In this and the following three chapters, we focus on the minimum cost flow problem, introduced in Section 1.2:

$$\text{minimize} \quad \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i, \qquad \forall\, i \in \mathcal{N},$$

$$b_{ij} \le x_{ij} \le c_{ij}, \qquad \forall\, (i,j) \in \mathcal{A},$$

where a_{ij}, b_{ij}, c_{ij}, and s_i are given scalars.

We begin by discussing several equivalent ways to represent the problem. These are useful because different representations lend themselves better or worse for various analytical and computational purposes. We then develop duality theory and the associated optimality conditions. This theory is fundamental for the algorithms of the following three chapters, and richly enhances our insight into the problem's structure.

4.1 TRANSFORMATIONS AND EQUIVALENCES

In this section, we describe how the minimum cost flow problem can be represented in several equivalent "standard" forms. This is often useful, because depending on the analytical or algorithmic context, a particular representation may be more convenient than the others.

4.1.1 Setting the Lower Flow Bounds to Zero

The lower flow bounds b_{ij} can be changed to zero by a translation of variables, that is, by replacing x_{ij} by $x_{ij} - b_{ij}$, and by adjusting the upper flow bounds and the supplies according to

$$c_{ij} := c_{ij} - b_{ij},$$

$$s_i := s_i - \sum_{\{j|(i,j)\in\mathcal{A}\}} b_{ij} + \sum_{\{j|(j,i)\in\mathcal{A}\}} b_{ji}.$$

Optimal flows and the optimal value of the original problem are obtained by adding b_{ij} to the optimal flow of each arc (i,j) and adding $\sum_{(i,j)\in\mathcal{A}} a_{ij}b_{ij}$ to the optimal value of the transformed problem, respectively. Working with the transformed problem saves computation time and storage, and for this reason most network flow codes assume that all lower flow bounds are zero.

4.1.2 Eliminating the Upper Flow Bounds

Once the lower flow bounds have been changed to zero, it is possible to eliminate the upper flow bounds, obtaining a problem with just a nonnegativity constraint on all the flows. This can be done by introducing an additional nonnegative variable z_{ij} that must satisfy the constraint

$$x_{ij} + z_{ij} = c_{ij}.$$

(In linear programming terminology, z_{ij} is known as a *slack variable*.) The resulting problem is a minimum cost flow problem involving for each arc (i,j), an extra node with supply c_{ij}, and two outgoing arcs, corresponding to the flows x_{ij} and z_{ij}; see Fig. 4.1.

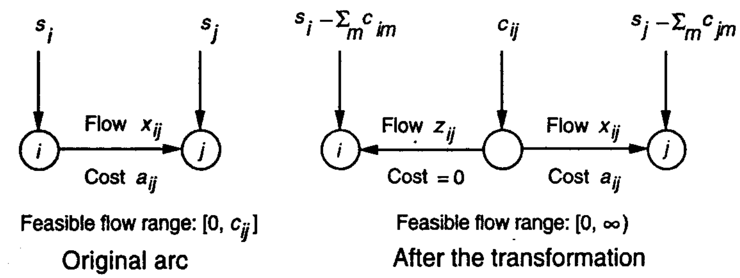

Figure 4.1: Eliminating the upper capacity bound by replacing each arc with a node and two outgoing arcs. Since for feasibility we must have $z_{ij} = c_{ij} - x_{ij}$, the upper bound constraint $x_{ij} \le c_{ij}$ is equivalent to the lower bound constraint $0 \le z_{ij}$. Furthermore, in view again of $x_{ij} = c_{ij} - z_{ij}$, the conservation of flow equation

$$-\sum_j z_{ij} - \sum_j x_{ji} = s_i - \sum_j c_{ij}$$

for the modified problem is equivalent to the conservation of flow equation

$$\sum_j x_{ij} - \sum_j x_{ji} = s_i$$

for the original problem. Using these facts, it can be seen that the feasible flow vectors (x,z) of the modified problem can be paired on a one-to-one basis with the feasible flow vectors x of the original problem, and that the corresponding costs are equal. Thus, the modified problem is equivalent to the original problem.

Eliminating the upper flow bounds simplifies the statement of the problem, but complicates the use of some algorithms. The reason is that problems with upper (as well as lower) flow bounds are guaranteed to have at least one optimal solution if they have at least one feasible solution, as we will see in Chapter 5. However, a problem with just nonnegativity constraints may be *unbounded*, in the sense that it may have feasible solutions of arbitrarily small cost. This is one reason why most network flow codes require that upper and lower bound restrictions be placed on all the flow variables.

4.1.3 Reduction to a Circulation Format

The problem can be transformed into the *circulation format*, in which all node supplies are zero. One way to do this is to introduce an artificial "accumulation" node t and an arc (t,i) for each node i with nonzero supply s_i. We may then introduce the constraint $s_i \le x_{ti} \le s_i$ and an arbitrary cost for the flow x_{ti}. Alternatively, we may introduce an arc (t,i) and a constraint $0 \le x_{ti} \le s_i$ for all i with $s_i > 0$, and an arc (i,t) and a constraint $0 \le x_{it} \le -s_i$ for all i with $s_i < 0$. The cost of these arcs should be very small (i.e., large negative) to force the corresponding flows to be at their upper bound at the optimum; see Fig. 4.2.

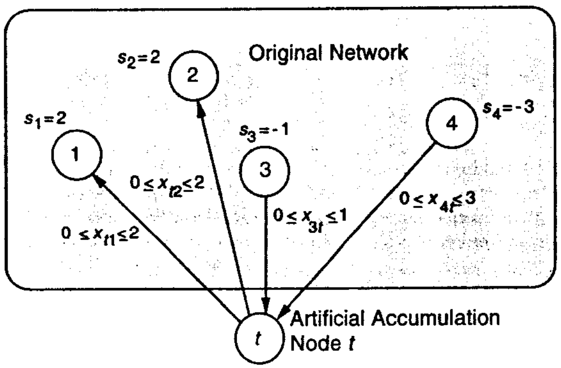

Figure 4.2: A transformation of the minimum cost flow problem into a circulation format by using an artificial "accumulation" node t and corresponding artificial arcs connecting t with all the nodes as shown. These arcs have very large negative cost, to force the corresponding flows to their upper bounds at the optimum.

4.1.4 Reduction to an Assignment Problem

Finally, the minimum cost flow problem may be transformed into a transportation problem of the form

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{\{j\mid(i,j)\in\mathcal{A}\}} x_{ij} = \alpha_i, \qquad \forall\, i = 1,\ldots,m, \\
& \sum_{\{i\mid(i,j)\in\mathcal{A}\}} x_{ij} = \beta_j, \qquad \forall\, j = 1,\ldots,n, \\
& 0 \le x_{ij} \le \min\{\alpha_i,\beta_j\}, \qquad \forall\,(i,j)\in\mathcal{A};
\end{aligned}$$

see Fig. 4.3. This transportation problem can itself be converted into an assignment problem by creating α_i unit supply sources (β_j unit demand sinks) for each transportation problem source i (sink j, respectively). For this reason, any algorithm that solves the assignment problem can be extended into an algorithm for the minimum cost flow problem. This motivates a useful way to develop and analyze new algorithmic ideas; apply them to the simpler assignment problem and generalize them using the construction just given to the minimum cost flow problem.

4.2 DUALITY

We have already introduced some preliminary duality ideas in the context of the assignment problem in Section 1.3.2. In this section, we consider the general minimum cost flow problem, and we obtain a dual problem using a procedure that is standard in duality theory. We introduce a Lagrange multiplier, also called a price p_i for the conservation of flow constraint for node i and we form the corresponding Lagrangian function

$$\begin{aligned}
L(x,p) &= \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} + \sum_{i\in\mathcal{N}} \left(s_i - \sum_{\{j\mid(i,j)\in\mathcal{A}\}} x_{ij} + \sum_{\{j\mid(j,i)\in\mathcal{A}\}} x_{ji} \right) p_i \\
&= \sum_{(i,j)\in\mathcal{A}} (a_{ij} + p_j - p_i)x_{ij} + \sum_{i\in\mathcal{N}} s_i p_i. \qquad (4.1)
\end{aligned}$$

Here, we use p to denote the vector whose components are the prices p_i.

Let us now fix p and consider minimizing $L(x,p)$ with respect to x *without the requirement to meet the conservation of flow constraints.* It is seen that p_i may be viewed as a penalty per unit violation of the conservation of flow constraint. If p_i is too small (or too large), there is an incentive for positive (or negative, respectively) violation of the constraint. This suggests that we should search for the correct values p_i for which,

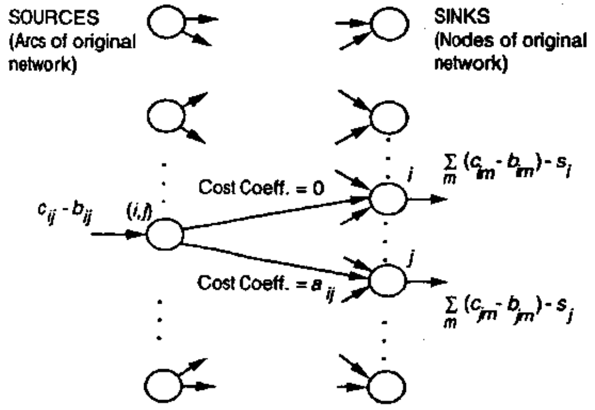

Figure 4.3: Transformation of a minimum cost flow problem into a transportation problem. The idea is to introduce a new node for each arc and introduce a slack variable for every arc flow; see Fig. 4.1. This not only eliminates the upper bound constraint on the arc flows, as in Fig. 4.1, but also creates a bipartite graph structure. In particular, we take as sources of the transportation problem the arcs of the original network, and as sinks of the transportation problem the nodes of the original network. Each transportation problem source has two outgoing arcs with cost coefficients as shown. The supply of each transportation problem source is the feasible flow range length of the corresponding original network arc. The demand of each transportation problem sink is the sum of the feasible flow range lengths of the outgoing arcs from the corresponding original network node minus the supply of that node, as shown. An arc flow x_{ij} in the minimum cost flow problem corresponds to flows equal to x_{ij} and $c_{ij} - b_{ij} - x_{ij}$ on the transportation problem arcs $\big((i,j),j\big)$ and $\big((i,j),i\big)$, respectively.

when $L(x,p)$ is minimized over all capacity-feasible x, there is no incentive for either positive or negative violation of all the constraints.

We are thus motivated to introduce the dual function value $q(p)$ at a vector p, defined by

$$q(p) = \min_x \big\{ L(x,p) \mid b_{ij} \le x_{ij} \le c_{ij}, (i,j) \in \mathcal{A} \big\}. \tag{4.2}$$

Because the Lagrangian function $L(x,p)$ is separable in the arc flows x_{ij}, its minimization decomposes into a separate minimization for each arc (i,j). Each of these minimizations can be carried out in closed form, yielding

$$q(p) = \sum_{(i,j)\in\mathcal{A}} q_{ij}(p_i - p_j) + \sum_{i\in\mathcal{N}} s_i p_i, \tag{4.3}$$

where

$$\begin{aligned} q_{ij}(p_i - p_j) &= \min_{b_{ij} \le x_{ij} \le c_{ij}} (a_{ij} + p_j - p_i) x_{ij} \\ &= \begin{cases} (a_{ij} + p_j - p_i) b_{ij} & \text{if } p_i \le a_{ij} + p_j, \\ (a_{ij} + p_j - p_i) c_{ij} & \text{if } p_i > a_{ij} + p_j. \end{cases} \end{aligned} \tag{4.4}$$

Consider now the problem

maximize $q(p)$

subject to no constraint on p,

where q is the dual function given by Eqs. (4.3) and (4.4). We call this the *dual problem*, and we refer to the original minimum cost flow problem as the *primal problem*. We also refer to the dual function as the *dual cost function* or *dual cost*, and we refer to the optimal value of the dual problem as the *optimal dual cost*.† We will see that solving the dual problem provides the correct values of the prices p_i, which will allow the optimal flows to be obtained by minimizing $L(x,p)$.

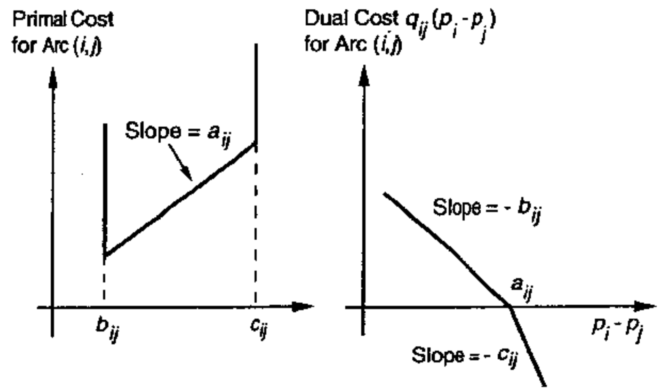

Figure 4.4: Form of the dual cost function q_{ij} for arc (i,j).

Figure 4.4 illustrates the form of the functions q_{ij}. Since each q_{ij} is piecewise linear, the dual function q is also piecewise linear. The dual function also has some additional interesting structure. In particular, suppose

† There is a slight abuse of terminology here, since in a dual context we are not minimizing a cost but rather maximizing a value, but there is some uniformity advantage in referring to cost in both the primal and the dual context. Besides, some problems such as the assignment problem in Section 1.3, are cast as maximization problems and their duals become minimization problems, so using the term "dual value" rather than "dual cost" would be inappropriate.

that all node prices are changed by the same amount. Then the values of the functions q_{ij} do not change, since these functions depend on the price differences $p_i - p_j$. If in addition we have $\sum_{i\in\mathcal{N}} s_i = 0$, as we must if the problem is feasible, we see that the term $\sum_{i\in\mathcal{N}} s_i p_i$ also does not change. Thus, the dual function value does not change when all node prices are changed by the same amount, implying that the equal cost surfaces of the dual cost function are unbounded. Figure 4.5 illustrates the dual function for a simple example.

We now turn to the development of the basic duality results for the minimum cost flow problem. To this end we appropriately generalize the notion of complementary slackness, introduced in Section 1.3 within the context of the assignment problem:

Definition 4.1: We say that a flow-price vector pair (x, p) satisfies *complementary slackness* (or CS for short) if x is capacity-feasible and

$$p_i - p_j \le a_{ij}, \quad \forall\, (i,j) \in \mathcal{A} \text{ with } x_{ij} < c_{ij}, \tag{4.5}$$

$$p_i - p_j \ge a_{ij}, \quad \forall\, (i,j) \in \mathcal{A} \text{ with } b_{ij} < x_{ij}. \tag{4.6}$$

Note that the CS conditions imply that

$$p_i = a_{ij} + p_j, \quad \forall\, (i,j) \in \mathcal{A} \text{ with } b_{ij} < x_{ij} < c_{ij}.$$

An equivalent way to write the CS conditions is that, for all arcs (i, j), we have $b_{ij} \le x_{ij} \le c_{ij}$ and

$$x_{ij} = \begin{cases} c_{ij} & \text{if } p_i > a_{ij} + p_j, \\ b_{ij} & \text{if } p_i < a_{ij} + p_j. \end{cases}$$

Another equivalent way to state the CS conditions is that x_{ij} attains the minimum in the definition of q_{ij}

$$x_{ij} = \arg \min_{b_{ij} \le z_{ij} \le c_{ij}} (a_{ij} + p_j - p_i) z_{ij} \tag{4.7}$$

for all arcs (i, j). Figure 4.6 provides a graphical interpretation of the CS conditions.

The following proposition is an important duality theorem, and will later form the basis for developing a more complete duality analysis with the aid of the simplex-related algorithmic developments of Chapter 5.

Proposition 4.1: A feasible flow vector x^* and a price vector p^* satisfy CS if and only if x^* and p^* are optimal primal and dual solutions, respectively, and the optimal primal and dual costs are equal.

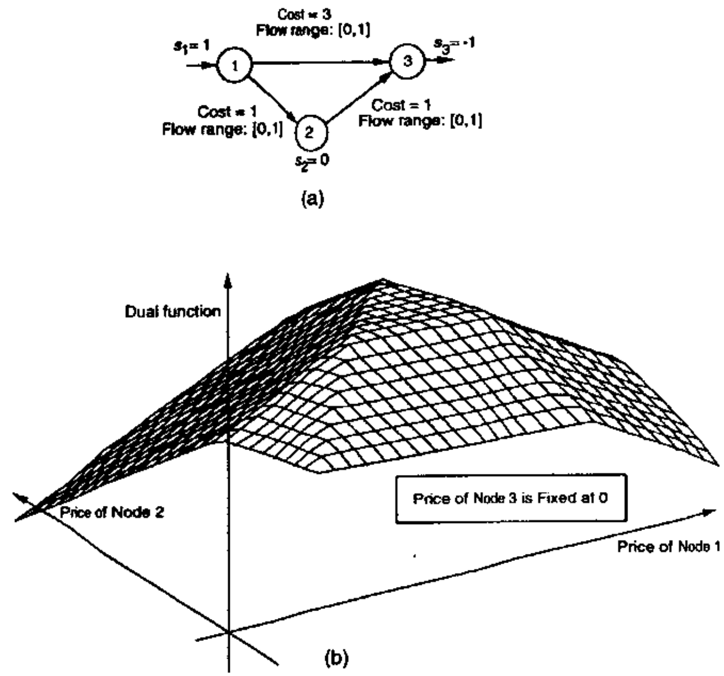

Figure 4.5: Form of the dual cost function q for the 3-node problem in (a). The optimal flow is $x_{12} = 1$, $x_{23} = 1$, $x_{13} = 0$. The dual function is

$$q(p_1, p_2, p_3) = \min\{0, 1 + p_2 - p_1\} + \min\{0, 1 + p_3 - p_2\}$$
$$+ \min\{0, 3 + p_3 - p_1\} + p_1 - p_3.$$

Diagram (b) shows the graph of the dual function in the space of p_1 and p_2, with p_3 fixed at 0. For a different value of p_3, say γ, the graph is "translated" by the vector (γ, γ); that is, we have $q(p_1, p_2, 0) = q(p_1 + \gamma, p_2 + \gamma, \gamma)$ for all (p_1, p_2). The dual function is maximized at the vectors p that satisfy CS together with the optimal x. These are the vectors of the form $(p_1 + \gamma, p_2 + \gamma, \gamma)$, where

$$1 \le p_1 - p_2, \qquad p_1 \le 3, \qquad 1 \le p_2.$$

Proof: We first show that for any feasible flow vector x and any price vector p, the primal cost of x is no less than the dual cost of p. Indeed, we

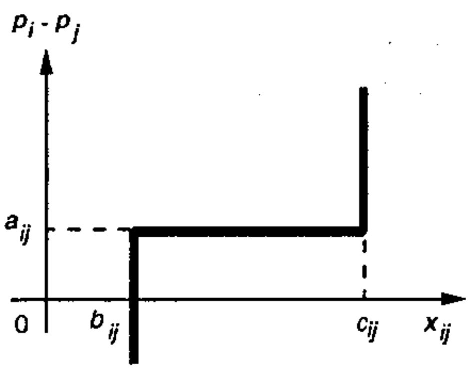

Figure 4.6: Illustration of CS for a flow-price pair (x,p). For each arc (i,j), the pair $(x_{ij}, p_i - p_j)$ should lie on the graph shown.

have from the definitions (4.1) and (4.2) of L and q, respectively,

$$\begin{aligned} q(p) &\le L(x,p) \\ &= \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} + \sum_{i\in\mathcal{N}} \left(s_i - \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} + \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} \right) p_i \qquad (4.8) \\ &= \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}, \end{aligned}$$

where the last equality follows from the feasibility of x.

If x^* is feasible and satisfies CS together with p^*, we have by the definition (4.2) of q

$$\begin{aligned} q(p^*) &= \min_x \{ L(x,p^*) \mid b_{ij} \le x_{ij} \le c_{ij}, (i,j) \in \mathcal{A} \} \\ &= L(x^*,p^*) \qquad (4.9) \\ &= \sum_{(i,j)\in\mathcal{A}} a_{ij}x^*_{ij}, \end{aligned}$$

where the second equality is true because

(x^*,p^*) satisfies CS if and only if

x^*_{ij} minimizes $(a_{ij} + p^*_j - p^*_i)x_{ij}$ over all $x_{ij} \in [b_{ij}, c_{ij}]$, $\forall\ (i,j) \in \mathcal{A}$,

[cf. Eq. (4.7)], and the last equality follows from the Lagrangian expression (4.1) and the feasibility of x^*. Therefore, Eq. (4.9) implies that x^* attains

the minimum of the primal cost on the right-hand side of Eq. (4.8), and p^* attains the maximum of $q(p)$ on the left-hand side of Eq. (4.8), while the optimal primal and dual values are equal.

Conversely, suppose that x^* and p^* are optimal primal and dual solutions, respectively, and the two optimal costs are equal, that is,

$$q(p^*) = \sum_{(i,j)\in\mathcal{A}} a_{ij}x^*_{ij}.$$

We have by definition

$$q(p^*) = \min_x \{ L(x,p^*) \mid b_{ij} \le x_{ij} \le c_{ij}, (i,j) \in \mathcal{A} \},$$

and also, using the Lagrangian expression (4.1) and the feasibility of x^*,

$$\sum_{(i,j)\in\mathcal{A}} a_{ij}x^*_{ij} = L(x^*,p^*).$$

Combining the last three equations, we obtain

$$L(x^*,p^*) = \min_x \{ L(x,p^*) \mid b_{ij} \le x_{ij} \le c_{ij}, (i,j) \in \mathcal{A} \}.$$

Using the Lagrangian expression (4.1), it follows that for all arcs (i,j), we have

$$x^*_{ij} = \arg \min_{b_{ij} \le x_{ij} \le c_{ij}} (a_{ij} + p^*_j - p^*_i)x_{ij}.$$

This is equivalent to the pair (x^*,p^*) satisfying CS. **Q.E.D.**

There are also several other important duality results. In particular, in Prop. 5.8 of Chapter 5 we will use a constructive algorithmic approach to show the following:

> **Proposition 4.2:** If the minimum cost flow problem (with upper and lower bounds on the arc flows) is feasible, then there exist optimal primal and dual solutions, and the optimal primal and dual costs are equal.

Proof: See Prop. 5.8 of Chapter 5. **Q.E.D.**

By combining Props. 4.1 and 4.2, we obtain the following variant of Prop. 4.1, which includes no statement on the equality of the optimal primal and dual costs:

Proposition 4.3: A feasible flow vector x^* and a price vector p^* satisfy CS if and only if x^* and p^* are optimal primal and dual solutions.

Proof: The forward statement is part of Prop. 4.1. The reverse statement, is obtained by using the equality of the optimal primal and dual costs (Prop. 4.2) and the reverse part of Prop. 4.1. **Q.E.D.**

4.2.1 Interpretation of CS and the Dual Problem

The CS conditions have a nice economic interpretation. In particular, think of each node i as choosing the flow x_{ij} of each of its outgoing arcs (i,j) from the range $[b_{ij}, c_{ij}]$, on the basis of the following economic considerations: For each unit of the flow x_{ij} that node i sends to node j along arc (i,j), node i must pay a transportation cost a_{ij} plus a storage cost p_j at node j; for each unit of the residual flow $c_{ij} - x_{ij}$ that node i does not send to j, node i must pay a storage cost p_i. Thus, the total cost to node j is $(a_{ij} + p_j)x_{ij} + (c_{ij} - x_{ij})p_i$, or

$$(a_{ij} + p_j - p_i)x_{ij} + c_{ij}p_i.$$

It can be seen that the CS conditions (4.5) and (4.6) are equivalent to requiring that node i act in its own best interest by selecting the flow that minimizes the corresponding costs for each of its outgoing arcs (i,j); that is,

(x,p) satisfies CS if and only if

$$x_{ij} \text{ minimizes } (a_{ij} + p_j - p_i)z_{ij} \text{ over all } z_{ij} \in [b_{ij}, c_{ij}], \quad \forall\ (i,j) \in \mathcal{A},$$

[cf. Eq. (4.7)].

To interpret the dual function $q(p)$, we continue to view a_{ij} and p_i as transportation and storage costs, respectively. Then, for a given price vector p and supply vector s, the dual function

$$q(p) = \min_{\substack{b_{ij} \le x_{ij} \le c_{ij} \\ (i,j)\in\mathcal{A}}} \left\{ \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij} + \sum_{i\in\mathcal{N}} \left(s_i - \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} + \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} \right) p_i \right\}$$

is the minimum total transportation and storage cost to be incurred by the nodes, by choosing flows that satisfy the capacity constraints.

Suppose now that we introduce an organization that sets the node prices, and collects the transportation and storage costs from the nodes. We see that if the organization wants to maximize its total revenue (given that the nodes will act in their own best interest), it must choose prices that solve the dual problem optimally.

4.2.2 Duality and CS for Nonnegativity Constraints

We finally note that there are variants of CS and Props. 4.1-4.3 for the versions of the minimum cost flow problem where $b_{ij} = -\infty$ and/or $c_{ij} = \infty$ for some arcs (i,j). In particular, in the case where in place of the capacity constraints $b_{ij} \le x_{ij} \le c_{ij}$, there are only nonnegativity constraints $0 \le x_{ij}$, the CS conditions take the form

$$p_i - p_j \le a_{ij}, \qquad \forall\ (i,j) \in \mathcal{A},$$

$$p_i - p_j = a_{ij}, \qquad \forall\ (i,j) \in \mathcal{A} \text{ with } 0 < x_{ij},$$

(see Fig. 4.7).

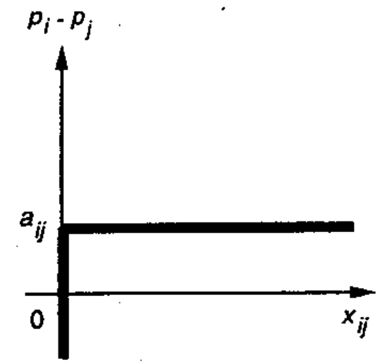

Figure 4.7: Illustration of CS for a flow-price pair (x,p) in the case of nonnegativity constraints $0 \le x_{ij}$ for the flow of each arc (i,j). The pair $(x_{ij}, p_i - p_j)$ should lie on the graph shown.

Some of the modifications needed to prove counterparts of the duality results for nonnegativity constraints are outlined in Exercise 4.3. In particular, Prop. 4.1 holds for this case as stated. However, showing a counterpart of Prop. 4.2 involves a slight complication. In the case of nonnegativity constraints, it is possible that there exist feasible flow vectors of arbitrarily small cost; a problem where this happens will be called *unbounded* in Chapter 5. Barring this possibility, the existence of primal and dual optimal solutions with equal cost (cf. Prop. 4.2) will be shown in Prop. 5.6 of Section 5.2.

4.3 NOTES, SOURCES, AND EXERCISES

The minimum cost flow problem was formulated in the early days of linear programming. There has been extensive research on the algorithmic solution of the problem, much of which will be the subject of the following three chapters. This research has followed two fairly distinct directions. On one hand there has been intensive development of practically efficient algorithms. These algorithms were originally motivated by general linear programming methods such as the primal simplex, dual simplex, and primal-dual methods, but gradually other methods, such as auction algorithms, were proposed, which have no general linear programming counterparts. The focus of research in these algorithms was to establish their validity through a proof of guaranteed termination, to analyze their special properties, and to establish their practical computational efficiency through experimentation with "standard" test problems.

On the other hand there have been efforts to explore the worst-case complexity limits of the minimum cost flow problem using polynomial algorithms. Edmonds and Karp [1972] developed the first polynomial algorithm, using a version of the out-of-kilter method (a variant of the primal-dual method to be discussed in Chapter 6) that employed cost and capacity scaling. Subsequently, in the late 70s, polynomial algorithms for the general linear programming problem started appearing, and these were of course applicable to the minimum cost flow problem. All of these polynomial algorithms are not strongly polynomial because their running time depends not just on the number of nodes and arcs, but also on the arc costs and capacities. A strongly polynomial algorithm for the minimum cost flow problem was given by Tardos [1985]. The existence of a strongly polynomial algorithm distinguishes the minimum cost flow problem from the general linear programming problem, for which there is no known algorithm with running time that depends only on the number of variables and constraints. However, a point made earlier in Section 1.3.4 should be repeated: a polynomial running time does not guarantee good practical performance. For example, Tardos' algorithm has not been seriously considered for algorithmic solution of practical minimum cost flow problems. Thus, to select an algorithm for a practical problem one must typically rely on criteria other than worst-case complexity.

Duality theory is of central importance in linear programming, and is similarly important in network optimization. It has its origins in the work of von Neuman on zero sum games, and was first formalized by Gale, Kuhn, and Tucker [1951]. Similar to linear programming, there are several possible dual problems, depending on which of the constraints are "dualized" (assigned a Lagrange multiplier). The duality theory of this chapter, where the conservation of flow constraints are dualized, is the most common and useful for the minimum cost flow problem. We will develop alternative forms of duality when we discuss other types of network optimization problems in Chapters 8-10.

We finally note that one can illustrate the relation between the primal and the dual problems in terms of an intuitive geometric interpetation (see Fig. 4.8). This interpretation is directed toward the advanced reader and will not be needed later. It demonstrates why the cost of any feasible flow vector is no less than the dual cost of any price vector (later, in Chapter 8, this will be called the *weak duality theorem*), and why thanks to the linearity of the cost function and the constraints, the optimal primal and dual costs are equal.

EXERCISES

4.1 (Reduction to One Source/One Sink Format)

Show how the minimum cost flow problem can be transformed to an equivalent problem where all node supplies are zero except for one node that has positive supply and one node that has negative supply.

4.2 (Duality for Assignment Problems)

Consider the assignment problem of Example 1.2. Derive the dual problem and the CS conditions, and show that they are mathematically equivalent to the ones introduced in Section 1.3.2.

4.3 (Duality for Nonnegativity Constraints)

Consider the version of the minimum cost flow problem where there are nonnegativity constraints

$$\text{minimize} \quad \sum_{(i,j)\in\mathcal{A}} a_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i, \qquad \forall\, i \in \mathcal{N},$$

$$0 \le x_{ij}, \qquad \forall\, (i,j) \in \mathcal{A}.$$

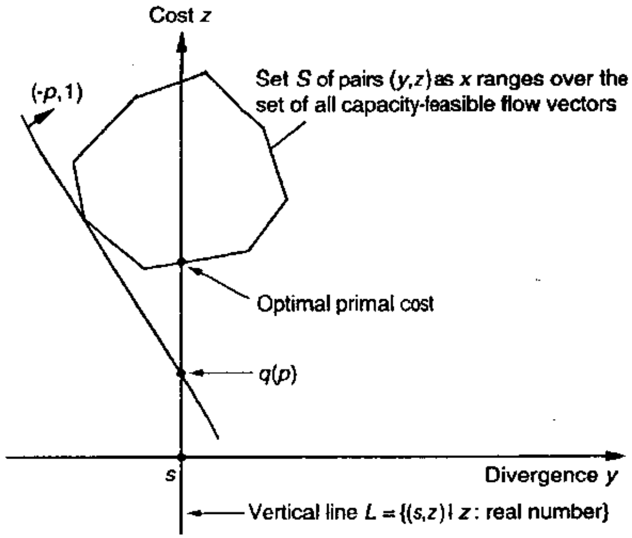

Figure 4.8: Geometric interpretation of duality for the reader who is familiar with the notion and the properties of hyperplanes in a vector space. Consider the (polyhedral) set S consisting of all pairs (y, z), where y is the divergence vector corresponding to x and z is the cost of x, as x ranges over all capacity-feasible flow vectors. Then feasible flow vectors correspond to common points of S and the vertical line

$$L = \{(s, z) \mid z : \text{ real number}\}.$$

The optimal primal cost corresponds to the lowest common point.

On the other hand, for a given price vector p, the dual cost $q(p)$ can be expressed as [cf. Eq. (4.2)]

$$q(p) = \min_{x:\ \text{capacity feasible}} L(x, p) = \min_{(y,z)\in S} \left\{ z - \sum_{i\in\mathcal{N}} y_i p_i \right\} + \sum_{i\in\mathcal{N}} s_i p_i.$$

Based on this expression, it can be seen that $q(p)$ corresponds to the intersection point of the vertical line L with the hyperplane

$$\left\{ (y, z) \,\middle|\, z - \sum_{i\in\mathcal{N}} y_i p_i = q(p) - \sum_{i\in\mathcal{N}} s_i p_i \right\},$$

which supports from below the set S, and is normal to the vector $(-p, 1)$. The dual problem is to find a price vector p for which the intersection point is as high as possible. The figure illustrates the equality of the lowest common point of S and L (optimal primal cost), and the highest point of intersection of L by a hyperplane that supports S from below (optimal dual cost).

Show that a feasible flow vector x^* and a price vector p^* satisfy the following CS conditions

$$p_i^* - p_j^* \le a_{ij}, \qquad \forall\ (i,j) \in \mathcal{A},$$
$$p_i^* - p_j^* = a_{ij}, \qquad \forall\ (i,j) \in \mathcal{A} \text{ with } 0 < x_{ij}^*,$$

if and only if x^* is primal optimal, p^* is an optimal solution of the following dual problem:

$$\text{maximize } \sum_{i\in\mathcal{N}} s_i p_i$$
$$\text{subject to } \ p_i - p_j \le a_{ij}, \qquad \forall\ (i,j) \in \mathcal{A},$$

and the optimal primal and dual costs are equal. *Hint*: Complete the details of the following argument. Define

$$q(p) = \begin{cases} \sum_{i\in\mathcal{N}} s_i p_i & \text{if } p_i - p_j \le a_{ij},\ \forall\ (i,j) \in \mathcal{A}, \\ -\infty & \text{otherwise,} \end{cases}$$

and note that

$$q(p) = \sum_{(i,j)\in\mathcal{A}} \min_{0\le x_{ij}} (a_{ij} + p_j - p_i) x_{ij} + \sum_{i\in\mathcal{N}} s_i p_i$$
$$= \min_{0\le x} \left\{ \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij} + \sum_{i\in\mathcal{N}} \left(s_i - \sum_{\{j\mid(i,j)\in\mathcal{A}\}} x_{ij} + \sum_{\{j\mid(j,i)\in\mathcal{A}\}} x_{ji} \right) p_i \right\}.$$

Thus, for any feasible x and any p, we have

$$q(p) \le \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij} + \sum_{i\in\mathcal{N}} \left(s_i - \sum_{\{j\mid(i,j)\in\mathcal{A}\}} x_{ij} + \sum_{\{j\mid(j,i)\in\mathcal{A}\}} x_{ji} \right) p_i \tag{4.10}$$
$$= \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij}.$$

On the other hand, we have

$$q(p^*) = \sum_{i\in\mathcal{N}} s_i p_i^* = \sum_{(i,j)\in\mathcal{A}} (a_{ij} + p_j^* - p_i^*) x_{ij}^* + \sum_{i\in\mathcal{N}} s_i p_i^* = \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij}^*,$$

where the second equality holds because the CS conditions imply that $(a_{ij} + p_j^* - p_i^*) x_{ij}^* = 0$ for all $(i,j) \in \mathcal{A}$, and the last equality follows from the feasibility of x^*. Therefore, x^* attains the minimum of the primal cost on the right-hand side of Eq. (4.10). Furthermore, p^* attains the maximum of $q(p)$ on the left-hand side of Eq. (4.10), which means that p^* is an optimal solution of the dual problem.