\left\{\sum_{i=0}^{\infty} U(t-i)\right\} = \sum_{i=0}^{\infty} \frac{e^{-is}}{s} = \frac{1}{s}\sum_{i=0}^{\infty}\left(e^{-s}\right)^i$$

Recordando la suma geométrica:

$S_T = a^0 + a^1 + a^2 + \ldots + a^n$

$aS_T = a^1 + a^2 + a^3 + \ldots + a^{n+1}$

luego

$S_T = \frac{1-a^{n+1}}{1-a} \Rightarrow \frac{1}{s}\sum_{i=0}^{\infty}\left(e^{-s}\right)^i = \frac{1}{s}\lim_{n\to\infty}\frac{1-\left(e^{-s}\right)^{n+1}}{1-e^{-s}}$

$\Rightarrow$

$$Ł\{f(t)\} = \frac{1}{s(1-e^{-s})}$$

Ahora se aplica transformada de Laplace a la ecuación diferencial

$$Ł\{x''\} + k^2Ł\{x\} = Ł\{f(t)\}$$

sea $X(s) = Ł\{x(t)\}$

luego

$$s^2X(s) - sx(0) - x'(0) + k^2X(s) = \frac{1}{s(1-e^{-s})}$$

$\Rightarrow$

$$X(s) = \frac{1}{(s^2+k^2)s(1-e^{-s})} = \frac{1}{k}\frac{k}{(s^2+k^2)s(1-e^{-s})}$$

Se conocen las transformadas de Laplace:

$$Ł^{-1}\left\{\frac{1}{s(1-e^{-s)}}\right\} = f(t) = \sum_{0}^{\infty} U(t-i)$$

$$Ł^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$$

Ahora se aplica el teorema de Convolución:

$$Ł^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$x(t) = \frac{1}{k}\int_0^t \sum_{i=0}^{\infty} U(\tau-i)\sin(k(t-\tau))d\tau = \frac{1}{k}\sum_{i=0}^{\infty}\int_0^t U(\tau-i)\sin(k(t-\tau))d\tau$$

$$x(t) = \frac{1}{k}\sum_{i=0}^{\infty}\left(\int_i^t \sin(k(t-\tau))d\tau\right)U(t-i) = \frac{1}{k^2}\sum_{i=0}^{\infty}(-1+\cos(k(t-i)))U(t-i)$$

finalmente

$$x(t) = \frac{1}{k^2}\sum_{i=0}^{\infty}(-1+\cos(k(t-i)))U(t-i)$$

1.7. (a) Considere el problema con valores iniciales:

$$x'' + 2x' + x = \sum_{n=0}^{\infty}\delta_{n\pi}(t) \qquad ; \qquad x(0) = x'(0) = 0$$

Determine la solución usando transformada de Laplace, para esto suponga que la transformada de la serie es la serie de las transformadas y similarmente para la transformada inversa.

(b) Si $t \in [j\pi, (j+1)\pi]$ demuestre que $x(t) = e^{-t}(t\alpha_j + \beta_j)$, para ciertas constantes α_j y β_j.

Solución:

parte (a):

$$x'' + 2x' + x = \sum_{n=0}^{\infty}\delta_{n\pi}(t)$$

$\Rightarrow$

$$s^2X(s) + 2sX(s) + X(s) = \sum_{n=0}^{\infty}e^{-n\pi s}$$

$\Rightarrow$

$$X(s) = \sum_{n=0}^{\infty}\frac{e^{-n\pi s}}{(s+1)^2}$$

$\Rightarrow$

$$F(s) = \frac{1}{(s+1)^2} \quad \Rightarrow \quad f(t) = Ł^{-1}\{F(s)\} = te^{-t}$$

$\Rightarrow$

$$Ł^{-1}\left\{\frac{e^{-n\pi s}}{(s+1)^2}\right\} = f(t-n\pi)U(t-n\pi) = [t-n\pi]e^{-(t-n\pi)}U(t-n\pi)$$

luego

$$x(t) = Ł^{-1}\left\{X(s)\right\} = \sum_{n=0}^{\infty} U(t-n\pi)[t-n\pi]e^{-(t-n\pi)}$$

parte (b):

si $\quad t \in [j\pi, (j+1)\pi]$

entonces

$$x(t) = \sum_{n=0}^{j} U(t-n\pi)[t-n\pi]e^{-(t-n\pi)} \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{j} [t-n\pi]e^{-(t-n\pi)}$$

$$x(t) = e^{-t}\left[t\sum_{n=0}^{j} e^{n\pi} - \sum_{n=0}^{j} n\pi e^{n\pi}\right]$$

denotando

$$\alpha_j = \sum_{n=0}^{j} e^{n\pi} \qquad ; \qquad \beta_j = \sum_{n=0}^{j} n\pi e^{n\pi}$$

finalmete

$$x(t) = e^{-t}(t\alpha_j + \beta_j)$$

1.8. Usando transformada de Laplace resuelva el siguiente sistema de ecuaciones diferenciales ordinarias.

$$x_1' = x_1 - x_2 + e^t \cos(t)$$
$$x_2' = x_1 + x_2 + e^t \operatorname{sen}(t)$$
$$x_1(0) = 0 \qquad x_2(0) = 0$$

Solución:

Se aplica transformada de Laplace a ambas ecuaciones. Sea $X_1(s) = \text{Ł}\{x_1(t)\}$ y $X_2(s) = \text{Ł}\{x_2(t)\}$. Entonces se cumple:

$$sX_1(s) - x_1(0) = X_1(s) - X_2(s) + \frac{(s-1)}{(s-1)^2+1}$$
$$sX_2(s) - x_2(0) = X_1(s) + X_2(s) + \frac{1}{(s-1)^2+1}$$

Aplicando las condiciones iniciales y escribiendo el problema como un sistema de ecuaciones algebraicas resulta:

$$\begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(s-1)}{(s-1)^2+1} \\ \frac{1}{(s-1)^2+1} \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{((s-1)^2+1)^2} \begin{bmatrix} s-1 & -1 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} (s-1) \\ 1 \end{bmatrix} = \frac{1}{((s-1)^2+1)^2} \begin{bmatrix} (s-1)^2-1 \\ 2(s-1) \end{bmatrix}$$

Entonces:

$$X_1(s) = \frac{(s-1)^2-1}{((s-1)^2+1)^2}, \quad X_2(s) = \frac{2(s-1)}{((s-1)^2+1)^2}$$

Ahora se calculan las transformadas inversas:

$$x_1(t) = \text{Ł}^{-1}\left\{\frac{(s-1)^2-1}{((s-1)^2+1)^2}\right\} = e^t\text{Ł}^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = e^t\text{Ł}^{-1}\left\{\frac{-d(\frac{s}{(s^2+1)})}{ds}\right\}$$

$$x_1(t) = e^t t\text{Ł}^{-1}\left\{\frac{s}{(s^2+1)}\right\} = e^t t\cos(t)$$

Lo mismo para $x_2(t)$

$$x_2(t) = Ł^{-1}\left\{\frac{2(s-1)}{((s-1)^2+1)^2}\right\} = e^t Ł^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = e^t Ł^{-1}\left\{\frac{-d\left(\frac{1}{(s^2+1)}\right)}{ds}\right\}$$

$$x_2(t) = e^t t Ł^{-1}\left\{\frac{1}{(s^2+1)}\right\} = e^t t\sin(t)$$

Finalmente la solución del sistema de ecuaciones diferenciales es:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^t t\cos(t) \\ e^t t\sin(t) \end{bmatrix}$$

1.9. (i) Encuentre las transformadas inversas de Laplace en los siguientes casos:

(a)

$$L^{-1}\left\{\frac{1}{s^2+s-20}\right\}$$

(b)

$$L^{-1}\left\{\frac{s-1}{s^2(s^2+1)}\right\}$$

(ii) Encuentre las siguientes transformadas de Laplace:

(a)

$$L\left\{e^t cos(3t)\right\}$$

(b)

$$L\left\{t^2 cos^2(t)\right\}$$

Solución

(i) (a)

$$L^{-1}\left\{\frac{1}{s^2+s-20}\right\} = L^{-1}\left\{\frac{A}{s+5}+\frac{B}{s-4}\right\} = A\cdot L^{-1}\left[\frac{1}{s+5}\right] + B\cdot L^{-1}\left[\frac{1}{s-4}\right]$$

$$= -\frac{1}{9}\cdot L^{-1}\left[\frac{1}{s+5}\right] + \frac{1}{9}\cdot L^{-1}\left[\frac{1}{s-4}\right] = -\frac{1}{9}\cdot e^{-5t} + \frac{1}{9}\cdot e^{4t}$$

(i) (b)

$$L^{-1}\left[\frac{s-1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C\cdot s+D}{s^2+1}\right]$$

$$= A\cdot L^{-1}\left[\frac{1}{s}\right] + B\cdot L^{-1}\left[\frac{1}{s^2}\right] + C\cdot L^{-1}\left[\frac{s}{s^2+1}\right] + D\cdot L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$A = 1 \longrightarrow B = -1 \longrightarrow C = -1 \longrightarrow D = 1$$

$$= 1 - t - cos(t) + sen(t)$$

(ii) (a)

Por trigonometria

$$\cos^2(x) = \frac{\cos 2x + 1}{2}$$

$$\Longrightarrow L\left\{e^t cos(3t)\right\} = L\left\{e^t(cos6t+1)\frac{1}{2}\right\} = \frac{1}{2}\left(L\{e^t cos6t\} + L\{e^t\}\right)$$

$$L\{e^t cos(3t)\} = \frac{1}{2}\left(\frac{s-1}{(s-1)^2+36} + \frac{1}{s-1}\right)$$

(ii) (b)

$$L\left\{t^2 cos^2(t)\right\} = L\left\{t^2(cos2t+1)\frac{1}{2}\right\} = \frac{1}{2}\left(L\{t^2 cos2t\} + L\{t^2\}\right)$$

$$= \frac{1}{2}\left((-1^2)\frac{d^2}{ds^2}(\frac{s}{s^2+4}) + \frac{2}{s^3}\right) = \frac{1}{2}\left(\frac{d}{ds}\left(\frac{s^2+4-2s^2}{(s^2+4)^2}\right) + \frac{2}{s^3}\right)$$

$$= \frac{1}{2}\left(\frac{-2s(s^2+4)^2 - 4s(s^2+4)(4-s^2)}{(s^2+4)^4} + \frac{2}{s^3}\right) = \frac{-s(s^2+4+8-2s^2)}{(s^2+4)^3} + \frac{1}{s^3}$$

$$L\{t^2 cos^2(t)\} = s\frac{(s^2-12)}{(s^2+4)^3} + \frac{1}{s^3}$$

1.10. Encuentre las siguientes transformadas de Laplace:

(a)

$$L\{e^{2t}(t-1)^2\}$$

(b)

$$L\{e^t U(t-5)\}$$

(c)

$$L\{te^{-3t}cos(3t)\}$$

(en (b) U es la función Escalón Unitario)

Solución

(a)

$$L\{e^{2t}(t-1)^2\} = L\{e^{2t}(t^2-2t+1)\} = L\{e^{2t}t^2\} - 2L\{e^{2t}t\} + L\{e^{2t}\}$$

$$= \frac{2!}{(s-2)^3} - 2\frac{1!}{(s-2)^2} + \frac{1}{(s-2)} = L\{e^{2t}(t-1)^2\} = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{(s-2)}$$

(b)

$$L\{U(t-a)\} = \frac{e^{-as}}{s} \Rightarrow L\{U(t-5)\} = \frac{e^{-5s}}{s}$$

$$L\{e^t U(t-5)\} = \frac{e^{-5(s-1)}}{s-1}$$

(c)

$$\Rightarrow L\{\cos(3t)\} = \frac{s}{s^2+9}$$

$$\Rightarrow L\{t\cos(3t)\} = \frac{s^2+9-2s^2}{(s^2+9)^2} = \frac{9-s^2}{(s^2+9)^2} = L\{g(t)\} = F(s)$$

$$\Rightarrow L\{e^{-2t}g(t)\} = F(s+2) = \frac{9-(s+2)^2}{((s+2)^2+9)^2}$$

$$L\{te^{-3t}cos(3t)\} = -(\frac{(s+2)^2-9}{((s+2)^2+9^2)})$$

1.11. Resuelva el siguiente sistema con condiciones iniciales.

$$X' = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} X + \begin{bmatrix} \delta(t-\omega) + U(t-2\omega) - U(t-3\omega) \\ 0 \end{bmatrix}$$

$$X(0) = 0$$

Solución:

Se aplica transformada de Laplace a ambas ecuaciones. Sea $X_1(s) = Ł\{x_1(t)\}$ y $X_2(s) = Ł\{x_2(t)\}$. Entonces se cumple:

$$sX_1(s) - x_1(0) = \omega X_2(s) + e^{-\omega s} + \frac{1}{s}e^{-2\omega s} - \frac{1}{s}e^{-3\omega s}$$

$$sX_2(s) - x_2(0) = -\omega X_1(s)$$

Aplicando las condiciones iniciales y escribiendo el problema como un sistema de ecuaciones algebraicas resulta:

$$\begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} e^{-\omega s} + \frac{1}{s}e^{-2\omega s} - \frac{1}{s}e^{-3\omega s} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{s^2+\omega^2} \begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix} \begin{bmatrix} e^{-\omega s} + \frac{1}{s}e^{-2\omega s} - \frac{1}{s}e^{-3\omega s} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{s^2+\omega^2} \begin{bmatrix} se^{-\omega s} + e^{-2\omega s} - e^{-3\omega s} \\ -\omega e^{-\omega s} + \frac{-\omega}{s}(e^{-2\omega s} - e^{-3\omega s}) \end{bmatrix}$$

Entonces:

$$X_1(s) = \frac{se^{-\omega s}}{s^2+\omega^2} + \frac{e^{-2\omega s} - e^{-3\omega s}}{s^2+\omega^2}$$

$$X_2(s) = \frac{-\omega e^{-\omega s}}{s^2+\omega^2} - \frac{\omega(e^{-2\omega s} - e^{-3\omega s})}{s(s^2+\omega^2)}$$

$$Ł^{-1}\left\{\frac{se^{\omega s}}{s^2+\omega^2}\right\} = U(t-\omega)f(t-\omega) = U(t-\omega)\cos(\omega(t-\omega))$$

$$F(s) = \frac{s}{s^2+\omega^2} \rightarrow f(t) = \cos\omega t$$

$$Ł^{-1}\left\{\frac{\omega(e^{-2\omega s} - e^{-3\omega s})}{(s^2+\omega^2)}\right\}\frac{1}{\omega}$$

$$= \frac{1}{\omega}[U(t-2\omega)\sin(\omega(t-2\omega)) - U(t-3\omega)\sin(\omega(t-3\omega))]$$

$$Ł^{-1}\left\{\frac{\omega e^{-\omega s}}{s^2+\omega^2}\right\} = -U(t-\omega)\sin(\omega(t-\omega))$$

$$Ł^{-1}\left\{\frac{\omega}{s(s^2+\omega^2)}\right\} = Ł^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+\omega^2}\right\} = Ł^{-1}\left\{\frac{As^2 + A\omega^2 + Bs^2 + Cs}{s(s^2+\omega^2)}\right\}$$

$$C = 0 \longrightarrow A = \frac{1}{\omega} \longrightarrow A + B = 0 \longrightarrow B = -\frac{1}{\omega}$$

$$Ł^{-1}\left\{\frac{1}{\omega s} - \frac{1}{\omega}\frac{s}{s^2+\omega^2}\right\} = \frac{1}{\omega} - \frac{1}{\omega}\cos\omega t = f(t)$$

$$Ł^{-1}\left\{\frac{\omega(e^{-2\omega s} - e^{-3\omega s})}{(s(s^2+\omega^2))}\right\} = U(t-2\omega)\left[\frac{1}{\omega} - \frac{1}{\omega}\cos\omega(t-2\omega)\right] - U(t-3\omega)\left[\frac{1}{\omega} - \frac{1}{\omega}\cos(\omega(t-3\omega))\right]$$

Finalmente las soluciones son:

$$x(t) = U(t-\omega)\cos(\omega(t-\omega)) + \frac{1}{\omega}\left[U(t-2\omega)\sin(\omega(t-2\omega)) - U(t-3\omega)\sin(\omega(t-3\omega))\right]$$

$$y(t) = -U(t-\omega)\sin(\omega(t-\omega)) + U(t-2\omega)\left[\frac{1}{\omega} - \frac{1}{\omega}\cos\omega(t-2\omega)\right] - U(t-3\omega)\left[\frac{1}{\omega} - \frac{1}{\omega}\cos(\omega(t-3\omega))\right]$$

1.12. (i) Resuelva la ecuación:

$$x' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} x + \begin{bmatrix} U(t-a) \\ \delta(t-b) \end{bmatrix} \qquad a, b \in \Re^+$$
$$x(0) = 0$$

Solución: Por Laplace:

$$x'(t) = ax(t) + by(t) + U(t-a)$$
$$y'(t) = -bx(t) + ay(t) + \delta(t-b)$$

Sea

$$x(t=0) = x_0 = 0$$
$$y(t=0) = y_0 = 0$$

$$\Rightarrow sX(s) - x_0 = aX(s) + bY(s) + e^{-sa}\frac{1}{s}$$
$$sY(s) - y_0 = -bX(s) + aY(s) + e^{-bs}$$

$$(s-a)X(s) - bY(s) = x_0 + e^{-sa}$$
$$bX(s) + (s-a)Y(s) = y_0 + e^{-bs}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{(s-a)^2 + b^2}\begin{bmatrix} s-a & b \\ -b & s-a \end{bmatrix}\begin{bmatrix} x_0 + \frac{1}{s}e^{-as} \\ y_0 + e^{-bs} \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{(s-a)}{s}e^{-as} + be^{-bs} \\ -\frac{b}{s}e^{-as} + (s-a)e^{-bs} \end{bmatrix}\frac{1}{(s-a)^2 + b^2}$$

$$\Rightarrow X(s) = \frac{be^{-bs}}{(s-a)^2 + b^2} + \frac{e^{-as}}{(s-a)^2 + b} - \frac{ae^{-as}}{s((s-a)^2 + b^2)}$$

$$Y(s) = -\frac{be^{-as}}{s((s-a)^2 + b^2)} + \frac{se^{-bs}}{(s-a)^2 + b^2} - \frac{ae^{-bs}}{(s-a)^2 + b}$$

$$Ł^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}=e^{at}Ł^{-1}\left\{\frac{b}{s^2+b^2}\right\}=e^{at}\sin bt$$

$$Ł^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=\frac{1}{b}e^{at}\sin bt$$

$$Ł^{-1}\left\{\frac{1}{s((s-a)^2+b^2)}\right\}=Ł^{-1}\left\{\frac{A}{s}+\frac{B}{(s-a)^2+b^2}\right\}=A+\frac{Be^{at}}{b}\sin bt$$

$$Ł^{-1}\left\{\frac{(s-a)+a}{(s-a)^2+b^2}\right\}=e^{at}Ł^{-1}\left\{\frac{s}{s^2+b^2}+\frac{a}{s^2+b^2}\right\}=e^{at}\cos(bt)+\frac{e^{at}a}{b}\sin(bt)$$

$$\Rightarrow x(t)=U(t-b)e^{a(t-b)}\sin(b(t-b))+U(t-a)\frac{1}{b}e^{a(t-a)}\sin(b(t-a))$$
$$-aU(t-a)\left\{A+\frac{B}{b}e^{a(t-a)}\sin(b(t-a))\right\}$$
$$Y(t)=-bU(t-a)\left\{A+\frac{B}{b}e^{a(t-a)}\sin(b(t-a))\right\}$$
$$+U(t-b)\left\{e^{a(t-b)}\cos(b(t-b))+\frac{a}{b}e^{a(t-b)}\sin(b(t-b))\right\}$$
$$-aU(t-b)\left\{\frac{1}{b}e^{a(t-b)}\sin(b(t-b))\right\}$$

$$y(t)=-bU(t-a)\left\{A+\frac{B}{b}e^{a(t-a)}\sin(b(t-a))\right\}+U(t-b)\left\{e^{a(t-b)}\cos(b(t-b))\right\}$$

1.13. Usando transformada de Laplace encuentre la solución de la ecuación integral:

$$y(t)=\cos t+\int_o^t e^{-s}y(t-s)ds$$

Solución:

$$L\left\{y(t)\right\}=Y(s)$$
$$L\{\cos t\}=\frac{s}{s^2+1}$$
$$L\left\{\int_o^t e^{-s}y(t-s)ds\right\}=L\{e^{-t}\}Y(s)=\frac{Y(s)}{s+1}$$

$$\Rightarrow Y(s) = \frac{s}{s^2+1} + Y(s)\frac{1}{s+1} \Rightarrow Y(s)[1 - \frac{1}{s+1}] = Y(s)\frac{s}{s+1} = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$y(t) = \cos(t) + \sin(t)$$

1.14. Calcule la transformada de Laplace de la onda cuadrada:

$$w(x) = \begin{cases} 1 & \text{si } 2n \leq t < 2n+1 \qquad , n \in N \\ -1 & \text{si } 2n+1 \leq t < 2n+2 \qquad , n \in N \end{cases} \tag{1.1}$$

Solución:

Forma 1

$$w(t) = U(t) - 2U(t-1) + 2U(t-2) - 2U(t-3) + \dots$$

$$w(t) = U(t) + 2\sum_{i=1}^{\infty}(-1)^i U(t-i)$$

$$\Rightarrow L\{w(t)\} = L\{U(t)\} + 2\sum_{i=1}^{\infty}(-1)^i L\{U(t-i)\}$$

$$= \frac{1}{s} + 2\sum_{i=1}^{\infty}(-1)^i \frac{1}{s}e^{-is} = \frac{1}{s}[1 + 2\sum_{i=1}^{\infty}(-e^{-s})^i]$$

$$\sum_{i=1}^{n} p^i = \frac{p - p^{n+1}}{1-p}, \lim_{n\to\infty}\sum_{i=1}^{n} p^i = \sum_{i=1}^{\infty} p^i = \frac{p}{1-p}, \qquad (|p| < 1)$$

$$= \frac{1}{2}\left[1 + 2\frac{(-e^{-s})}{1+e^{-s}}\right] = \frac{1}{2}\frac{1-e^{-s}}{1+e^{-s}} / \frac{e^{\frac{s}{2}}}{e^{\frac{s}{2}}} = \frac{1}{s}\tan h\left(\frac{s}{2}\right)$$

Forma 2

$$F(t+w) = F(t) \Rightarrow L\{F(t)\} = \frac{\int_o^w e^{-s\beta}F(\beta)d\beta}{1-e^{-sw}}$$

$w = 2$

$$\int_o^2 e^{-s\beta} w(\beta) d\beta = \int_o^1 e^{-s\beta} 1 d\beta + \int_1^2 e^{-s\beta}(-1) = \left(\frac{-e^{-s\beta}}{s} \right) | + \left(\frac{e^{-s\beta}}{s} \right) |$$

$$= \frac{-e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}$$

$$L\{F(t)\} = \frac{1}{s} \left[\frac{1 - 2e^{-s} + e^{-2s}}{1 - e^{-2s}} \right] = \frac{1}{s} \left[\frac{(1 - e^{-s})}{(1 + e^{-s}} \right] / \frac{e^{\frac{1}{2}s}}{e^{\frac{1}{2}s}}$$

$$\Rightarrow L\{F(t)\} = \frac{1}{s} \tan h \left(\frac{s}{2} \right)$$

1.15. (i) Dibuje la función e(t)dada por:

$$e(t) = t - j \quad , para \quad j \leq t < j + 1, \qquad j \in \{0\} \cup N$$

Demuestre que su transformada de Laplace es dada por:

$$\frac{1}{s} \left(\frac{1}{s} + \frac{e^{-s}}{e^{-s} - 1} \right)$$

(ii) Considere ahora el circuito electrico RLC en serie

El voltaje $v(t)$ en el condensador C satisface la ecuación diferencial

$$C \frac{d^2 v}{dt^2} + \frac{RC}{L} \frac{dv}{dt} + \frac{1}{L} v = \frac{e(t)}{L}$$

Calcule el voltaje $v(t)$por el condensador de la corriente $i(t) = C \frac{dv}{dt}(t)$, si la entrada de voltaje $e(t)$ esta dado por la función de la parte (i), y las condiciones iniciales son $v(0) = v_0, i(0) = i_0$

Las constantes quedan dadas por $LC = 1, RC = 2$

Solución:

(i)

$$V_i(t) = t - U(t-1) - U(t-2) - \ldots - U(t-i) - \ldots$$

$$V_i(t) = t - \sum_{i=1}^{\infty} U(t-i)$$

$$L\{V_i(t)\} = L\{t\} - L\left\{\sum_{i=1}^{\infty} U(t-i)\right\}$$

$$= \frac{1}{s^2} - \sum_{i=1}^{\infty} e^{-si}\frac{1}{s} = \frac{1}{s^2} - \frac{1}{s}\sum_{i=1}^{\infty}(e^{-s})^i = \frac{1}{s^2} - \frac{1}{s}\frac{e^{-s}}{1-e^{-s}} = \frac{1}{s}\left(\frac{1}{s} - \frac{e^{-s}}{1-e^{-s}}\right)$$

(ii)

$$LCv'' + RCv' + v = v_i$$
$$v'' + 2v' + v = v_i \setminus L$$
$$s^2V(s) - sv(0) - v'(0) + 2sV(s) - 2v(0) + V(s) = V_i(s)$$
$$V(s) = \frac{v_0}{s+1} + \frac{v_0 + v'(0)}{(s+1)^2} + \frac{v_i(s)}{(s+1)^2}$$
$$v'(0) = \frac{1}{C}i_0$$

$$L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad ; \qquad L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}t$$

$$L^{-1}\left\{\frac{V_i}{s+1}\right\} = \int_0^t V_i(\tau)e^{-(t-\tau)}(t-\tau)d\tau = V_3(t)$$

$$\Rightarrow v(t) = v_0e^{-t} + \left(v_0 + \frac{1}{C}\cdot i_0\right)te^{-t} + \underbrace{\int_0^t V_i(\tau)e^{-(t-\tau)}(t-\tau)d\tau}$$

$$\Rightarrow v(t) = v_0e^{-t} + \left(v_0 + \frac{1}{C}\cdot i_0\right)te^{-t} + V_3(t)$$

$$i(t) = C\cdot\frac{dv}{dt}$$

$$i(t) = i_0\cdot e^{-t} - (C\cdot v_0 + i_0)te^{-t} + \frac{dV_3}{dt}$$

$$\frac{dV_3}{dt} = \frac{d}{dt}\left(e^{-t}\left(-t\int_0^t V_i(\tau)e^{\tau}d\tau - \int_0^t V_i(\tau)e^{\tau}\tau d\tau\right)\right)$$

$$\frac{dV_3}{dt} = e^{-t}\left(-t\int_0^t V_i(\tau)e^{\tau}d\tau + \int_0^t (1-\tau)e^{\tau}V_i(\tau)d\tau\right)$$

$$i(t) = i_0\cdot e^{-t} - (C\cdot v_0 + i_0)te^{-t} + e^{-t}\left(\int_0^t (1-\tau)e^{\tau}V_i(\tau)d\tau - t\int_0^t V_i(\tau)e^{\tau}d\tau\right)$$

Otra forma:

$$L\{V_i(t)\} = \frac{1}{s^2} - \frac{1}{s}\sum_{i=1}^{\infty}(e^{-s})^i$$

$$L^{-1}\left\{\frac{V_i(s)}{(s+1)^2}\right\} = L^{-1}\left\{\frac{1}{s^2(s+1)^2} - \frac{1}{(s+1)^2}\sum_{i=1}^{\infty}e^{-si}\right\}$$

$$L^{-1}\left\{\frac{-2}{s} + \frac{1}{s^2} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2} - \sum_{i=1}^{\infty}\left(\frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}\right)e^{-si}\right\}$$

$$= \left(-2 + t + 2e^{-t} + te^{-t}\right) - \sum_{i=1}^{\infty}\left(1 - e^{-(t-i)} - (t-i)e^{-(t-i)}\right)U(t-i)$$

$$v(t) = -2+t+(2+v_0)e^{-t}+\left(1 + v_0 + \frac{i_0}{C}\right)te^{-t}-\sum_{i=1}^{\infty}\left(1 - e^{-(t-i)} - (t-i)e^{-(t-i)}\right)U(t-i)$$

$$i(t) = C + (i_0 - 2C)e^{-t} - (C + v_0C + i_0)te^{-t}$$

$$-\sum_{i=1}^{\infty}\left[(t-i)e^{-(t-i)}U(t-i) + \left(1 + (t+1-i)e^{-(t-i)}\delta(t-i)\right)\right]$$

1.16. Encuentre la transformada inversa de las siguientes funciones:

(a)

$$L^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$$

(b)

$$L^{-1}\left\{ln\left(\frac{s^2+1}{s^2+4}\right)\right\}$$

(c)

$$L^{-1}\left\{\frac{sF(s)}{s^2+4}\right\}$$

Suponga que $f(t) = L^{-1}(F(s))$

Solución:

(a) Fracciones Parciales :

$$\frac{(s+1)^2}{(s+2)^4} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$A(s+2)^3 + B(s+2)^2 + C(s+2) + D = s^2 + 2s + 1$$

$$A(s^3 + 6s^2 + 12s + 8) + B(s^2 + 4s + 4) + C(s+2) + D = s^2 + 2s + 1$$

$$A = 0 \longrightarrow B = 1 \longrightarrow C = -2 \longrightarrow D = 1$$

$$L^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} + L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= te^{-2t} - 2t^2\frac{e^{-2t}}{2} + t^3\frac{e^{-2t}}{6}$$

$$L^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = te^{-2t}\left[1 - t + \frac{t^2}{6}\right]$$

(b)

$$L^{-1}\left\{ln\left(\frac{s^2+1}{s^2+4}\right)\right\} = L^{-1}\left\{ln\left(s^2+1\right)\right\} - L^{-1}\left\{ln\left(s^2+4\right)\right\}$$

$$L(f(t)) = F(s) = \left\{ln\left(s^2+1\right)\right\} \qquad L(g(t)) = G(s) = \left\{ln\left(s^2+4\right)\right\}$$

$$\frac{dF(s)}{ds} = \frac{2s}{s^2+1} \qquad \frac{dG(s)}{ds} = \frac{2s}{s^2+4}$$

$$\rightarrow 2L^{-1}\left\{F'(s)\right\} = 2\cos t \qquad \rightarrow 2L^{-1}\left\{G'(s)\right\} = 2\cos 2t$$

Además que:

$$-L\left\{tf(s)\right\} = \frac{2s}{s^2+1} \qquad /L^{-1}()$$

$$-tf(t) = 2\cos t$$

$$f(t) = -\frac{2}{t}\cos t$$

$$-L\left\{tg(s)\right\} = \frac{2s}{s^2+4} \qquad /L^{-1}()$$

$$f(t) = -\frac{2}{t}\cos 2t$$

$$\Rightarrow L^{-1}\left\{ln\left(\frac{s^2+1}{s^2+4}\right)\right\} = \frac{2}{t}[\cos(2t) - \cos(t)]$$

(c)

$$L^{-1}\left\{\frac{sF(s)}{s^2+4}\right\}$$

$$L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$L^{-1}\left\{F(s)\right\} = f(t)$$

Por teorema de convolución

$$\Rightarrow L^{-1}\left\{\frac{sF(s)}{s^2+4}\right\} = \int_o^t f(t-\beta)\cos 2\beta d\beta$$

1.17. Usando transformada de Laplace resuelva los siguientes problemas.

(i) Encuentre la solución de:

$$EI\frac{d^4y}{dx^4} = \omega_0\left[1 - U\left(x - \frac{L}{2}\right)\right]$$

que satisface $y(0) = y'(0) = 0 \quad , \quad y(L) = y''(L) = 0.$

Aqui E, I, ω_0 son constantes positivas y $L > 0$.

(ii) Encuentre la solución de:

$$y'' - 7y' + 6y = e' + \delta(t - 10\pi) + \delta(t - 20\pi)$$
$$y(0) = y'(0) = 0$$

Solución:

(i)

$$EI\frac{d^4y}{dx^4} = \omega_0\left[1 - U\left(x - \frac{L}{2}\right)\right] \qquad /L()$$

$$EIL(y^{(iv)}) = \omega_0 L(1) - \omega_0 L\left(U\left(x - \frac{L}{2}\right)\right)$$

Sea $F(s) = L(y(x))$

$$EI\left[s^4F(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right] = \frac{\omega_0}{s} - \omega_0\frac{e^{-\frac{L}{2}s}}{s}$$

Sea $y''(0) = C_1 \qquad y'''(0) = C_2$

$$s^4F(s) - sC_1 - C_2 = \frac{\omega_0}{EIs}\left[1 - e^{-\frac{L}{2}s}\right]$$

$$s^4F(s) = \frac{\omega_0}{EI}\left[\frac{1}{s} - \frac{e^{-\frac{L}{2}s}}{s}\right] + sC_1 + C_2$$

$$F(s) = \frac{\omega_0}{EI}\left[\frac{1}{s^5} - \frac{e^{-\frac{L}{2}s}}{s^5}\right] + \frac{C_1}{s^3} + \frac{C_2}{s^4} \qquad /L^{-1}()$$

$$y(x) = \frac{\omega_0}{EI}\left[L^{-1}\left\{\frac{1}{s^5}\right\} - L^{-1}\left\{\frac{e^{-\frac{L}{2}s}}{s^5}\right\}\right] + C_1L^{-1}\left\{\frac{1}{s^3}\right\} + C_2L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$\to L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}L^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{x^4}{24}$$

$$\to L^{-1}\left\{e^{-\frac{L}{2}s}\frac{1}{s^5}\right\} = U\left(x - \frac{L}{2}\right)G\left(x - \frac{L}{2}\right) = U\left(x - \frac{L}{2}\right)\frac{\left(x - \frac{L}{2}\right)^4}{24}$$

$$\to L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{3!}L^{-1}\left\{\frac{2}{s^3}\right\} = \frac{x^2}{2}$$

$$\to L^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}L^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{x^3}{6}$$

$$\Rightarrow y(x) = \frac{\omega_0}{EI}\left[\frac{x^4}{24} - \frac{\left(x - \frac{L}{2}\right)^4}{24}U\left(x - \frac{L}{2}\right)\right] + C_1\frac{x^2}{2} + C_2\frac{x^3}{6}$$

Evaluando en L:

$$y(L) = 0 = \frac{\omega_0}{EI}\left[\frac{L^4}{24} - \frac{L^4}{16\cdot 24}\right] + C_1\frac{L^2}{2} + C_2\frac{L^3}{6}$$

$$0 = \frac{\omega_0}{EI}\left[\frac{L^2}{12} - \frac{L^2}{16 \cdot 12}\right] + C_1 + \frac{C_2 L}{3}$$

$$0 = \frac{5}{64}\frac{\omega_0}{EI}L^2 + C_1 + \frac{C_2 L}{3}$$

$$y'(x) = \frac{\omega_0}{EI}\left(\frac{x^3}{6} - \frac{1}{24}\left[4\left(x - \frac{L}{2}\right)^3 U\left(x - \frac{L}{2}\right) + \left(x - \frac{L}{2}\right)^4 \delta\left(\frac{L}{2} - x\right)\right]\right) + C_1 x + C_2\frac{x^2}{2}$$

$$y''(x) = \frac{\omega_0}{EI}\left(\frac{x^2}{2} - \frac{1}{24}\left[12\left(x - \frac{L}{2}\right)^2 U\left(x - \frac{L}{2}\right) + 4\left(x - \frac{L}{2}\right)^3 \delta\left(\frac{L}{2} - x\right)\right]\right)$$

$$+\frac{\omega_0}{EI}\left(\frac{x^2}{2} - \frac{1}{24}\left[4\left(x - \frac{L}{2}\right)^3 \delta\left(\frac{L}{2} - x\right) + \left(x - \frac{L}{2}\right)^4 \left(\delta\left(\frac{L}{2} - x\right)\right)'\right]\right)$$

$$+C_1 x + C_2\frac{x^2}{2}$$

$$y''(L) = \frac{\omega_0}{EI}\left(\frac{L^2}{2} - \frac{1}{24}\left[12\left(\frac{L}{2}\right)^2\right]\right) + C_1 + C_2 L = 0$$

OJO Considere que:

$$\delta\left(\frac{L}{2} - x\right) = 0 \qquad para \quad X = L$$

Se comportan igual:

$$\left(\delta\left(\frac{L}{2} - x\right)\right)' = 0 \qquad para \quad X = L$$

$$\left(U\left(x - \frac{L}{2}\right)\right)' = \delta\left(\frac{L}{2} - x\right)$$

$$\Rightarrow \frac{3}{8}\frac{\omega_0}{EI}L^2 + C_1 + C_2 L = 0$$

$$\Rightarrow \frac{5}{64}\frac{\omega_0}{EI}L^2 + C_1 + C_2\frac{L}{3} = 0$$

$$\frac{19}{64}\frac{\omega_0}{EI}L^2 + \frac{2}{3}C_2 L = 0$$

$$-\frac{57}{128}\frac{\omega_0}{EI}L = C_2$$

$$C_1 = \frac{57}{128}\frac{\omega_0}{EI}L^2 - \frac{3}{8}\frac{\omega_0}{EI}L^2$$

$$C_1 = \frac{9}{128}\frac{\omega_0}{EI}L^2$$

$$\Rightarrow y(x) = \frac{\omega_0}{EI}\left[\frac{x^4}{24} - \frac{\left(x-\frac{L}{2}\right)^2}{24}U\left(x-\frac{L}{2}\right) + \frac{9}{256}L^2x^2 - \frac{19}{256}Lx^3\right]$$

(ii)

$$y'' - 7y' + 6y = e' + \delta(t-10\pi) + \delta(t-20\pi) \qquad /L$$

$$L(y'') - 7L(y') + 6L(y) = L(e') + L\left(\delta(t-10\pi)\right) + L\left(\delta(t-20\pi)\right)$$

Sea $F(s) = L(y)$

$$s^2F(s) + sy(0) + y'(0) - 7sF(s) - 7y(0) + 6F(s) = \frac{1}{s-1} + e^{-10\pi s} + e^{-20\pi s}$$

$$F(s)(s-1)(s-6) = \frac{1}{s-1} + e^{-10\pi s} + e^{-20\pi s}$$

$$F(s) = \frac{1}{(s-1)^2(s-6)} + \frac{e^{-10\pi s}}{(s-1)(s-6)} + \frac{e^{-20\pi s}}{(s-1)(s-6)} \qquad /L^{-1}()$$

Fracciones Parciales :

$$\frac{1}{(s-1)^2(s-6)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-6}$$

$$A(s^2+7s+6) + B(s-6) + C(s^2-2s+1) = 1$$

$$\begin{array}{ccc} A + C = 0 & \rightarrow & A = -C \\ -7A + B - 2C = 0 & \rightarrow & 5A = B \\ 6A - 6B + C = 1 & \rightarrow & 6A - 30A - A = 1 \end{array}$$

$$A = -\frac{1}{25} \longrightarrow B = -\frac{1}{5} \longrightarrow C = \frac{1}{25}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s-1)^2(s-6)}\right\} = -\frac{1}{25}L^{-1}\left\{\frac{1}{(s-1)}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{25}L^{-1}\left\{\frac{1}{s-6}\right\}$$

$$= -\frac{1}{25}e^{t} - \frac{1}{5}te^{t} + \frac{1}{25}e^{6t}$$

Fracciones Parciales :

$$\frac{1}{(s-1)(s-6)} = \frac{A}{s-1} + \frac{B}{s-6}$$

$$A = -\frac{1}{5} \longrightarrow B = \frac{1}{5}$$

$$L^{-1}\left\{\frac{1}{(s-1)(s-6)}\right\} = -\frac{1}{5}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{5}L^{-1}\left\{\frac{1}{s-6}\right\}$$

$$= -\frac{1}{5}e^{t} + \frac{1}{5}e^{6t}$$

$$L^{-1}\left\{\frac{e^{-10\pi s}}{(s-1)(s-6)}\right\} = -\frac{1}{5}L^{-1}\left\{\frac{e^{-10\pi s}}{s-1}\right\} + \frac{1}{5}L^{-1}\left\{\frac{e^{-10\pi s}}{s-6}\right\}$$

$$= -\frac{1}{5}U(t-10\pi)e^{t-10\pi} + \frac{1}{5}U(t-10\pi)e^{6(t-10\pi)}$$

$$L^{-1}\left\{\frac{e^{-20\pi s}}{(s-1)(s-6)}\right\} = -\frac{1}{5}L^{-1}\left\{\frac{e^{-20\pi s}}{s-1}\right\} + \frac{1}{5}L^{-1}\left\{\frac{e^{-20\pi s}}{s-6}\right\}$$

$$= -\frac{1}{5}U(t-20\pi)e^{t-20\pi} + \frac{1}{5}U(t-20\pi)e^{6(t-20\pi)}$$

$$y(t) = e^{t}\left[-\frac{1}{5}t - \frac{1}{25}\right] + \frac{1}{25}e^{6t} + \frac{1}{5}U(t-10\pi)\left[e^{6(t-10\pi)} - e^{t-10\pi}\right] + \frac{1}{5}U(t-20\pi)\left[e^{6(t-20\pi)} - e^{t-20\pi}\right]$$

1.18. Resuelva por medio de la transformada de Laplace el siguiente problema con condición:

$$y'' - 4y' + 3y = 1 - U(t-2) - U(t-4) + U(t-6)$$
$$y(0) = 0 \quad , \quad y'(0) = 0$$

Solución:

$$y'' - 4y' + 3y = 1 - U(t-2) - U(t-4) + U(t-6) \qquad /L()$$

$$L(y'') - 4L(y') + 3L(y) = L(1) - L(U(t-2)) - L(U(t-4)) + L(U(t-6))$$

Sea $F(s) = L(y)$

$$s^2F(s) - sy(0) - y'(0) + 4(sF(s) - y(0)) + 3F(s) = \frac{1}{s} - \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-6s}}{s}$$

$$F(s)\left[s^2 + 4s + 3\right] = \frac{1}{s}\left[1 - e^{-2s} - e^{-4s} + e^{-6s}\right]$$

Calculemos

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s+3)} \cdot \frac{1}{(s+1)}\right\}$$

$$AL^{-1}\left\{\frac{1}{s}\right\} + BL^{-1}\left\{\frac{1}{(s+3)}\right\} + CL^{-1}\left\{\frac{1}{(s+1)}\right\}$$

$$A + Be^{-3t} + Ce^{-t}$$

$$A(s+3)(s+1) + Bs(s+1) + Cs(s+3) = 1$$

$$A = \frac{1}{3} \longrightarrow B = \frac{1}{6} \longrightarrow C = -\frac{1}{2}$$

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s+3)} \cdot \frac{1}{(s+1)}\right\} = \frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-2s}}{s(s+3)(s+1)}\right\} = F(t-2)U(t-2) == -\left[\frac{1}{3} + \frac{1}{6}e^{-3(t-2)} - \frac{1}{2}e^{-(t-2)}\right]U(t-2)$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-4s}}{s(s+3)(s+1)}\right\} = -\left[\frac{1}{3}+\frac{1}{6}e^{-3(t-4)}-\frac{1}{2}e^{-(t-4)}\right]U(t-4)$$
$$\Rightarrow L^{-1}\left\{\frac{e^{-6s}}{s(s+3)(s+1)}\right\} = \left[\frac{1}{3}+\frac{1}{6}e^{-3(t-6)}-\frac{1}{2}e^{-(t-6)}\right]U(t-6)$$

1.19. Resuelva la siguiente edo:

$$y''+4y'+13y=\delta(x-\pi)+\delta(x-3\pi)$$
$$y(0)=1 \quad , \quad y'(0)=0$$

Solución:

$$s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+13Y(s)=e^{-\pi s}+e^{-3\pi s}$$

$$Y(s)=\frac{(s+2)}{(s+2)^2+9}+\frac{2}{3}\cdot\frac{3}{(s+2)^2+9}+\frac{1}{3}\cdot\frac{3}{(s+2)^2+9}\left(e^{-\pi s}+e^{-3\pi s}\right)$$

$$y(t)=e^{-2t}\cos 3t+\frac{2}{3}\cdot e^{-2t}\sin 3t+\frac{1}{3}\left(U(t-\pi)\cdot e^{-2(t-\pi)}\sin(3(t-\pi))+U(t-3\pi)\cdot\sin(3(t-3\pi))\right)$$

1.20. Se muestra en la figura un modelo no lineal de una caldera.

Usted podra hacer variar la seal de entrada $g(t)$ (seal de combustible) con el fin de mantener dentro de ciertos rangos de operación a la variable de salida $h(t)$ (presión del vapor sobrecalentado).

Una buena aproximación del modelo esta dada por:

$$\frac{dr}{dt}=a\cdot m(t)-b\cdot r(t)+b\cdot h(t)$$
$$\frac{dh}{dt}=c\cdot r(t)-d\cdot h(t)$$
$$\frac{dm}{dt}=e\cdot q(t)-e\cdot m(t)$$
$$\frac{dq}{dt}=-f\cdot q(t)+f\cdot g(t)$$

a , b , c , d , e y f son constantes positivas. $q(t)$es el calor del horno, $m(t)$ es el flujo masico de vapor generado y $r(t)$ es la presión del vapor saturado.

(a) Encuentre la función de trasferencia $W(s) = \frac{H(s)}{G(s)}$.

(Observación: las condiciones iniciales son cero).

(b) Si se aplica $g(t) = 4U(t)$, obtenga una salida del sistema $h(t)$.

Dato: $a = b = c = d = e = f = 1$.

Solución:

(a)

(1) $SR(s) = aM(s) - bR(s) + bH(s)$
(2) $SH(s) = cR(s) - dH(s)$
(3) $SM(s) = eQ(s) - eM(s)$
(4) $SQ(s) = -fQ(s) + fG(s)$

(4)

$$\Rightarrow Q(s) = \frac{fG(s)}{s+f}$$

(3)

$$\Rightarrow M(s) = \frac{eQ(s)}{s+e} = \frac{e \cdot f \cdot G(s)}{(s+e)(s+f)}$$

(1)

$$\Rightarrow R(s) = \frac{a \cdot e \cdot f \cdot G(s)}{(s+b)(s+e)(s+f)} + \frac{bH(s)}{s+b}$$

$$\Rightarrow \frac{H(s)}{G(s)} = \frac{a \cdot e \cdot f \cdot c}{(s+e)(s+f)((s+d)(s+b) - c \cdot b)}$$

(b)

$$g(t) = 4U(t) \Rightarrow G(s) = \frac{4}{s}$$

$$H(s) = \frac{G(s)}{(s+1)^2(s^2+2s)} = \frac{4}{(s+1)^2(s+2)s^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} + \frac{D}{s} + \frac{E}{s^2}$$

$$H(s) = \frac{4}{s+1} + \frac{4}{(s+1)^2} + \frac{1}{s+2} + \frac{-5}{s} + \frac{2}{s^2}$$

$$h(t) = 4(1+t)e^{-t} + e^{-2t} + 2t - 5$$

1.21. Encuentre la solución del problema con condiciones iniciales nulas $x(0) = 0 \quad , \quad x'(0) = 0$.

$$x^n + k^2 x = k \sum_{n=0}^{\infty} (-1)^n \delta\left(t - \frac{n\pi}{k}\right)$$

A continuación, encontrar la solución si $t \in \left[\frac{(n-1)\pi}{k}, \frac{n\pi}{k}\right]$

Solución:

$$x(s) = \sum_{n=0}^{\infty} (-1)^n \frac{k}{s^2+k^2} \cdot e^{-\frac{n\pi}{k}s}$$

$$x(t) = \sum_{n=0}^{\infty} (-1)^n \cdot U\left(t - \frac{n\pi}{k}\right) \sin\left(k\left(t - \frac{n\pi}{k}\right)\right)$$

si $t \in \left[\frac{(i-1)\pi}{k}, \frac{j\pi}{k}\right] \Rightarrow U\left(t - \frac{n\pi}{k}\right) = 1 \quad , \quad n = 0, 1, \ldots, j-1$

$$U\left(t - \frac{n\pi}{k}\right) = 0 \quad , \quad n = j, \ldots, \infty$$

$$\Rightarrow x(t) = \sum_{n=0}^{j-1} \sin(kt - n\pi)$$

1.22. Considere la ecuación diferencial:.

$$ay' + y = b \cdot f(t-c)$$
$$y(0) = 0$$

$a > 0 \quad , b > 0 \quad , c > 0 \quad , c \ll 1$ constantes

(a) Sea $f(t) = \delta(t)$ la función delta de Dirac. Encuentre $y(t)$ y grafique.

(b) Sea $f(t) = \sum_{i=0}^{\infty} \delta(t-i)$. Encuentre $y(t)$ y grafique.

(c) Para la solución de la parte (b) evalúe $y(N+c)$ con $N \in N$. Indique las condiciones necesarias para que $y(N+c)$ converja cuando $N \to \infty$, y demuestre que cuando converje:

$$\lim_{N \to \infty} y(N+c) = b\left(1 + \frac{1}{e^{\frac{1}{a}} - 1}\right)$$

(d) Sea $f(t) = U(t)$ el escalón unitario. Encuentre $y(t)$ y bosqueje.

(e) Sea $f(t) = \sum_{i=0}^{\infty} U(t-i)$. Encuentre $y(t)$ y bosqueje.

Solución:

(a)

$$ay' + y = b \cdot f(t-c) \quad , \quad y(0) = 0 \quad \Rightarrow Y(s) = \frac{be^{-cs}}{as+1}F(s)$$

$$f(t) = \delta(t) \Rightarrow F(s) = 1$$

$$Y(s) = \frac{be^{-cs}}{as+1} = \frac{\frac{b}{a}e^{-cs}}{s+\frac{1}{a}} \Rightarrow y(t) = \frac{b}{a} \cdot e^{-\frac{1}{a}(t-c)}U(t-c)$$

pues

$$L^{-1}\left\{\frac{b}{a} \cdot \frac{1}{s+\frac{1}{a}}\right\} = \frac{b}{a} \cdot e^{-\frac{1}{a}t} \Rightarrow L^{-1}\left\{\frac{b}{a} \cdot \frac{1}{s+\frac{1}{a} \cdot e^{-cs}}\right\} = \frac{b}{a} \cdot e^{-\frac{1}{a}(t-c)}U(t-c)$$

(b)

$$f(t)=\sum_{i=0}^{\infty}\delta(t-i)\Rightarrow F(s)=\sum_{i=0}^{\infty}e^{-is}$$

$$Y(s)=\sum_{i=0}^{\infty}\frac{b\cdot e^{-(i+c)s}}{as+1}\Rightarrow y(t)=\sum_{i=0}^{\infty}\frac{b}{a}\cdot e^{-\frac{1}{a}(t-i-c)}\cdot U(t-i-c)$$

(c)

$$y(N+c)=\sum_{i=0}^{\infty}\frac{b}{a}\cdot e^{-\frac{1}{a}(N-i)}\cdot U(N-i)=\sum_{i=0}^{N}\frac{b}{a}\cdot e^{-\frac{1}{a}(N-i)}=\frac{b}{a}\cdot e^{-\frac{N}{a}}\sum_{i=0}^{N}\left(e^{\frac{1}{a}}\right)^{i}$$

$$y(N+c)=\frac{b}{a}\cdot e^{-\frac{N}{a}}\left(\frac{1-e^{\frac{(N+1)}{a}}}{1-e^{\frac{1}{a}}}\right)=\frac{b}{a}\cdot\left(\frac{e^{-\frac{N}{a}}-e^{\frac{1}{a}}}{1-e^{\frac{1}{a}}}\right)$$

$$\lim_{N\to\infty}y(N+c)=\frac{b}{a}\cdot\left(\frac{\lim_{N\to\infty}(e^{-\frac{1}{a}})^{N}-e^{\frac{1}{a}}}{1-e^{\frac{1}{a}}}\right)=\frac{b}{a}\cdot\frac{e^{\frac{1}{a}}}{e^{\frac{1}{a}}-1}\Leftrightarrow e^{\frac{1}{a}}<1 \qquad /Ln$$

$$-\frac{1}{a}<0\Rightarrow a>0$$

Dado que $a>0\Rightarrow$ siempre converje.

(d)

$$F(s)=\frac{1}{s}$$

$$f(t)=U(t)$$

$$\Rightarrow Y(s)=\frac{be^{-cs}}{s(as+1)}=\left(\frac{b}{s}\cdot\frac{ab}{(as+1)}\right)e^{-cs}$$

$$L^{-1}\left\{\frac{b}{s(as+1)}\right\}=L^{-1}\left\{\frac{A}{s}+\cdot\frac{b}{as+1}\right\}=L^{-1}\left\{\frac{Aas+A+Bs}{s(as+1)}\right\}\Rightarrow A=b\rightarrow B=-ab$$

$$\left(A+\frac{B}{a}e^{-\frac{1}{a}}t\right)U(t)\Rightarrow y(t)=b\left(1-e^{-\frac{1}{a}(t-c)}\right)U(t)$$

(e)

$$f(t)=\sum_{i=0}^{\infty}U(t-i)\Rightarrow F(s)=\sum_{i=0}^{\infty}\frac{1}{s}\cdot e^{-is}$$

$$\Rightarrow Y(s)=\sum_{i=0}^{\infty}\frac{b\cdot e^{-(c+i)s}}{s(as+1)}\Rightarrow y(t)=\sum_{i=0}^{\infty}b\left(1-e^{-\frac{1}{a}(t-c-i)}\right)U(t-c-i)$$

1.23. La función de transferencia que caracteriza a un motor de corriente continua y que relaciona la frecuencia que gira el motor $y(t)$ y el voltaje de armadura $v(t)$ viene dada por:

$$H(s)=\frac{Y(s)}{V(s)}=\frac{k}{(R_a+L_a s)(Js+b)+k^2}$$

$$y(0)=y'(0)=0 \quad , \quad k=i_f G$$

i_f es la corriente de campo constante caracteristica del motor,J es la inercia del motor,b es el coeficiente de fricción viscosa y R_a, L_a resistencia e inductancia de armadura respectivamente.

Calcular $y(t)$ ocupando el teorema de convolución.

Solución:

$$Y(s)=H(s)\cdot V(s)\Rightarrow y(t)=\int_0^t v(\tau)h(t-\tau)d\tau$$

$$H(s)=\frac{Y(s)}{V(s)}=\frac{k}{JL_a s^2+(JR_a+bL_a)s+sR_a+k^2}$$

$$=\frac{\frac{k}{JL_a}}{s^2+\left(\frac{R_a}{L_a}+\frac{b}{J}\right)s+\frac{bR_a+k^2}{JL_a}}$$

$$a=\frac{k}{JL_a} \quad , \quad b=\left(\frac{R_a}{L_a}+\frac{b}{J}\right)\frac{1}{2} \quad , \quad c=\frac{bR_a+k^2}{JL_a}$$

$$H(s)=\frac{Y(s)}{V(s)}=\frac{a}{s^2+2bs+c}=\frac{a}{(s+b)^2+c-b^2}$$

Caso 1: $c - b^2 = 0$

$$h(t) = L^{-1}\left\{\frac{a}{(s+b)^2}\right\} = e^{-bt}L^{-1}\left\{\frac{a}{s^2}\right\} = a \cdot e^{-bt} \cdot t$$
$$\Rightarrow y(t) = \int_0^t v(\tau) a \cdot e^{-b(t-\tau)}(t-\tau)d\tau$$

Caso 2: $c - b^2 = w^2 > 0$

$$h(t) = L^{-1}\left\{\frac{a}{(s+b)^2 + w^2}\right\} = \frac{a}{w^2} \cdot e^{-bt} L^{-1}\left\{\frac{w}{s^2 + w^2}\right\} = \frac{a}{\sqrt{c-b^2}} \cdot e^{-bt} \sin\left(\sqrt{c-b^2} \cdot t\right)$$
$$\Rightarrow y(t) = \int_0^t v(\tau) \frac{a}{\sqrt{c-b^2}} \cdot e^{-b(t-\tau)} \sin\left(\sqrt{c-b^2} \cdot (t-\tau)\right) d\tau$$

Caso 3: $c - b^2 = -w^2 < 0$

$$h(t) = L^{-1}\left\{\frac{a}{(s+b)^2 - w^2}\right\} = a \cdot e^{-bt} L^{-1}\left\{\frac{1}{(s-w)(s+w)}\right\} = \frac{a \cdot e^{-bt}}{2w} \cdot L^{-1}\left\{\frac{1}{(s-w)} - \frac{1}{(s+w)}\right\}$$
$$= \frac{a \cdot e^{-bt}}{2\sqrt{b^2 - c}} \cdot \left(e^{wt} - e^{-wt}\right) = \frac{a \cdot e^{-bt}}{\sqrt{b^2 - c}} \cdot \cos h(\sqrt{b^2 - c} \cdot t)$$
$$\Rightarrow y(t) = \int_0^t v(\tau) \frac{a}{\sqrt{b^2 - c}} \cdot e^{-b(t-\tau)} \cdot \cos h\left(\sqrt{b^2 - c} \cdot (t-\tau)\right) d\tau$$