SEREP: System Equivalent Reduction-Expansion Process

Assume the equation of motion of the form

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}$$

We want to reduce the size of the system to a smaller number so that the model only includes p modes or the m degrees of freedom that are necessary to the model. Here p is defined based on the expected bandwidth of the excitation and m is defined based on the number of degrees of freedom that are acted upon by an external load, or are necessary as output states in the model. The number of retained coordinates is constrained to be equal to the number of retained modes, and is chosen to be the larger of p or m.

We then partition the coordinate vector into two parts, x_r and x_t, the retained and truncated coordinates, and reorganize the equations of motion in the form (considering only the undamped case for now)

$$\begin{bmatrix} M_{rr} & M_{rt} \\ M_{tr} & M_{tt} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_r \\ \ddot{\mathbf{x}}_t \end{bmatrix} + \begin{bmatrix} K_{rr} & K_{rt} \\ K_{tr} & K_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{F}_r \\ \mathbf{0} \end{bmatrix}$$

Consider the eigensolution for the mass normalized eigenvectors, $\Phi = [\Phi_{ar}\Phi_{at}]$ where r and t stand for retained and truncated, and a represents that all coordinates are retained in the vector. The modes to be retained, Φ_{ar}, is an $n \times m$ matrix, and the modes to be truncated, Φ_{at}, is an $n \times (n-m)$ matrix. Substituting

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_t \end{bmatrix} = \Phi\mathbf{r} = [\Phi_{ar}\Phi_{at}] \begin{bmatrix} \mathbf{r}_r \\ \mathbf{r}_t \end{bmatrix} = \begin{bmatrix} \Phi_{rr}\Phi_{rt} \\ \Phi_{tr}\Phi_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{r}_r \\ \mathbf{r}_t \end{bmatrix}, \tag{1}$$

and pre-multiplying by

$$\Phi^T = [\Phi_{ar}\Phi_{at}]^T$$

yields the equations of motion in modal coordinates

$$I\ddot{\mathbf{r}} + \lambda\mathbf{r} = \Phi^T\mathbf{F} \tag{2}$$

with a transformed mass matrix of

$$\begin{bmatrix} \Phi_{ar}^T \\ \Phi_{at}^T \end{bmatrix} \begin{bmatrix} M_{rr} & M_{rt} \\ M_{tr} & M_{tt} \end{bmatrix} \left[\Phi_{ar}\Phi_{at}\right] = \begin{bmatrix} \Phi_{ar}^T M \Phi_{ar} & \Phi_{ar}^T M \Phi_{at} \\ \Phi_{at}^T M \Phi_{ar} & \Phi_{at}^T M \Phi_{at} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_t \end{bmatrix}$$

and a transformed stiffness matrix of

$$\begin{bmatrix} \Phi_{ar}^T \\ \Phi_{at}^T \end{bmatrix} \begin{bmatrix} K_{rr} & K_{rt} \\ K_{tr} & K_{tt} \end{bmatrix} \left[\Phi_{ar}\Phi_{at}\right] = \begin{bmatrix} \Phi_{ar}^T K \Phi_{ar} & \Phi_{ar}^T K \Phi_{at} \\ \Phi_{at}^T K \Phi_{ar} & \Phi_{at}^T K \Phi_{at} \end{bmatrix} = \begin{bmatrix} \Lambda_r & 0 \\ 0 & \Lambda_t \end{bmatrix}$$

We then truncate the modal vector $\mathbf{r} = [\mathbf{r}_r\mathbf{r}_t]^T$ by assuming $\mathbf{r}_t = \mathbf{0}$. Equation (2) then reduces to

$$I_r\ddot{\mathbf{r}}_r + \lambda_r\mathbf{r}_r = \Phi_{ar}^T\mathbf{F}_r \tag{3}$$

Since now $\mathbf{x}_r = \Phi_{rr}\mathbf{r}_r$ from equation (1), we can substitute $\mathbf{r}_r = \Phi_{rr}^{-1}\mathbf{x}_r$ into equation (3) yielding

$$\Phi_{rr}^{-1^T}\Phi_{ar}^T M \Phi_{ar}\Phi_{rr}^{-1}\ddot{\mathbf{x}}_r + \Phi_{rr}^{-1^T}\Phi_{ar}^T K \Phi_{ar}\Phi_{rr}^{-1}\mathbf{x}_r = \Phi_{rr}^{-1^T}\Phi_{ar}^T\mathbf{F}$$

We can then define a coordinate transformation matrix T where

$$T = \Phi_{ar}\Phi_{rr}^{-1} = \begin{bmatrix}\Phi_{rr}\\ \Phi_{tr}\end{bmatrix}\Phi_{rr}^{-1} = \begin{bmatrix} I \\ \Phi_{tr}\Phi_{rr}^{-1}\end{bmatrix}$$

so that

$$\tilde{M}\ddot{\mathbf{x}}_r + \tilde{C}\dot{\mathbf{x}}_r + \tilde{K}\mathbf{x}_r = \tilde{\mathbf{F}}$$

where $\tilde{M} = T^T M T$, $\tilde{C} = T^T C T$, $\tilde{K} = T^T K T$ and $\tilde{\mathbf{F}} = T^T\mathbf{F}$ yields our reduced order model.