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USE OF LOAD-DEPENDENT VECTORS FOR DYNAMIC ANALYSIS OF LARGE SPACE STRUCTURES

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SUMMARY

Structural models of large space structures have a substantial number of degrees of freedom (DOF) and possess semi-positive-definite stiffness matrices. The paper presents an efficient co-ordinate reduction procedure for structural dynamic analysis of large space structures. The method is based on the superposition of load-dependent Ritz vectors, which are computed in block form using a shifted stiffness matrix. Comparative transient dynamic analyses are performed on a 2803 DOF model of the space station Freedom using the load-dependent method (LDM) and the mode-displacement method (MDM) based on the superposition of eigenvectors. It is shown that the LDM is able to provide convergence of displacements with a small number of vectors. The acceleration response is found to be more sensitive to vector truncation than the displacement response. Error norms based on the representation of the dynamic load by the vector basis are developed to provide an indication of the effect of vector truncation on the structural response.

INTRODUCTION

Finite-element dynamic analysis of complex structural systems such as the space shuttle and the planned space station Freedom requires detailed structural models with a large number of degrees of freedom (DOF). The standard procedure for performing a transient dynamic analysis of these models utilizes the mode-displacement method (MDM) based on the superposition of eigenvectors. For structural models with large numbers of DOF, the eigenvector basis is generally truncated because of the enormous computational effort and time required to calculate all eigenvectors and eigenvalues of the finite-element model. The constraint of having to use a truncated modal basis, and the fact that the computational effort to calculate vibration characteristics based on an 'exact' eigensolution is costly compared to Ritz-based methods gives motivation and justification for considering other procedures for generating an orthogonal vector basis suitable for dynamic response computations.[1]

A new method of dynamic analysis for structural systems subjected to fixed spatial distribution of the dynamic load was introduced by Wilson *et al.*[2,3] as an economic alternative to classical mode-superposition techniques. The 'load-dependent' method (LDM) is based on a transformation to a reduced system of generalized Ritz co-ordinates using load-dependent transformation vectors generated from the specified spatial distribution of the dynamic loads.

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These vectors can be generated at a fraction of the cost of eigenvectors, and directly include in the basis the static correction effects for the truncation of higher modes. Many computational variants and extensions of the LDM have been reported in the literature.[4-9] Arnold *et al.*[10] were the first to report the use of the LDM as proposed by Wilson *et al.*[2] for the dynamic analysis of 'large' structures. Numerical applications for the dynamic analysis of a 1000 DOF structural model of an optical laser tracking system indicated that the LDM 'modal' extraction time was approximately one tenth of the eigenvalue extraction procedure while maintaining the convergence of the displacement response. Similar conclusions were reached by Léger in the dynamic analysis of a 1944 DOF dam-foundation finite-element model.[8]

The rate of convergence of the LDM has been studied on smaller systems where reference solutions can be computed from complete eigenvectors bases.[6,7,9] Comparisons of forces and displacement responses were made between the LDM and the MDM, as well as with the MDM supplemented by a static correction to account for the truncation of higher modes in the form of what is known as the mode acceleration method (MAM).[11] It was shown that the LDM possesses convergence characteristics that are similar to those of the MAM in terms of the number of vectors to be included in the superposition to obtain accurate results. However, since the load-dependent (LD) vectors are much faster to generate than the 'exact' eigenvectors, the LDM is more efficient than the MAM, especially for the analysis of large structures.

The purpose of this paper is to present a comparative study of transient dynamic analysis of a 2803 DOF model of the space station Freedom using the LDM and the MDM. A shifted block form of the LDM is developed to deal effectively with semi-positive-definite stiffness matrices, closely spaced modes and loading histories with arbitrary spatial variations in time. Parametric analyses using a different number of vectors in a block, a different number of vectors in the summation, and different magnitude of the shift constant are used to study the sensitivity of the free-vibration response and the displacement response, as well as the acceleration response, to various modelling assumptions. It should be noted that the characteristics of the acceleration response based on the LDM have not been studied in the past. The performance of error norms based on the representation of the dynamic load by the vector bases are studied to provide an indication of the effect of vector truncation on the structural response.

ALGORITHM FOR GENERATION OF LOAD-DEPENDENT VECTORS

Shifted block form of the LDM

The algorithm for generating an orthogonal LD vector basis is based on using the static amplitudes of the dynamic loads at selected times. The vector basis generated aligns itself with the spatial distribution of the loads. Consequently the vector basis has the potential of a high participation with respect to the response to the dynamic loading. A summary of the algorithm is given in Appendix I. The LD vectors are generated in blocks, each block having several vectors, to deal with closely spaced vibration frequencies, and a spatial variation of the loading distribution in time that can often exist in space structures.[5,7-9] The algorithm requires the factorization of the stiffness matrix $\mathbf{K}$. Since space structures have rigid body modes, $\mathbf{K}$ is singular and a procedure to solve the semi-positive-definite system must be implemented. Displacement constraints, equilibrium constraints, and self-equilibrating force methods have been proposed to deal with the eigenvalue problem of a rank deficient matrix system.[12-14] However, these are cumbersome to implement and often destroy the sparseness of the matrix system. Usually, a mass-shifted stiffness matrix, $(\mathbf{K} + \sigma\mathbf{M})$, that preserves the eigenvectors and shifts the eigenvalues by a constant, is employed to solve this class of problem. Yiu[14] noted that if a large value of σ is used, the vector basis will not span the rigid body modes unless a large number of vectors is computed since spurious low-frequency, non-zero strain energy, modes appear in the basis. The shift σ should therefore be small enough to permit an accurate recovery of the rigid body modes and large enough to avoid ill-conditioning of the shifted stiffness matrix. Thus, the algorithm begins with the mass-shifting of the stiffness matrix $\mathbf{K}$, followed by the calculation of the set of displacements $\mathbf{U}_0$, reflecting the response to the block loading $\mathbf{F}(s)$. Any structural vibration frequencies near the shift point are well represented in the shifted LD vector basis.[15-16] Next, the *a priori* knowledge of the rigid body modes is used to remove rigid body displacement components from the static response $\mathbf{U}_0$. This step is performed in order to avoid having any arbitrarily large rigid body components contained in the shifted stiffness matrix which would dominate the elastic deformation components calculated using finite precision computation. It also permits the calculation of error norms related to the elastic deformation response under the specified loading. As the algorithm proceeds, the vectors $\mathbf{X}$ are orthonormalized with respect to the mass matrix $\mathbf{M}$, and contain a static residual to reduce the effects of truncation of higher frequencies. A final step of orthogonalizing the vectors $\mathbf{X}$ with respect to $\mathbf{K}$ is necessary to produce the vector basis ${}^0\mathbf{X}$ having the frequencies $\hat{\omega}$ to uncouple the equations of motion.

The influence of the frequency content of the loads on the response can be assessed by considering the relative contribution of the elastic and the inertia forces that are resisting the applied dynamic loads. For vectors with structural frequencies about three times higher than the frequency of the applied loading, the resistance is essentially elastic since the inertia and damping forces can be neglected. The contribution of higher vectors to the response can thus be computed from a static analysis, as described in step 2 of Appendix I. To improve the performance of the LDM for systems with dynamic load history containing predominant high-frequency components, a combination of low-frequency LD vectors computed with a small shift point, and high-frequency LD vectors computed with a shift point close to the predominant frequency of the loading, can be used to produce a vector basis as described by Hong Xia and Humar.[17] This approach is easily implemented from the algorithm presented in Appendix I.

Error norms

During the generation of vectors $\mathbf{X}_i$, the norm ε_{ui} is computed to indicate the participation of the starting static displacement vector to the solution.[6] This norm is computed using the displacement $\mathbf{U}_i$ at each cycle and the initial displacement set $\mathbf{U}_0$, where at cycle i

$$\varepsilon_{ui} = \frac{\| \mathbf{MU}_0 \|_\infty}{\| \mathbf{MU}_0 \|_\infty} \tag{1}$$

The behaviour of ε_{ui} was studied to determine whether it could be used as a criterion to judge when to terminate the generation of vectors. A second norm, $\varepsilon_L(t)$, also intended to represent the degree of participation of the spatial load distribution,[16-18] was calculated to determine whether it is reliable to judge the quality of the vector basis $\mathbf{X}$ in terms of its ability to represent the applied transient load. This norm is computed by the following formula, where for k

$$\varepsilon_L(t) = \frac{|\mathbf{P}(t)^{\mathrm{T}}\mathbf{e}_L|}{\mathbf{P}(t)^{\mathrm{T}}\mathbf{P}(t)} \tag{2}$$

in which $\mathbf{P}(t)$ is the applied load vector at time t, and $\varepsilon_L(t)$ the error in the representation of the load in LD co-ordinates, with

$$\mathbf{e}_L(t) = \mathbf{P}(t) - \sum_{j=1}^{k} \mathbf{M}^0\mathbf{X}_j^0\mathbf{X}_j^{\mathrm{T}}\mathbf{P}(t) \tag{3}$$

Computer implementation

A computer program[19] was developed to generate $\mathbf{X}$, $\hat{\omega}$, ε_u and $\varepsilon_L(t)$. The algorithm was coded on a CRAY X-MP/EA 464 Supercomputer using the Fortran computer language. The program interfaces with the bulk Nastran data set and Nastran Output files[20] containing the structural mass and stiffness matrices of the model. Either LD vectors or Nastran modal eigenvectors can be used in the transient dynamic analysis.

SYSTEM ANALYSED

The space station Freedom was analysed for a simulated docking with the space shuttle. A view of the station is given in Figure 1. In the analyses performed, the McDonald Douglas Nastran MB15YZ model of the Station with 2803 DOF was used along with docking load case 915L, consisting of the set of three transient translational forces shown in Figure 2, and an additional set of three rotational forces (not shown) applied to the end of the docking arm.[19] The position of the photovoltaic (PV) arrays in the analysis had an orientation in the $Y-Z$ plane. Vector bases with 30, 60 and 90 vectors were generated to compute the elastic displacements and total accelerations of selected DOFs based on a constant damping ratio of 0·01. The vector bases were generated using a block size of six vectors, with $\mathbf{U}_0$ consisting of six displacement vectors corresponding to the response of the six individual docking loads acting on the structure at time $t = 1{\cdot}93$ s, which coincided with the peak magnitude of applied forces. The response at the tip of an outboard PV array (node 8022 in Figure 1) is used to study the performance of

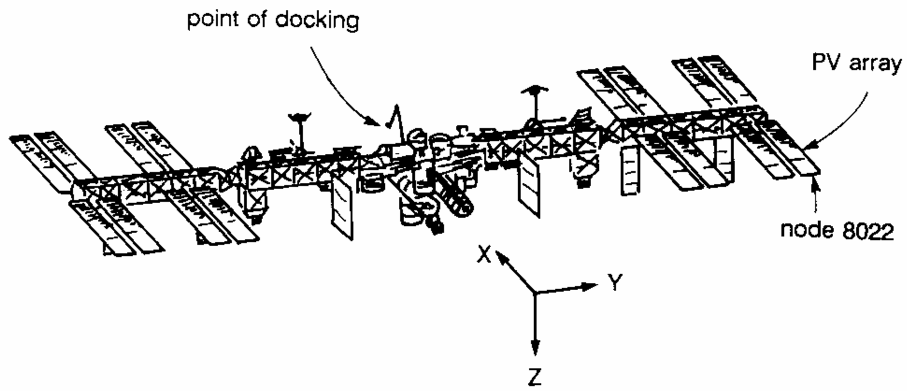

Figure 1. Space station Freedom

the proposed solution strategy. Neither a complete solution based on a numerical integration of the coupled equations of motion, nor a complete modal vector basis was available to compare with the LDM. For the purpose of this study, a comparison was made with a solution obtained from the MDM using a truncated modal basis consisting of 210 eigenvectors. The frequency of the modes ranged from 0 to 6·0 Hz. Figure 3 shows a comparison of the frequency range of the modal vectors and selected LD vector bases. The 60 LD vector basis with a static residual (identified as SR in the legend of Figure 3) has approximately the same

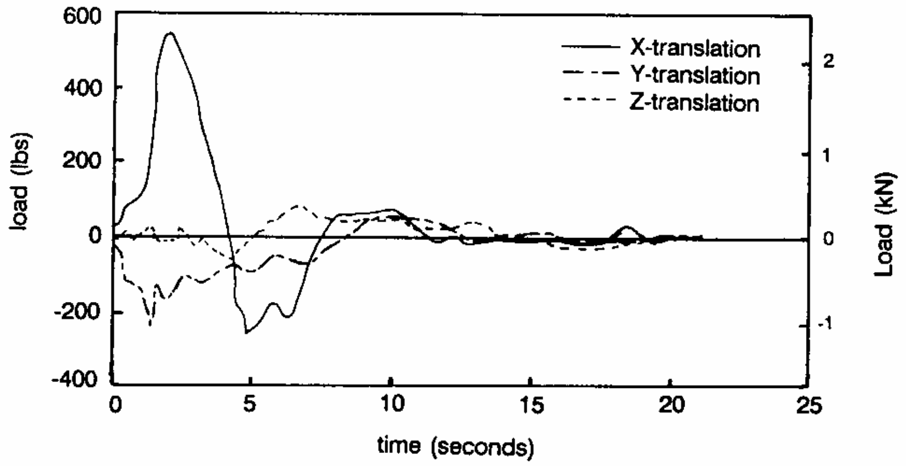

Figure 2. Translational forces for docking load case

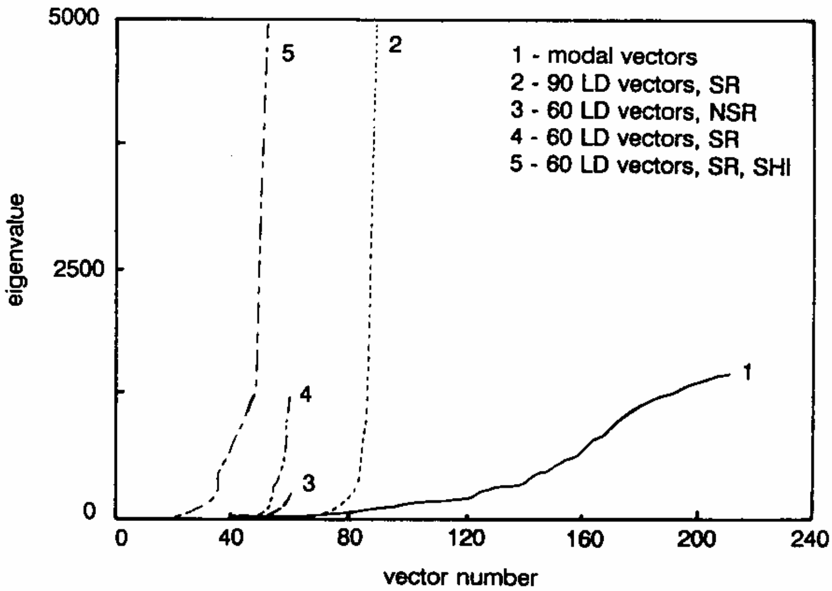

Figure 3. Span of eigenvalue (frequency) range of modal (eigenvectors) and load-dependent (LD) vector bases (SR: with static residual, NSR: without static residual, SHI: shift constant $\sigma = 200$

frequency range as the modal basis. A larger LD vector basis has a greater span, and this span is extended for those bases containing the static residual as opposed to those not having it (identified as NSR in Figure 3's legend). All LD vector bases used in this study were generated with a shift constant of $\sigma = 1 \cdot 0$, except for the 60 vector basis with $\sigma = 200 \cdot 0$ identified in Figure 3 as SHI. The effect of using a larger value for σ produces a basis that has more distantly spaced frequencies and a greater span.

DYNAMIC RESPONSE ANALYSIS

Transient displacements and accelerations

Time history plots of the translational displacements and accelerations in the X-direction are shown in Figures 4 and 5 for the 90 vector solution using the LDM and the 210 eigenvector solution using the MDM. The LD based displacements are shown to agree closely with the results of the MDM, whereas there exists a greater discrepancy between the accelerations computed by the two methods. The maximum discrepancy in the transient translational displacements and accelerations along the X-, Y- and Z-axes based on the results from the MDM with 210 eigenvectors, and LDM analyses with 90, 60, 30 vector bases are shown in Figure 6. It is apparent that the discrepancy between the MDM and LDM decreases as the number of vectors in the LD basis is increased from 30 to 90 vectors. The LDM with a 60 vector basis is able to achieve the same result as the 210 MDM vector basis for displacements. The LDM for accelerations shows greater discrepancies with the MDM, and requires a larger number of vectors in the basis to achieve the same level of accuracy as that found in the displacements with 60 LD vectors.

Error norm analysis

The norm ε_u is plotted against the number of blocks of LD vectors in Figure 7 to establish whether it can give a good indication of when to terminate the generation of vectors. An

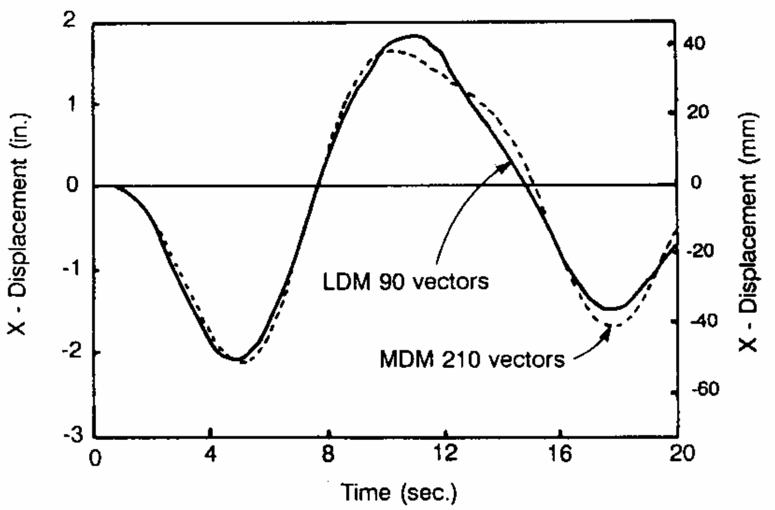

Figure 4. Displacement response at node 8022 (LDM: load-dependent method, MDM: mode displacement method)

examination of the data reveals that for 30 LD vectors (five blocks of six vectors), the value of ε_u is approximately 50 times greater than that for 60 LD vectors (ten blocks of six vectors), at which the displacements were in very close agreement with the MDM results. The fact that ε_u has become quite small at 60 LD vectors suggests that there is not much to be gained by 30 additional vectors to achieve a 90 vector basis. This was found to be true only for the displacements. Thus, ε_u could be considered more reliable to indicate displacement convergence than acceleration convergence.

The behaviour of ε_u in Figure 7 shows that a fluctuation in the error norm can exist where a local maximum develops. This phenomenon is quite noticeable for the case of five blocks with six vectors per block. This behaviour is associated with the structure responding more to the forces related to the displacements $\mathbf{U}_i$ as opposed to $\mathbf{U}_0$. Numerical experimentation indicated that this occurs when a block of loads used to calculate $\mathbf{U}_0$ is not uniformly displacing the mass, leading to displacement patterns $\mathbf{U}_i$ in some of the later cycles which 'accelerate' more the mass. These fluctuations tend to occur in the initial cycles of vector generation; as the number of vectors is increased the phenomenon appears to decay as shown in Figure 7. This phenomenon is not as pronounced in the vector basis generated using fewer vectors per block, as indicated by the results shown in Figure 7 for a block size of one vector. Asymptotic convergence of ε_u to zero can thereby be expected as the number of vectors is increased. A specific threshold value for ε_u to terminate the process of generation of vectors cannot be assigned at this time, because sensitivity to models with a more uniformly distributed

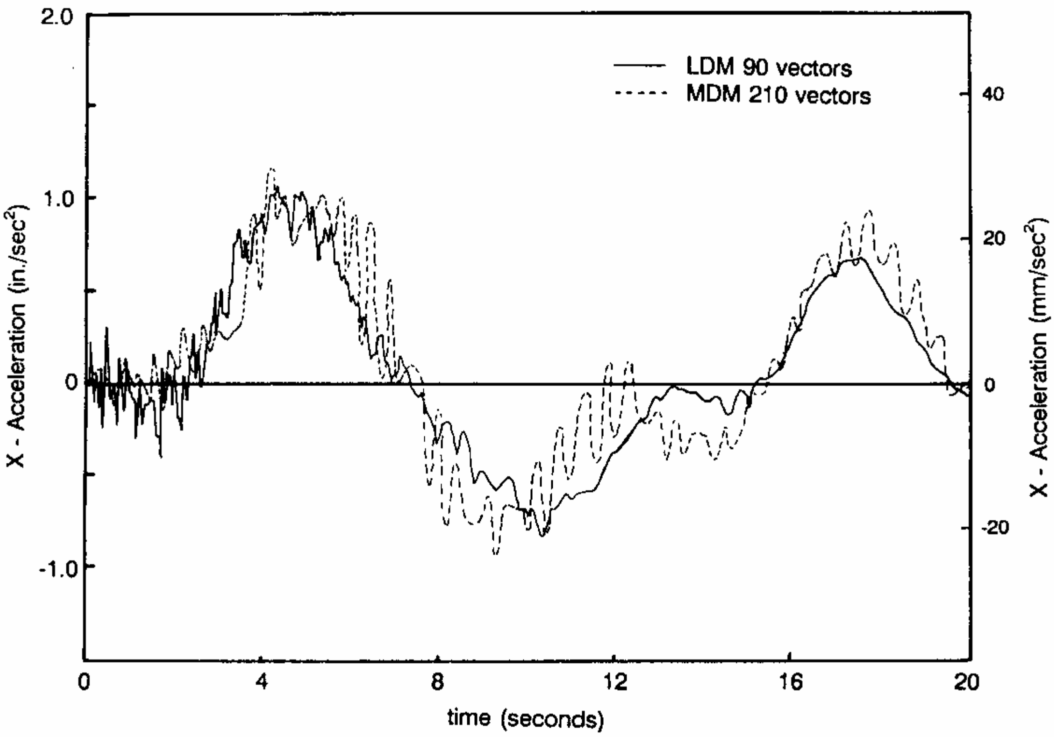

Figure 5. Acceleration response at node 8022 (LDM: load-dependent method, MDM: mode displacement-method)

mass is not known, as well as the fact that the results presented are from only one loading condition.

A presentation of the norm $\varepsilon_L(t)$ associated with the Z-component of the translational docking load at time $t = 1{\cdot}93$ s is given in Figure 8. The representation of the loading obtained by the LD vector bases is better than that obtained from the 210 modal vector basis. This representation is improved as a static residual is added, and further improved as the number of LD vectors in the basis is increased. The fact that fewer LD vectors were found to be required than modal vectors to achieve comparable results for $\varepsilon_L(t)$ gives an indication that the LD vector bases align more with the loading than the modal vector basis, such that a product of LD vectors with the load vector is larger than the product using modal and load

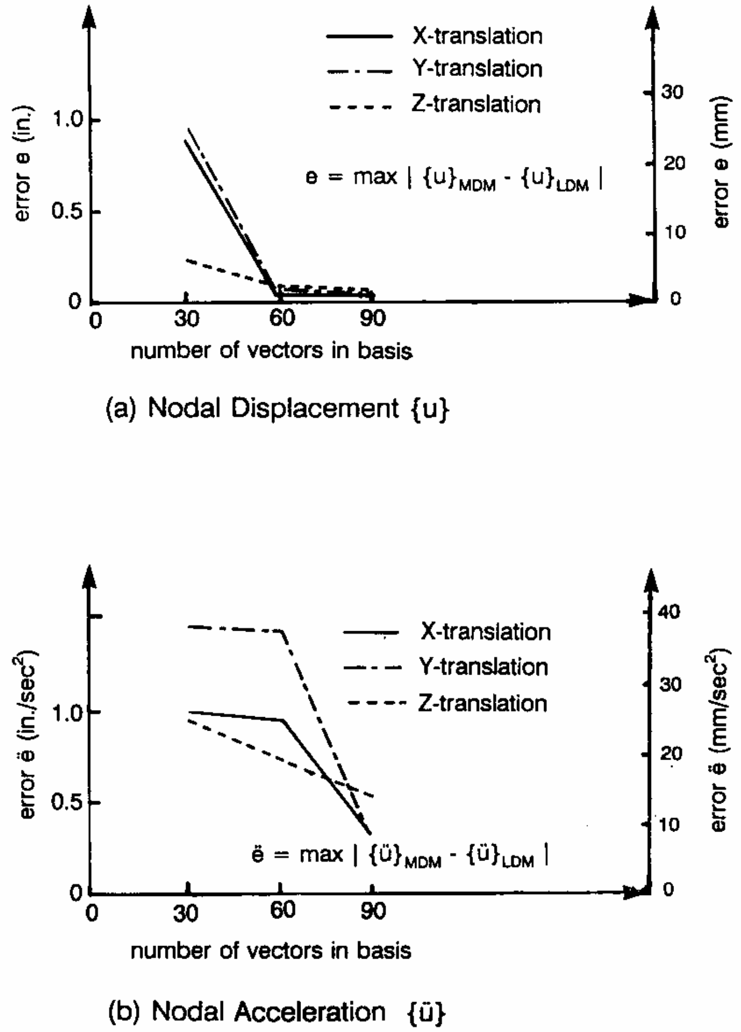

Figure 6. Error between the mode displacement method (MDM) and the load-dependent method (LDM) analyses for node 8022

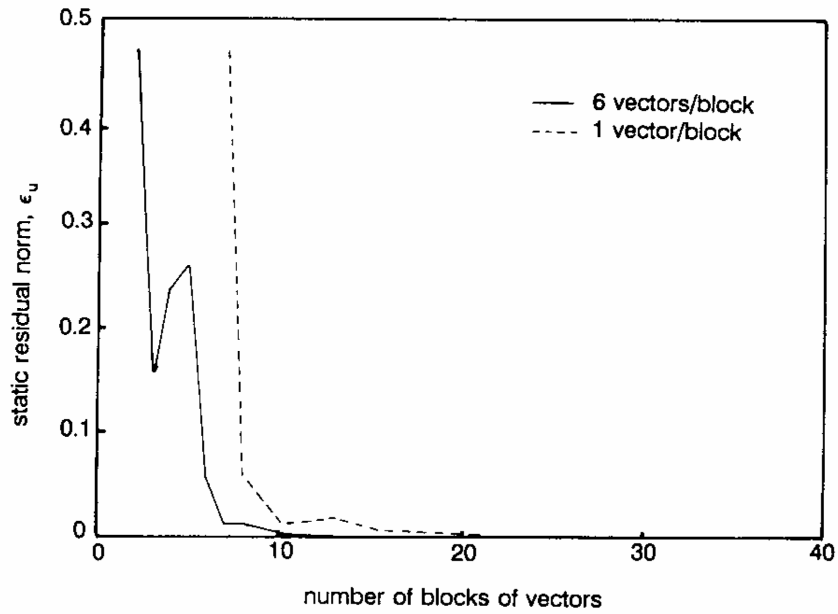

Figure 7. Static residual error norm ε_u as a function of the number of blocks generated

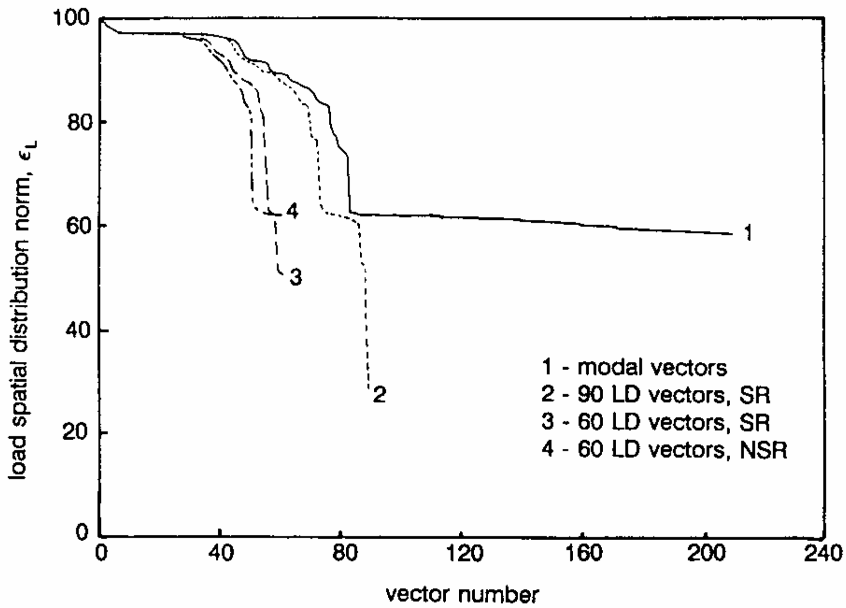

Figure 8. Loading error norm $\varepsilon_L(t)$ for Z-translational docking forces at time 1·93 s using modal (eigenvectors) and load-dependent (LD) vector bases (SR: with static residual, NSR: without static residual)

vector. The LDM is thus more effective to compute the dynamic response, as compared to the possibility of having the vectors orthogonal to the loading, as is sometimes the case in the MDM. A close examination of Figure 8 shows that most of the modal vectors' contribution to the load representation occurs in the mid-frequency range (the vector number in Figure 8 corresponds to the ordering of the frequencies). Also, the modal vectors which are orthogonal to the loading are spread throughout the spectrum. This orthogonality is identified by the parts of the curve joining the values of $\varepsilon_L(t)$ where the slope is zero.

A comparison of the behaviour of ε_u with that of $\varepsilon_L(t)$ indicates that, although ε_u may become small, $\varepsilon_L(t)$ may not. Since convergence for accelerations was observed to require more LD vectors, more emphasis should be placed on $\varepsilon_L(t)$ for analysis in which the acceleration response is of prime interest when attempting to judge the quality of a vector basis for the selected loading conditions. This could be extended to situations where the LD vectors were generated based on a particular load case and there is the desire to use them for another load condition.

CONCLUSIONS

Based on the results of the transient dynamic analysis of the space station Freedom using the superposition of load-dependent (LD) vector algorithm presented herein, the following conclusions are noted:

1. LD vectors provide accurate displacement solutions using fewer vectors than eigenvectors computed from an 'exact' eigenanalysis. Moreover, past investigations have shown that the computer execution time to generate an LD vector basis is approximately one tenth of the execution time required to compute an 'exact' eigenbasis. The use of ε_u for criteria to stop the generation of the vector process appears to be satisfactory to assess a priori the quality of the vector basis in computing the displacement response.
2. More LD vectors are required when computing accelerations than displacements. Criteria for judging the quality of the vector basis using $\varepsilon_L(t)$ needs to be more strict when the acceleration response is of interest as opposed to displacements.
3. LD vectors have a broader frequency range than the equivalent number of eigenvectors, and are able to better represent the loading function. These features of the LD vectors are enhanced by including the static residual in the basis.
4. Further work is required to study the basic behaviour and calibration of ε_u and $\varepsilon_L(t)$ for different load cases and structures, respectively. The optimization of the algorithm to take advantage of vectorization and concurrent multiprocessors procedures should be considered for the solution of very large structural models.[21–22]

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APPENDIX I. BLOCK FORM OF LD VECTOR ALGORITHM FOR SEMI-POSITIVE-DEFINITE SYSTEMS

1. Dynamic equilibrium equations:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(s)\alpha(t)$$

2. Initial calculations:

(a) Shift and factorize stiffness matrix:

$$\mathbf{K}^* = \mathbf{K} + \sigma\mathbf{M}$$

$$\mathbf{K}^* = \mathbf{LDL}^{\mathrm{T}}$$

(b) Solve for static response to block loading, $\mathbf{U}_0$:

$$\mathbf{K}^*\mathbf{U}_0 = \mathbf{F}(s)$$

3. Calculate rigid body modes:

(a) Use geometric description and DOF relationship to describe rigid body motions $\mathbf{X}_1$
(b) Generate first block of orthonormalized Ritz vectors:

$$\mathbf{X}_1 = \bar{\mathbf{X}}_1\boldsymbol{\beta}$$

$$\boldsymbol{\beta} = (\bar{\mathbf{X}}_1^{\mathrm{T}}\mathbf{M}\bar{\mathbf{X}}_1)^{-1/2}$$

(c) Remove rigid body modes from static block $\mathbf{U}_0$ (Gram–Schmidt orthogonalization):

$$\mathbf{U}_1 = \mathbf{U}_0 - \mathbf{X}_1(\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{U}_0)$$

4. Generate additional Ritz vectors X_i, $i = 2, \ldots, n-1$:

(a) Solve for $\mathbf{X}_i$

$$\mathbf{K}^*\bar{\mathbf{X}}_i = \mathbf{M}\mathbf{U}_{i-1}$$

(b) M-orthogonalize $\mathbf{X}_i$ against previous blocks (Gram–Schmidt):

$$\mathbf{X}_i^* = \bar{\mathbf{X}}_i - \sum_{j=m}^{i-1} \mathbf{X}_j\mathbf{X}_j^{\mathrm{T}}\mathbf{M}\bar{\mathbf{X}}_i, \qquad 1 \leqslant m \leqslant i-2$$

(c) M-orthogonalize vectors in block $\mathbf{X}_i^*$ by modified Gram–Schmidt to obtain $\mathbf{X}_i$
(d) Remove new LD block $\mathbf{X}_i$ from static block $\mathbf{U}_{i-1}$ (Gram–Schmidt):

$$\mathbf{U}_i = \mathbf{U}_{i-1} - \mathbf{X}_i(\mathbf{X}_i^{\mathrm{T}}\mathbf{M}\mathbf{U}_{i-1})$$

5. Add static block residual U_{n-1} as static correction terms X_n:

6. Make LD vectors X stiffness orthogonal (optional–uncouples equations of motion):

(a) Form and solve the reduced $n \times n$ eigenvalue problem

$$\mathbf{K}^{**} = \mathbf{X}^{\mathrm{T}}\mathbf{K}^{*}\mathbf{X}$$

$$\mathbf{K}^{**} - \hat{\omega}^{2}\mathbf{I}\mathbf{\Psi} = \mathbf{0}$$

(b) Compute final LD vectors ${}^{0}\mathbf{X} = \mathbf{X}\mathbf{\Psi}$

REFERENCES

1. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, 1982.
2. E. L. Wilson, M. W. Yuan and J. M. Dickens, 'Dynamic analysis by direct superposition of Ritz vectors', *Earthquake eng. struct. dyn.*, **10**, 813–824 (1982).
3. E. L. Wilson, 'A new method of dynamic analysis for linear and nonlinear systems', *Finite Elements Anal. and Des.*, **1**, 21–23 (1985).
4. B. Nour-Omid and R. W. Clough, 'Dynamic analysis of structures using Lanczos coordinates', *Earthquake eng. struct. dyn.*, **12**, 565–577 (1984).
5. B. Nour-Omid and R. W. Clough, 'Block Lanczos method for dynamic analysis of structures', *Earthquake eng. and struct. dyn.*, **13**, 271–275 (1985).
6. P. Léger and E. L. Wilson, 'Generation of load dependent Ritz transformation vectors in structural dynamics', *Eng. Comput.*, **4**, 309–318 (1987).
7. P. Léger, 'Load dependent subspace reduction methods for structural dynamic computations', *Comput. Struct.*, **29**, 993–999 (1988).
8. P. Léger, 'Microcomputer solution of large structural dynamic problems', *Microcomput. Civil Eng.*, **4**, 127–132 (1989).
9. P. Léger, J. M. Ricles and L. J. Robayo, 'Reducing modal truncation error in the wave response of offshore structures', *Commun. Appl. Numer. Methods*, **6**, 7–16 (1990).
10. R. R. Arnold, R. L. Citerley, M. Chargin and D. Galant, 'Application of Ritz vectors for dynamic analysis of large structures', *Comput. Struct.*, **21**, 461–467 (1985).
11. P. Léger and E. L. Wilson, 'Modal summation methods for structural dynamic computations', *Earthquake eng. struct. dyn.*, **16**, 23–27 (1988).
12. R. W. Cough and J. Penzien, *Dynamics of Structures*, McGraw Hill, 1975.
13. R. R. Craig, Jr., *Structural Dynamics*, Wiley, 1981.
14. Y.-C. Yiu, 'Reduced vector basis for dynamic analysis of large damped structures', SEMM Report No.90–6, Division of Structural Eng. and Struct. Mech., Univ. of Calif., Berkeley, 1990.
15. K. J. Bathe and S. Ramaswamy, 'An accelerated subspace iteration method', *Comput. Methods Appl. Mech. Eng.*, **23**, 313–331 (1980).
16. A. Ibrahimbegovic and E. L. Wilson, 'Automated truncation of Ritz vector basis in modal transformation', *J. Eng. Mech. ASCE*, **116**, 2506–2520 (1990).
17. Hong Xia and J. L. Humar, 'Frequency dependent Ritz vectors', *Earthquake eng. struct. dyn.*, **21**, 215–231 (1992).
18. K. J. Joo, E. L. Wilson and P. Léger, 'Ritz vectors and generation criteria for mode superposition analysis', *Earthquake eng. struct. dyn.*, **18**, 149–167 (1989).
19. J. M. Ricles, 'Development of load-dependent Ritz vector method for structural dynamic analysis of large space structures', NASA/JSC, Contract No. NGT-44-005-803, 1990.
20. Computerized Structure Analysis and Research Corporation, *CSA*/NASTRAN Users Manual, Vol. 1, 1988.
21. C. Farhat and E. L. Wilson, 'Modal superposition dynamic analysis on concurrent multiprocessors', *Eng. Comput.*, **3**, 305–311 (1986).
22. E. L. Wilson and C. Farhat, 'Linear and nonlinear finite element analysis on multiprocessor computer systems', *Commun. Appl. Numer. Methods*, **4**, 425–434 (1988).