Braneworlds

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Abstract. This course is an introduction to the physics of braneworlds. We concentrate on braneworlds with only one extra-dimension and discuss their gravity. We derive the gravitational equations on the brane from the bulk Einstein equation and explore some limits in which they reduce to 4-dimensional Einstein gravity. We indicate how cosmological perturbations from braneworlds are probably very different from usual cosmological perturbations and give some examples of the preliminary results in this active field of research.
For completeness, we also present an introduction to 4-dimensional cosmological perturbation theory and, especially its application to the anisotropies of the cosmic microwave background.

1. INTRODUCTION

During recent years, cosmology has become one of the most successful fields in physics. The precise measurements of the anisotropies in the cosmic microwave background have confirmed a simple 'concordance model': The Universe is spatially flat. Its energy density is dominated by vacuum energy (or a cosmological constant) which contributes about 70% to the expansion of the Universe, $\Omega_\Lambda \simeq 0.7$. The next important contribution is cold dark matter (CDM) with $\Omega_{CDM} \simeq 0.3$. The expansion velocity of the Universe is given by the Hubble constant $H_0 = 100h$km/sMpc, with $h \simeq 0.7$. Baryons only contribute a small portion of $\Omega_b h^2 \simeq 0.02$. Massive neutrinos contribute similarly or less.

The structures in the Universe (galaxies, clusters, voids and filaments) have formed out of small initial fluctuations which have been generated during inflation and have an almost scale-invariant spectrum, $n = 1 \pm 0.1$. There is probably also a small amount of tensor fluctuations (gravity waves) generated during inflation, which however has not yet been detected.

All the above numbers are accurate to a few percent and will be measured even more precisely with ongoing and planned experiments. This situation is unprecedented in cosmology. About twenty years ago, these numbers where known at best within a factor of two or even only by their order of magnitude. The concordance model is in agreement with most cosmological data, most notably the CMB anisotropy measurements, supernova type Ia distances (see contribution by Varun Shani), statistical analysis of the galaxy distribution, constraints from cosmic nucleosynthesis, cluster abundance and evolution etc.

However, on a theoretical level our understanding has remained poor. We have no satisfactory answers to the questions:

- What is dark matter ?
- What is dark energy?
- What is the 'inflaton'? Or what is, more precisely, the physics of inflation?
- How can we resolve the Big Bang and other singularities of classical general relativity?

There is justified hope that the last question could be resolved within a theory of quantum gravity, which is anyway needed if we want to put all fundamental interactions on a common footing. At present, the most successful attempt towards a theory of quantum gravity is string theory. This theory is based on the assumption that the 'fundamental objects' are not particles but one dimensional strings. Particles then manifest as excitations, proper modes, of strings. It would lead us much too far to give an introduction to string theory at this point. The interested student will have to study the two volumes of Polchinski [1].

In this course we next give an introduction to braneworlds, or, more generally, to physical effects of extra dimensions. In Section 3, we derive the gravitational equations for braneworlds with one co-dimension from Einstein's equations in the bulk. We then discuss in detail the Randall–Sundrum II model, its background and its perturbations. In Section 5, we give an introduction to 4-dimensional cosmological perturbation theory and, especially to the CMB anisotropy spectrum. Only after this we are ready for braneworld cosmology in Section 6. We write down the most general brane cosmology in an empty bulk and we investigate some of its modifications w.r.t. 4-dimensional cosmology. In particular, we discuss the modification of the slow roll parameters during braneworld inflation. We also present one example of the modifications in the evolution of cosmological perturbations which are relevant in braneworlds. We end with some conclusions.

Notation: We use capital Latin indices $A, B, \cdots$ to denote bulk coordinates, lower case Greek indices $\mu, \nu, \cdots$ for coordinates on a four dimensional brane, and lower case Latin indices, $i, j, \cdots$ for spatial 3-dimensional quantities. We sometimes also use bold symbols to denote 3d spatial vectors. We use the metric signature $(-,+,\cdots,+)$. The 4-dimensional Minkowski metric is denoted by $(\eta_{\mu\nu})$.

Throughout we set $c = \hbar = k_{\rm Boltzmann} = 1$ so that time and length scales are measured in inverse energies (usually GeV's), and mass and temperature correspond to an energy. The four dimensional Newton constant is then given by $G_4 = 0.67 \times 10^{-38}\text{GeV}^{-2}$. Useful relations in this set of units are $1 = 0.2$GeV fm and 1 eV $= 1.16 \times 10^4$K. Here fm $=$ femtometer $= 10^{-15}$m.

2. BASICS OF BRANEWORLDS

2.1. What are Braneworlds?

The interest of string theory lies in the fact that it may provide a unified description of gauge interactions and gravity. Its weak point is, that it is extremely hard to make predictions from string theory which are testable at energies available in experiments. The reason for that is that string theory probably fully manifests itself only at very high energies of the order the Planck scale. The observed 4-dimensional Planck scale is given by Newton's constant, G_4. In our units with $\hbar = c = 1$ the Planck scale is $E_4 = M_4 = 1/\sqrt{4\pi G_4} \simeq 3 \times 10^{18}$GeV. This energy scale cannot be achieved by far at terrestrial accelerators (the LHC presently under construction at CERN will achieve about 7000GeV).

Nevertheless, string theory makes some relatively firm predictions which might lead to observational consequences at low energy. First of all, it predicts that spacetime is ten-dimensional with one time and nine spatial dimensions. Since the observed world has only four dimensions, one usually assumes that the other six are compact and very small, so that they cannot be resolved by any physical experiment available to us so far.

Furthermore, string theory predicts the existence of so called p-branes, $p+1$-dimensional sub-manifolds of the ten dimensional spacetime on which open strings end. Gauge fields and gauge fermions which correspond to string end points can only move along these p-branes, while gravitons which are represented by closed strings (loops) can propagate in the full spacetime, the 'bulk'.

This basic fact of string theory has led to the idea of braneworlds: it may be that our $3+1$-dimensional spacetime is such a 3-brane. If this is so, only gravity can probe the bulk and the additional dimensions can be much larger than the smallest length scale which we have probed so far, which is of the order of $(200\text{GeV})^{-1} \simeq 10^{-18}$m. Actually, Newton's law has been tested only down to scales of about 0.1mm [2]. Hence, in the braneworld picture where only gravity can probe the extra-dimensions, these can be as large as 0.1mm$= 10^{-3}$m. In the next subsection we show how this fact can be employed to address the hierarchy problem.

2.2. Lowering the fundamental Planck scale

The fact that the 4-dimensional Planck scale, $M_4 \sim 10^{19}$GeV is so much larger than the fundamental scale in elementary particle physics, the electroweak scale $E_{ew} \simeq 10^3$GeV is called the hierarchy problem. Apart from it seeming unnatural to have two so widely separated scales to describe fundamental physics, a more serious problem is the fact that as soon as we have a unified quantum theory which describes also gravity, the scale M_4 will enter in quantum corrections of all electroweak scale quantities which are not especially protected e.g. by symmetries and it will therefore completely spoil the so successful low energy standard model.

Here we show that within the braneworld picture, it is possible that the 4-dimensional Planck scale is not fundamental but only an effective scale which can become much larger than the fundamental Planck scale M_P if the extra-dimensions are much large than M_P^{-1}. Our argument goes back to Arkani-Hamed , Dimopoulos and Dvali (1998) [3].

Let M_P be the fundamental Planck scale and L the size of n extra dimensions. In addition there are 3 large spatial dimensions (and time). For simplicity we assume the n extra-dimensions to be rolled up as a cylinder, $(S^1)^{\otimes n} \otimes \mathbb{R}^3$. The gravitational constants $G_{(n+4)}$ and G_4 are defined by the force laws of gravity. Due to the Gauss constraint these must have the forms

$$F_{(n+4)} = G_{(n+4)} \frac{m_1 m_2}{r^{n+2}} \ , \quad \text{and} \qquad F_4 = G_4 \frac{m_1 m_2}{r^2} \ .$$

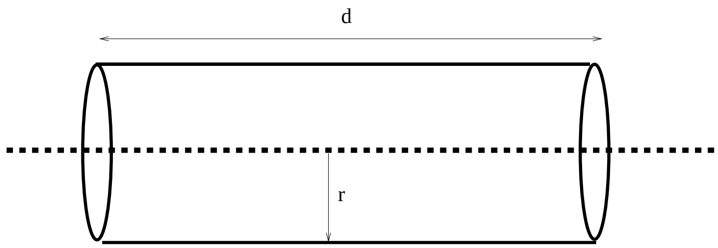

FIGURE 1. A $1+1$ dimensional cylinder around a 1-dimensional grid of mass points, corresponding to the de-compactification of a mass point on a circle.

On small scales, $r \ll L$, an observer on the brane sees $n+4$ dimensional gravity, while on large scales, $r \gg L$, the cylinder can no longer be resolved and simple 4 dimensional gravity is observed. In order to relate the constants G_4 and $G_{(n+4)}$, we de-compactify the compact dimensions leading to an n-dimensional lattice of masses which looks from far like a hypersurface of mass density m/L^n (see Fig. 1). Around the mass distribution, we now form a cylinder C of dimension $n+2$, length d and radius r (see Fig. 1). To satisfy Gauss' law, we require

$$\int_C F_\perp d\sigma = S_{(2+n)} G_{(4+n)} \times (\text{mass in } C).$$

Here, $F_\perp$ is the component of the force normal to the surface and S_j is the volume of a j dimensional sphere, $S_j = 2\pi^{(j+1)/2}/\Gamma([j+1]/2)$, where Γ denotes the Gamma-function, $\Gamma([j+1]/2) = ([j-1]/2)!$ for integer values of $(j+1)/2$.

The first integral is $4\pi r^2 d^n F(r)$ while the mass inside the cylinder is md^n/L^n. With the 4-dimensional force law this implies

$$G_4 = \frac{S_{(2+n)}}{4\pi} \frac{G_{(4+n)}}{L^n} \ . \tag{1}$$

In order to relate the gravitational constant to the Planck mass, we express Newtonian gravity in terms of an action principle. For a static weak gravitational field ϕ and a mass density ρ in $4+n$ dimensions, we can obtain the Poisson equation by varying the action

$$I = \int d^{(3+n)}x \left[\frac{M_P^{(2+n)}}{2} \phi \nabla^2_{(3+n)} \phi + \rho^{(4+n)} \phi + ... \right] \ .$$

Integrating out ϕ, we obtain again the Newtonian force law and the relation

$$M_P^{(2+n)} = \frac{G^{-1}_{(4+n)}}{S_{(2+n)}} \ . \tag{2}$$

Integrating the Lagrangian over the extra dimensions relates the 4- and $4+n$-dimensional Planck masses by

$$M_4^2 = M_P^{(2+n)} L^n \ .$$

Together with Eq. (2) this reproduces the result (1). For a sufficiently large length scale L, $M_4 \simeq 3 \times 10^{18}$GeV can therefore be much larger than the fundamental higher dimensional Planck scale M_P.

Experimental 'micro-gravity' bounds [2] require $L < 0.1$mm. For $n = 2$ and $L \sim 1$mm the fundamental Planck scale can be of the order of the electroweak scale, $M_P \sim (1 \ - \ 10)$TeV . This Planck scale seems to be in agreement with most other bounds (cooling of supernovae, evolution of the Universe, etc) but leads to the very interesting prospective that effects from string theory might be observable at the Large Hadron Collider (LHC) presently under construction at CERN [4].

Using 'large' extra-dimensions, $L \gg M_P^{-1}$, the fundamental Planck scale can therefore be of the same order as the electroweak scale. On the other hand, there is no explanation for the length scale $L \sim 0.1$mm$\sim 10^{-3}$eV. So the hierarchy problem has not actually been solved, but it has been moved from an energy hierarchy to an unexplained length scale. The hope is, that there would exist solutions of string theory which lead to such a scale dynamically.

2.3. New Physics from higher dimensions

Kaluza-Klein modes. We denote the four brane coordinates by x^μ and the additional n bulk coordinates by y^a. For simplicity, we consider a bulk where the extra dimensions are rolled up in a cylinder of circumference L. In the general situation where the compact extra dimensions form a non-flat Calabi-Yau manifold, $\mathscr{C}$, the exponentials below have to be replaced by the corresponding eigen-functions of the Laplacian on $\mathscr{C}$. The case of a warped geometry, where the extra dimensions may even be non-compact, will be discussed separately in Section 4.

Be now ϕ a massless scalar field in the 'bulk', $\phi(x,y)$ with $\phi(\cdots, y^a + L\cdots) = \phi(\cdots, y^a, \cdots)$. We can expand the y dependence of ϕ in Fourier series

$$\phi(x,y) = \sum_j \phi_j(x) \exp(i2\pi j \cdot y/L)$$

Since ϕ satisfies the massless wave equation, $\nabla^2_{4+n}\phi = 0$, in the bulk, for a 4d observer which cannot resolve the scale L, the modes $j \neq 0$ will become massive fields,

$$(\nabla_4^2 + m_j^2)\phi_j = 0 \qquad \text{with} \qquad m_j^2 = \frac{(2\pi j)^2}{L^2} \ .$$

These modes of the gravitational potential give raise to exponential corrections to Newton's law,

$$V(r) = \frac{G_4}{4\pi r}[1 + \frac{e^{-r/L}}{r^2} + \cdots] \ .$$

If the extra dimensions are large, the first few masses can be very low, but if the graviton is the only bulk field, its weak coupling to other bulk modes leaves the theory nevertheless viable [4].

As we shall see in Section 4, the mass spectrum can even be continuous, $0 < m < \infty$, if there are non-compact extra dimensions.

Higher dimensional spin modes. The spin states of massless particles in d space time dimensions are characterized by an irreducible representation of $SO(d-2)$. For $d = 4$, massless particles always carry a 1–dimensional representation of $SO(2)$ [1]. Taking into account also parity, this leads to the two helicity modes of all massless particles in 4 dimensions, independent of their spin. This situation changes drastically if we allow for extra dimensions. Let us consider $d = 5$: a massless particle of spin s in $d = 5$ spacetime dimensions, carries the representation D^s of $SO(3)$ and thus has $2s+1$ spin states, like a massive particle of spin s in 4 dimensions.

Let us, for example, consider the graviton. In $4+1$ dimensions it has the 5 helicity states of the tensor representation of SO(3). Projected onto a $3+1$ brane, two of them become the usual spin 2 graviton, two are a spin 1 particle, a gravi-vector and one has spin 0, the gravi-scalar.

The gravi-vector couples to the $\mu 4$ components of the energy momentum tensor. Interpreting these as the electromagnetic current J^μ and the gravi-vector as the electromagnetic potential A^μ, the five-dimensional Einstein equations lead to Maxwell's equations for A^μ and J^μ. This is the so called Kaluza–Klein miracle, which is also true if any, non–Abelian gauge group replaces the one-dimensional torus which plays here the role of the electrodynamic gauge group $U(1)$. This finding of Kaluza and Klein [6] has evoked an interest in extra-dimensions in the 20ties, long before string theory.

[1] The full little group for massless particles is actually $ISO(2)$, the group of two dimensional Euclidean motions and rotations. But for finite dimensional representations of $ISO(2)$ the translations are acting trivially. The reason why not all representations of the universal covering group of $SO(2)$, $\mathbb{R}$, have to be considered, namely e^{ikx}, $k \in \mathbb{R}$ is rather subtle. On the classical level, where $SO(2)$ is the relevant group, this problem disappears. A full discussion of the finite dimensional representations of the Poincaré group for massless particles in $d = 4$ can be found e.g. in [5].

The positive aspects of the vector sector are, however, over shaded by the problems coming from the gravi-scalar. It couples to the four-dimensional energy momentum tensor and modifies gravity. It leads to a scalar-tensor theory of gravity with several observable consequences. For example, one can calculate the modification in the slowing down of the binary pulsar [7] PSR1913+16 due to the radiation of gravi-scalars. In Ref. [8] it is shown that in the simple case of a 5-dimensional cylindrical bulk, this leads to a modification of the quadrupole formula by about 20%, while observations agree with the quadrupole formula to better than $\frac{1}{2}$%. If there are more than one extra-dimensions, there are several gravi-scalars and this problem is only enhanced.

Clearly, a modification of higher dimensional gravity is necessary to address the problem. Usually, one gives a mass to the gravi–scalar. There are several ways to do this and the resulting four dimensional theory in general depends on this choice. One proposal is the Goldberger–Wise mechanism [9].

Another solution is offered by non-compact extra-dimensions. As we shall see, in curved spacetimes, extra-dimensions can even be infinite. Due to a so called 'warp factor' they become very small when seen from the brane. For infinite extra dimensions it can happen that the gravi-scalar represents a non-normalizable and therefore unphysical mode. This is precisely what happens in the Randall–Sundrum model and we shall discuss it in this context in Section 4.

3. GEOMETRY OF FIVE DIMENSIONAL BRANEWORLD GEOMETRY

From now on we restrict ourselves to five dimensional braneworlds, *i.e.* braneworlds with only one extra-dimension. The idea here is still that spacetime has 10 (or for M-theory 11) dimensions, but 5 (or 6) of them are compactified to a static manifold of about Planck scale, while one of them is large. This picture is motivated mainly from 11-dimensional M-theory, *e.g.* the Horava-Witten model [10], but we shall not try to implement a realization of this model here. Nevertheless, it is important to note, that one co-dimension differs significantly from more than one. The main point is that the 3-brane splits space into two parts, the 'left' and the 'right' hand side of the brane. We shall see, that in this case it is possible to determine the gravitational equations on the brane by simply postulating Einstein's equations in the bulk. This is no longer possible in the case of two or more extra-dimensions.

In this section we derive and discuss these brane gravity equations. In the next section we shall apply them to the Randall–Sundrum model, which we consider as beeing so far the most promising braneworld model.

To determine the gravitational equations on the brane, we start from the basic hypothesis that string theory predicts Einstein gravity in the bulk,

$$G_{AB} = \kappa_5 T_{AB} \ . \tag{3}$$

We want to discuss in detail the situation of a 3- brane in a 5–dimensional bulk. We denote the bulk coordinates by $(x^A) = (x^\mu, y)$, where (x^μ) are coordinates along the brane and y is a transverse coordinate. We denote the brane position by $y = y_b$; in general y_b depends on the point on the brane, $y_b = y_b(x^\mu)$. A more general embedding of the brane will be discussed below. Very often we consider an energy momentum tensor of the form

$$T_{AB}((x^C) = (x^\lambda, y)) = \frac{\Lambda_5}{\kappa_5} g_{AB} + \delta^\mu_A \delta^\nu_B T_{\mu\nu}((x^\lambda))\delta(y - y_b) \ . \tag{4}$$

Here Λ_5 is a bulk cosmological constant, $T_{\mu\nu}$ is the energy momentum tensor on the brane and $\kappa_5 = 6\pi^2 G_5$ is the five-dimensional gravitational coupling constant. This is the most general ansatz if we do not allow for any matter fields in the bulk.

3.1. The second fundamental form

As above, g_{AB} is the bulk metric. We denote the projection operator onto the brane by q^A_μ. The induced metric on the brane, also called the first fundamental form, is then given by

$$g_{\mu\nu}(x^\lambda) = q^A_\mu(x^\lambda) q^B_\nu(x^\lambda) g_{AB}(x^\lambda, y_b(x^\lambda)). \tag{5}$$

We denote the covariant derivative in the bulk by ${}^b\nabla_A$ and covariant derivative on the brane (with respect to the induced metric) by ∇_μ. We also introduce the brane normal n, a vector field defined on the brane which is normal to all vectors parallel to the brane. Clearly, for an arbitrary vector field $X = X^\mu \partial_\mu$ along the brane, $\nabla_\mu X \neq {}^b\nabla_\mu X$. The difference of these two covariant derivatives is given by the extrinsic curvature also called the second fundamental form which we now introduce.

Be $X = X^\mu\partial_\mu$ and $Y = Y^\mu\partial_\mu$ two vector fields on the brane. Their covariant derivative on the brane is given by

$$\nabla_Y X = (Y^\mu\partial_\mu X^\nu + \Gamma^\nu_{\mu\beta}X^\mu Y^\beta)\partial_\nu \ ,$$

while the covariant derivative in the bulk is

$${}^b\nabla_Y X = (Y^\mu\partial_\mu X^\nu + \Gamma^\nu_{\mu\beta}X^\mu Y^\beta)\partial_\nu + \Gamma^4_{\mu\beta}\partial_4 \ .$$

Therefore, there exists a bi-linear form $K_{\mu\nu}$ on the brane such that

$${}^b\nabla_Y X = \nabla_Y X + K(X,Y)n \ . \tag{6}$$

Since ${}^b\nabla_Y X - {}^b\nabla_X Y = [Y,X]$ is tangent to the brane, $K(X,Y) - K(Y,X) = 0$, hence K is symmetric. K is called the 'second fundamental form' or 'extrinsic curvature' of the brane. Its sign is not uniquely defined in the literature; we shall use Eq. (6) as its definition.

Since n is the unit normal of the brane, $g(n, {}^b\nabla_Y X) = K(Y,X)$. But as n is normal to the brane vector field X we have $0 = {}^b\nabla_Y(g(n,X)) = g({}^b\nabla_Y n, X) + g(n, {}^b\nabla_Y X)$, so that

$$K(Y,X) = -g({}^b\nabla_Y n, X) = -\frac{1}{2}\left[g({}^b\nabla_Y n, X) + g({}^b\nabla_X n, Y)\right] \ .$$

In components, $K(X,Y) = K_{\mu\nu}X^\mu Y^\nu$, this becomes

$$K_{\mu\nu} = -\frac{1}{2}\left[{}^b\nabla_\mu n_\nu + {}^b\nabla_\nu n_\mu\right] \ . \tag{7}$$

Close to the brane we can choose coordinates such that

$$g_{AB}dx^A dx^B \equiv ds_b^2 = g_{\mu\nu}dx^\mu dx^\nu + dy^2 \ .$$

In these so called 'Gaussian normal coordinates', $n = \partial_y$ and ${}^b\nabla_\mu n_\nu = \Gamma^4_{\mu\nu} = -(1/2)g_{\mu\nu},_4$. The second fundamental form then becomes simply

$$K_{\mu\nu} = -\frac{1}{2}\partial_y g_{\mu\nu} \ .$$

For general coordinates (z^μ) on the brane we have to define a brane parameterization $x^A = X_b^A(z^\mu)$. The vector fields $(e_\mu) = (\partial_\mu X_b^A(z)\partial_A)$ then form a basis of tangent vectors on the brane. In terms of these one obtains by means of Eq. (7)

$$K_{\mu\nu} = -\frac{1}{2}\left[g_{AB}(e^A_\mu\partial_\nu n^B + e^A_\nu\partial_\mu n^B) + e^A_\nu e^B_\mu n^C g_{AB,C}\right] \ , \tag{8}$$

where a comma denotes an ordinary derivative, $f_{,C} = \frac{\partial f}{\partial x^C}$.

3.2. The junction conditions

Einstein's equations with a thin hyper-surface of matter become singular since there is a δ- function in the energy momentum tensor. Integrating them once across the brane leads to the so called junction or jump conditions (of Israel, Lancos, Darmois, Misner) [11, 12, 13, 14].

Before we come to the algebraically more complicated situation of general relativity, let us first recall the well known junction conditions of electrostatics: we consider a conducting boundary surface (e.g. capacitor plate) with surface charge density ρ and current density j along the surface.

The homogeneous Maxwell equations require that the tangential part of the electric field $E_\parallel$ and the normal component of the magnetic field, $B_\perp$, are continuous across the boundary. This is usually derived by the following argument: denoting the two sides of the capacitor plate by the super-scripts $+$ and $-$ the homogeneous Maxwell equations imply

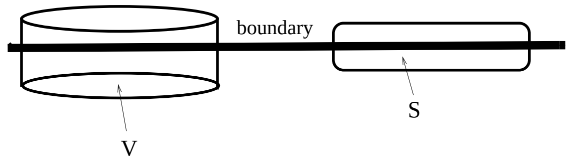

FIGURE 2. A conducting boundary (bold line). We indicate the integration surface S normal to it and the 'pill box' type volume V encompassing both sides. Our equations become exact in the limit where the height of the pill box and the width of the surfaces S approach zero.

for some surface S spanning from one side to the other of our boundary or for a volume V encompassing a little of both sides of the boundary (see Fig. 2)

$$\begin{array}{lllll} 0 & = & \int_S(\nabla\wedge E)d\sigma = & \int_{\partial S} E_\| ds = & L(E_\|^+ - E_\|^-) \quad \text{hence} \\ [E_\|] & \equiv & E_\|^+ - E_\|^- = & 0\ , & \quad \text{and} \\ 0 & = & \int_V(\nabla B)dv = & \int_{\partial V} Bd\sigma = & S(B_\perp^+ - B_\perp^-) \quad \text{hence} \\ [B_\perp] & \equiv & B_\perp^+ - B_\perp^- = & 0\ . & \end{array} \tag{9}$$

In the same manner, integrating the inhomogeneous Maxwell equations implies for the normal component of E and the tangential component of B

$$[E_\perp] \equiv E_\perp^+ - E_\perp^- = 4\pi\rho\ , \tag{10}$$

$$[B_\|] \equiv B_\|^+ - B_\|^- = 4\pi j\wedge n\ . \tag{11}$$

Similar junction conditions exist also for Einstein's equations. (Lanczos, 1922, Darmois 1927, Misner & Sharp 1964, Israel 1966, see Refs [11, 12, 13, 14]), the so called junction conditions. To obtain them, we have to split the geometrical quantities into components parallel and transverse to a given hyper–surface. This is often also done in 4d gravity (3+1 formalism or ADM formalism), where one wants to study the time evolution of the metric on a 3d spatial hypersurface. Examples are numerical relativity, or canonical quantization of gravity where the canonical fields are the spatial metric components q_{ij} and their canonical momenta are given by the extrinsic curvature, $\pi_{ij} = K_{ij}$.

There one considers spacelike hypersurfaces, *i.e.* , hypersurfaces with timelike normal n, $g(n,n) < 0$. In braneworlds we have a timelike hypersurface with spacelike normals, $g(n,n) > 0$.

Using a slicing of spacetime into 4d hyper surfaces, one can express the 5d Riemann curvature in terms of the 4d one and the extrinsic curvature. The equations are relatively simple if we write them in Gaussian coordinates (Gauss-Codazzi-Mainardi formulas, see *e.g.* [14])

$$^5R^\mu{}_{\nu\alpha\beta} = R^\mu_{\nu\alpha\beta} + K_{\nu\alpha}K^\mu_\beta - K_{\nu\beta}K^\mu_\alpha \qquad \text{(Gauss formula)} \tag{12}$$

$$^5R^4{}_{\mu\nu\alpha} = \nabla_\nu K_{\mu\alpha} - \nabla_\alpha K_{\mu\nu} \qquad \text{(Codazzi formula)} \tag{13}$$

$$^5R^4{}_{\mu4\nu} = \partial_y K_{\mu\nu} + K_{\mu\beta}K^\beta_\nu \qquad \text{(Mainardi formula)} \tag{14}$$

$$^5R = {}^5R^{\mu\nu}{}_{\mu\nu} + 2\,{}^5R^{4\mu}{}_{4\mu} = R + 2\partial_y K - K^2 - K_{\alpha\beta}K^{\alpha\beta}\ . \tag{15}$$

For the last equation we have used $\partial_y g^{\mu\nu} = 2K^{\mu\nu}$ so that

$$g^{\mu\nu}\partial_y K_{\mu\nu} = \partial_y(g^{\mu\nu}K_{\mu\nu}) - (\partial_y g^{\mu\nu})K_{\mu\nu} = \partial_y K - 2K^{\mu\nu}K_{\mu\nu}\ , \qquad K = K^\mu_\mu\ .$$

From these we can determine the 5-dimensional Einstein tensor,

$$^5G^4_4 = \frac{1}{2}\left[-R + K^2 - K^{\mu\alpha}K_{\alpha\mu}\right] \tag{16}$$

$$^5G^4_\mu = \nabla_\mu K - \nabla_\nu K^\nu_\mu \tag{17}$$

$$^5G^\mu_\nu = G^\mu_\nu + 2K^\mu_\alpha K^\alpha_\nu - KK^\mu_\nu + \partial_y K^\mu_\nu - \delta^\mu_\nu\partial_y K + \frac{1}{2}g_{\mu\nu}(K^2 + K_{\alpha\beta}K^{\alpha\beta})\ . \tag{18}$$

The derivatives wrt y indicate that in order to determine the 5-dimensional Einstein tensor, it is not sufficient to know the second fundamental form on the brane itself, but we also have to know it on both sides of the brane.

We now derive equations on the brane from the bulk Einstein equation, $G_{AB} = \kappa_5 T_{AB}$ which we assume to be valid as a low energy consequence from string theory. To identify the energy momentum tensor on the brane which contains a delta-function in y-direction, we define

$$S^A_B = \lim_{\varepsilon\to 0} \int_{y_b-\varepsilon}^{y_b+\varepsilon} T^A_B dy \ ,$$

so that

$$\lim_{\varepsilon\to 0} \int_{y_b-\varepsilon}^{y_b+\varepsilon} {}^5G^A_B dy = \kappa_5 S^A_B \ .$$

The 4d metric is continuous and also $K_{\mu\nu}$ has no delta-function in y, but possibly a jump across the brane. This means that ${}^5G^4_4$ and ${}^5G^4_\mu$ have no delta-function. Only ${}^5G_{\mu\nu}$ may have one stemming from the term $\partial_y K_{\mu\nu} - g_{\mu\nu}\partial_y K$ if $K_{\mu\nu}$ has a jump. Hence as consequence from the junction conditions, we obtain the following relations for S_{AB}

$$\begin{aligned} 0 &= S^4_4 & (19)\\ 0 &= S^4_\mu \qquad \text{and} & (20)\\ \kappa_5 S^\nu_\mu &= \left[K^\nu_\mu\right] - \delta^\nu_\mu \left[K\right] \qquad \text{or} & (21)\\ \left[K^\nu_\mu\right] &= \kappa_5 (S^\nu_\mu - \frac{1}{3}\delta^\nu_\mu S) \qquad \text{where} \quad S = S^\mu_\mu = S^A_A \ . & (22) \end{aligned}$$

3.3. Z_2 symmetry

In addition to Gaussian normal coordinates (which one can always choose, at least locally) we now assume Z_2 symmetry: the two sides of the brane are mirror images. For the metric components this implies

$$\begin{aligned} g_{\mu\nu}(x^\lambda, y_b + y) &= g_{\mu\nu}(x^\lambda, y_b - y) & (23)\\ g_{\mu 4}(x^\lambda, y_b + y) &= -g_{\mu 4}(x^\lambda, y_b - y) & (24)\\ g_{44}(x^\lambda, y_b + y) &= g_{44}(x^\lambda, y_b - y) & (25) \end{aligned}$$

The same symmetry is required for T_{AB} and any other bulk tensor field.

Under this condition we have $K_+ = -K_-$ so that the junction conditions (22) reduce to

$$2K^\mu_\nu = \kappa_5 (S^\mu_\nu - \frac{1}{3}\delta^\mu_\nu S) \ . \tag{26}$$

If the braneworld satisfies Z_2 symmetry, the brane energy momentum tensor determines the second fundamental form.

However, we now show that even with Z_2 symmetry, knowing the brane energy momentum tensor is not sufficient to determine the brane Einstein tensor. To demonstrate this we first rewrite the 4d Einstein tensor, for a general coordinate system in terms of the 5d Riemann tensor and the extrinsic curvature:

$$\begin{aligned} G_{\mu\nu} &= {}^5G_{AB} q^A_\mu q^B_\nu + {}^5R_{AB} n^A n^B g_{\mu\nu} - K^\alpha_\mu K_{\alpha\nu} + K K_{\mu\nu} \\ &\quad + \frac{1}{2} g_{\mu\nu}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \tilde{E}_{\mu\nu} \ . \end{aligned} \tag{27}$$

Here ${}^5R_{AB}n^A n^B$ corresponds to ${}^5R_{44} = {}^5R^\mu_{4\mu 4}$ in Gaussian coordinates and $\tilde{E}_{\mu\nu} \equiv {}^5R_{ABCD} n^A n^C q^D_\nu q^B_\mu$ corresponds to ${}^5R_{4\mu 4\nu}$. Eq. (27) in Gaussian coordinates is a simple consequence of the expressions (12,13,14) for the Riemann tensor. In a general coordinate system it can be found in Ref. [15, 16] (careful, the sign for $K_{\mu\nu}$ is different there). In 5 dimensions the Weyl tensor is given by

$$R_{ABCD} = \frac{2}{3}(g_{A[C}R_{D]B} - g_{B[C}R_{D]A}) - \frac{1}{6} g_{A[C}g_{D]B} R + C_{ABCD} \ . \tag{28}$$

Here $[AB]$ indicates anti-symmetrization in the indices A and B, and C_{ABCD} is the 5-dimensional Weyl tensor defined by Eq. (28). It is easy to verify that C_{ABCD} is traceless and obeys the same symmetries as the Riemann tensor, R_{ABCD}.

Inserting the 5-dimensional Einstein Eq. (3) for ${}^5G_{AB}$ and Eq. (28) in the expression containing the Ricci tensor R_{AB}, as well as in $\tilde{E}_{\mu\nu}$, we obtain the 4d brane gravity equation

$$\begin{aligned} G_{\mu\nu} &= \frac{2}{3}\kappa_5\left[T_{AB}q^A_\mu q^B_\nu + g_{\mu\nu}(T_{AB}n^An^B - \frac{1}{4}T)\right] - K^\alpha_\mu K_{\alpha\nu} + KK_{\mu\nu} \\ &\quad +\frac{1}{2}g_{\mu\nu}\left(K_{\alpha\beta}K^{\alpha\beta} - K^2\right) - E_{\mu\nu} \end{aligned} \tag{29}$$

where

$$E_{\mu\nu} = C_{ABCD}n^An^Cq^D_\nu q^B_\mu \tag{30}$$

is the 'projection' of the Weyl tensor along the brane normal. The Codazzi equation (13) gives in addition

$$\kappa_5 T_{AB}n^Aq^B_\mu = \nabla_\mu K - \nabla_\nu K^\nu_\mu \ . \tag{31}$$

Because of the last term in Eq. (29), it is not sufficient to know the bulk energy momentum tensor and initial conditions for n, g_{AB} and K_{AB} to solve the gravitational equations on the brane. In addition we need to know $E_{\mu\nu}$, components of the bulk Weyl tensor on the brane. The Weyl tensor, which is the part of the curvature which can be non-vanishing even if the energy momentum tensor vanishes, contains information on bulk gravity waves. Bulk gravity waves can flow onto, respectively be emitted from the brane and thereby affect its evolution. This information is encoded only in the full bulk initial conditions. Therefore, to determine the evolution of the brane matter and geometry, in principle we have to solve the full bulk equations! Only in situations with very special symmetries this can be avoided. However, as soon as we want to perturb such symmetric solutions we have to take into account all the bulk modes, and we do expect the solutions to differ significantly from the results of 4-dimensional perturbation theory.

3.4. Brane gravity with an empty bulk

We now exemplify the effect of the bulk Weyl tensor in the case of an empty bulk. This will be the situation which we study for the rest of these lectures. We assume that the bulk is empty up to a simple cosmological constant Λ_5.

$$T_{AB} = -\frac{\Lambda_5}{\kappa_5}g_{AB} + q_A{}^\mu q_B{}^\nu S_{\mu\nu}\delta(y-y_b) \ . \tag{32}$$

where $S_{\mu\nu}$ is the energy momentum tensor on the brane. It consists of a brane tension λ and the matter energy momentum tensor $\tau_{\mu\nu}$.

$$S_{\mu\nu} = \lambda g_{\mu\nu} + \tau_{\mu\nu} \ . \tag{33}$$

The junction conditions read

$$[g_{\mu\nu}] = 0 \qquad \text{first junction condition,} \tag{34}$$

$$[K_{\mu\nu}] = -\kappa_5\left(S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S\right) \qquad \text{second junction condition.} \tag{35}$$

Z_2 symmetry requires that

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = -\frac{\kappa_5}{2}\left(S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S\right) \ . \tag{36}$$

Inserting our ansatz for T_{AB} in the brane gravity equation (29) and using the second junction condition to eliminate the second fundamental form, we obtain

$$G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \kappa_4\tau_{\mu\nu} + \kappa_5^2\sigma_{\mu\nu} - E_{\mu\nu} \ , \tag{37}$$

with

$$\Lambda_4 = \frac{1}{2}(\Lambda_5 + \frac{\kappa_5^2}{6}\lambda^2) \ , \tag{38}$$

$$\kappa_4 = \kappa_5^2\lambda/6 = 2/M_4^2 \tag{39}$$

$$\sigma_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}\tau^2 \ , \tag{40}$$

and, as before $E_{\mu\nu}$ denotes the projected 5d Weyl tensor, evaluated on either side of the brane (but not exactly on the brane where it may be ill-defined). The quantities κ_4 and M_4 denote the 4–dimensional gravitational coupling constant and Planck mass respectively and $\tau = \tau^\mu_\mu$ is the trace of the matter energy momentum tensor. The relation between the 4- and 5-dimensional Planck mass in the braneworld approach is now obtained as follows: using $\kappa_5 \sim M_5^{-3}$ and $\kappa_4 \sim M_4^{-2}$, Eq. (39) shows that $M_4^2 \simeq M_5^6/\lambda$. Using that the 4-dimensional cosmological constant is small, $\Lambda_4 \ll |\Lambda_5|$, we have $\lambda \simeq \sqrt{-6\Lambda_5}/\kappa_5 \simeq \sqrt{-6\Lambda_5}M_5^3$, so that $M_4^2 \simeq M_5^3\sqrt{-\Lambda_5} = M_5^3 L$ with $L \simeq \sqrt{-\Lambda_5}$.

In the limit $\lambda\tau_{\mu\nu} \gg \tau_{\mu\alpha}\tau^\alpha_\nu$ we recover the 4–dimensional Einstein equation if $E_{\mu\nu}$ is negligible. The existence of this limit depends crucially on the existence of a 4–dimensional brane tension. In order for the 4d gravitational coupling constant to be positive, the brane tension must be positive, $\lambda > 0$. At high energy densities (in the early universe) the quadratic term $\sigma_{\mu\nu}$ can become dominant and modify the dynamics (the expansion law of the universe). In general, there is an additional part, $E_{\mu\nu}$, carrying information from the bulk geometry and evolution, which can affect the brane evolution in a crucial way.

3.5. Energy momentum conservation

The Codazzi equation (13) together with Z_2 symmetry implies

$$\nabla_\mu K - \nabla_\nu K^\nu_\mu = \kappa_5 T^4_\mu = 0 \,. \tag{41}$$

With the Gauss equation (12) and Z_2 symmetry this ensures energy and momentum conservation on the brane,

$$\nabla_\nu \tau^\nu_\mu = 0. \tag{42}$$

From the 4-dimensional contracted Bianchi identities we obtain in addition

$$\nabla_\nu E^\nu_\mu = \kappa_5^2 \nabla_\nu \sigma^\nu_\mu \,. \tag{43}$$

Hence, the longitudinal part of $E_{\mu\nu}$ is fully determined by the matter content of the brane, while the transverse traceless part is not specified:

$$E_{\mu\nu} = E^{(TT)}_{\mu\nu} + E^{(L)}_{\mu\nu} \tag{44}$$

where $\nabla_\nu E_\mu^{\nu\ (TT)} = 0$ and

$$E^{(L)}_{\mu\nu} = \frac{1}{2}(\nabla_\mu\theta_\nu + \nabla_\nu\theta_\mu) \quad \text{with} \quad \nabla_\mu\theta^\mu = 0 \,.$$

Inserting this in Eq. (43), we obtain

$$\nabla^2\theta_\mu = \frac{\kappa_5^2}{2}\left[-\tau_{\alpha\beta}(\nabla_\mu\tau^{\alpha\beta} + \nabla^\alpha\tau^\beta_\mu) + \frac{1}{3}(\nabla_\alpha\tau)(\tau^\alpha_\mu - q^\alpha_\mu\tau)\right] \,. \tag{45}$$

For given initial conditions, this equation has always a unique solution θ_μ on the brane which determines $E^{(L)}_{\mu\nu}$. However, the transverse part, $E^{(TT)}_{\mu\nu}$ is not determined by the brane energy momentum tensor; it comes from bulk gravity waves. Only if $E^{(TT)}_{\mu\nu} = 0$ does the brane energy momentum tensor determine the brane Einstein tensor. As we shall see, already in quite simple situations this is not the case.

4. THE RANDALL SUNDRUM MODEL

We now consider an Anti-de Sitter (AdS) bulk, $\Lambda_5 < 0$ and would like to obtain Minkowski space on the brane. Since AdS is conformally flat, $E_{\mu\nu} = 0$. A Minkowski brane can be achieved by setting $\tau_{\mu\nu} = 0$. If in addition the brane tension is related to the 5–dimensional coupling constant and the cosmological constant by

$$\lambda^2\kappa_5^2/6 = -\Lambda_5 \,, \tag{46}$$

Eq. (37) implies $G_{\mu\nu} = 0$. Eq. (46) is it the Randall–Sundrum (RS) fine tuning condition [17, 18]. Small deviations from the RS condition lead to an exponentially expanding/contracting brane. The 4-dimensional gravitational constant becomes

$$\kappa_4 = \frac{\lambda\kappa_5^2}{6} = -\frac{\Lambda_5}{\lambda} > 0 \,, \quad \text{or, equivalently} \quad \lambda = -\frac{\Lambda_5}{\kappa_4} \,, \quad \text{and} \quad \kappa_4 = \kappa_5\sqrt{\frac{-\Lambda_5}{6}} \,. \tag{47}$$

4.1. The metric

The following coordinates for (a part of) Anti-de Sitter will be useful for us:

$$ds^2 = e^{-2|z|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \quad \text{Gaussian coordinates} \tag{48}$$

$$ds^2 = \left(\frac{\ell}{y}\right)^2 \left(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\right) \qquad |y| > \ell \quad \text{conformal coordinates.} \tag{49}$$

Einstein's equations, $G_{AB} = -\Lambda_5 g_{AB}$, give $\Lambda_5 = -\frac{6}{\ell^2}$. The RS fine tuning requires $\lambda = \sqrt{-6\Lambda_5/\kappa_5^2}$.

In their first model [17] (RS1 model) Randall and Sundrum propose two branes, the first positioned at $z = 0$ is called the hidden brane, and the second positioned at $z = k\ell$ is called the visible brane and represents our Universe. The gravitational force on the visible brane is suppressed by the factor $\exp(-2k)$ w.r.t. the hidden brane, leading to an enhancement by a factor $\exp(k)$ of the apparent Planck mass. However, also this two brane model contains a gravi-scalar (also called radion) which has to obtain a mass by some non-gravitational mechanism.

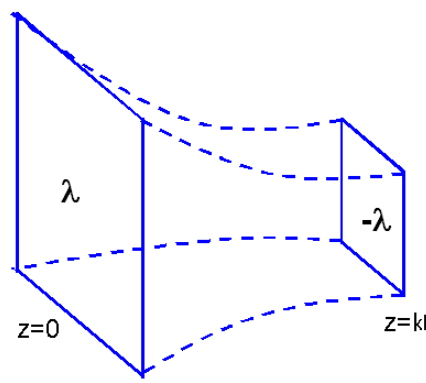

FIGURE 3. The RS1 model with two branes. The visible brane (our Universe) at $z = k\ell$ and the hidden brane at $z = 0$.

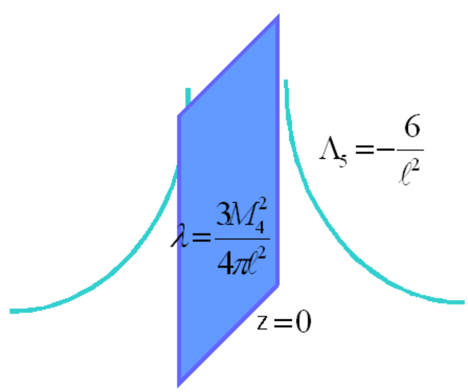

FIGURE 4. The RS2 model with only one brane at $z = 0$.

This problem is resolved in the second model [18] (RS2 model). There, our universe is located on the brane at $z = 0$ corresponding to $y = \ell$ and no second brane is present. The apparent 4-dimensional Planck mass as measured on the brane on scales much larger than ℓ is then again given by

$$\lambda = -\frac{\kappa_4}{\Lambda_5} = \frac{3M_4^2}{\ell^2}\,. \qquad \text{Furthermore,} \tag{50}$$

$$\kappa_4^2 = -\frac{\Lambda}{6}\kappa_5^2 = \ell^{-2}\kappa_5^2 \qquad \text{so that} \qquad M_4^2 = M_5^3\ell\,. \tag{51}$$

Hence in the RS2 model, the AdS curvature scale ℓ enters in the same way as L for cylindric Kaluza-Klein models. Since the scale ℓ is limited by present day micro gravity experiments which have not detected any deviation from Newton's law [2], we have

$$\ell < 0.1\ mm \quad \text{implying} \quad \lambda > (1\ TeV)^4, \text{ hence} \quad M_5 = (M_4^2/\ell)^{1/3} > 10^5\ TeV\ .$$

4.2. Gravity waves in the RS2 model

In order to see that the radion mode is absent in the non-compact RS2 model, we consider perturbations to the AdS metric. It is easy to show that one can always choose a gauge (local coordinate system) so that the perturbed metric is of the form

$$ds^2 = \left(\frac{\ell}{y}\right)^2 \left[-(1+2\Psi)dt^2 - 4\Sigma_i dt dx^i + ((1-2\Phi)\delta_{ij} + 2H_{ij})dx^i dx^j + 4\Xi_i dy dx^i \right. \tag{52}$$

$$\left. -4\mathscr{B} dt dy + (1+2\mathscr{C})dy^2\right] \tag{53}$$

Here H_{ij} and Σ_i, Ξ_i are transverse (*i.e.* divergence free) and H_{ij} is traceless. In other words, $\partial_i\Sigma_i = \partial_i\Xi_i = \partial_i H_{ij} = H_i^i = 0$. As we shall see there are 5 homogeneous modes in these 10 physical perturbation variables corresponding to the 5 gravity wave modes in 5 dimensions. We consider the homogeneous 5d wave equation in the bulk.

Since the 3-brane is homogeneous, scalar-, vector- and tensor degrees of freedom decouple and we can consider them in turn (for more details see Section 5).

4.2.1. Tensor perturbations

We first consider only the tensor H_{ij}, so that the perturbed metric is given by

$$ds^2 = \left(\frac{\ell}{y}\right)^2 \left(-dt^2 + (\delta_{ij} + 2H_{ij})dx^i dx^j + dy^2\right)\ . \tag{54}$$

We Fourier transform H_{ij} in the 3-dimensional $\mathbf{x}$ coordinates and consider one mode with fixed wave vector $\mathbf{k}$, so that $H_{ij}(t,y,\mathbf{x}) = H_{ij}(t,y)\exp(i\mathbf{k}\cdot\mathbf{x})$. Since spacetime is isotropic and homogeneous in $\mathbf{x}$, different $\mathbf{k}$–modes do not couple. The bulk Einstein equations, $\delta G_{AB} = -\Lambda\delta g_{AB}$, for the Fourier mode k then give

$$\left(\partial_t^2 + k^2 - \partial_y^2 + \frac{3}{y}\partial_y\right) H_{ij} = 0\ . \tag{55}$$

The general solution to this equation is of the form

$$H_{ij} = h_m e_{ij} \quad \text{with } h_m = e^{i\omega t}(my)^2\left[AJ_2(my) + BY_2(my)\right]\ . \tag{56}$$

Here e_{ij} is the polarization tensor, $k^i e_{ij} = e^i{}_j = 0$ and $\omega^2 = m^2 + k^2$. The separation constant m^2 is arbitrary and can, in principle also be negative. J_2 and Y_2 are the Bessel functions of order 2. They are oscillating and decaying. Bessel functions represent "δ–function normalizable" perturbations like harmonic waves in flat space, in the sense that [19, 20]

$$\int_0^\infty h_m h_{m'} \frac{dy}{m^2 y^3} = m\delta(m - m')\ . \tag{57}$$

These are just the ordinary gravity modes of 4-dimensional mass m without a mass gap which are discussed in the original RS paper [18]. To find the correct weight $1/y^3$, we use that h_m satisfies

$$\left(\square_4 + \partial_y^2 - \frac{3}{y}\partial_y\right) h_m = 0\ , \tag{58}$$

and thus $\check{h} = h_m/y^{3/2}$ satisfies the equation of motion of a scalar field in a flat 5-dimensional spacetime (with y–dependent mass term),

$$\left(\square_4 + \partial_y^2 - \frac{15}{4y^2}\right)\check{h} = 0\ . \tag{59}$$

This mode has to be normalizable w.r.t the Minkowski metric (no additional weight).

As we have mentioned above, m^2 is arbitrary and can also be chosen negative. However, if $m^2 < 0$ and therefore m is imaginary, it is more useful to decompose the two independent solutions in the form

$$h_m = e^{i\omega t}(|m|y)^2\left[CK_2(|m|y) + DI_2(|m|y)\right] \ , \tag{60}$$

where K_2 and I_2 are the modified Bessel functions of order 2. Considering the behavior of the Bessel functions, one sees that I_2 grows exponentially (see Fig. 5) and is clearly not normalizable (*i.e.* not square integrable with some weight which is a power law in y). Therefore, this mode is unphysical and we have to set $D = 0$. It is important to note that $\omega^2 = k^2 + m^2$ can become negative in this case leading to $\omega = \pm i|\omega|$. In other words, negative mass solutions become exponentially growing 'tachyonic' instabilities! It is still unclear whether these tachyonic modes are relevant for cosmological braneworlds.

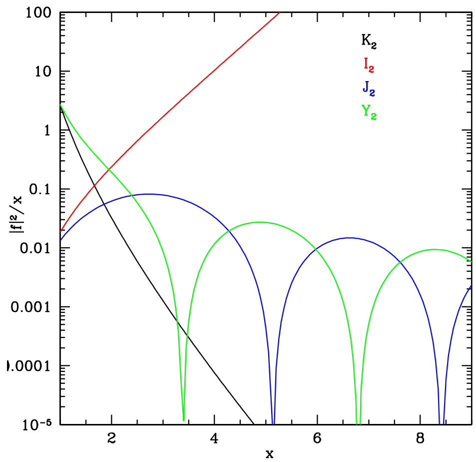

FIGURE 5. The Bessel functions of order 2.

In the limit $m \to 0$ only the Y_2–mode survives and we obtain $h_{(m=0)} = C\exp(\omega t)$ independent of the coordinate y. This zero-mode is normalizable with respect to the measure dy/y^3.

The general solution for a tensor perturbation is of the form

$$h = h_0 + \int_{-\infty}^{\infty} h_m dm^2 \ . \tag{61}$$

At the brane position, $y = y_b = \ell$, the perturbations must satisfy the junction condition (36). These represent boundary conditions for the perturbations H_{ij} in the bulk. On the right hand side of Eq. (36), we can in principle have an arbitrary perturbation of the matter energy momentum tensor. However, the only non-vanishing term of the tensor contribution to $\tau_{\mu\nu}$ is the traceless part of τ_{ij}, i.e. the anisotropic stress on the brane, $\Pi^{(T)}_{ij}$. A short computation shows

$$\begin{aligned} \delta K_{ij}\big|_{y_b} &= \left(\frac{2}{\ell}H_{ij} - \partial_y H_{ij}\right)\Big|_{y_b} \ , \quad \text{hence} \\ -2(\partial_y H_{ij})\big|_{y_b} &= \kappa_5 \Pi^{(T)}_{ij} \ , \end{aligned} \tag{62}$$

where $\Pi^{(T)}$ are tensor–type anisotropic stresses on the brane.

Let us first consider the homogeneous case $\Pi^{(T)} \equiv 0$. For $m^2 > 0$, the solutions are of the form

$$h = \exp(\pm i\omega t)(my)^2\left[AJ_2(my) + BY_2(my)\right] \ . \tag{63}$$

The junction condition (62) then requires

$$B = -A\frac{J_1(m\ell)}{Y_1(m\ell)} \simeq \frac{\pi}{4}(m\ell)^2 A \ , \tag{64}$$

where the last expression is a good approximation for $m\ell \ll 1$. This is precisely the result of Randall and Sundrum [18]. It is not modified even if we allow for the negative mass modes, $-m^2 > 0$, because a physical solution has to be of the form

$$h = C\exp(\pm t\sqrt{-m^2-k^2})(|m|y)^2 K_2(|m|y) \;, \quad \text{with} \quad \partial_y h = |m|C\exp(\pm t\sqrt{-m^2-k^2})(|m|y)^2 K_1(|m|y) \;, \tag{65}$$

and since K_1 has no zero, the junction condition (62) requires $C = 0$.

But in a realistic brane universe, $\Pi^{(T)}$ is not exactly zero. In cosmology, it is typically just a factor 10 smaller than other perturbations of the energy momentum tensor on the brane. We therefore cannot require $C \equiv 0$. However, as long as $\Pi^{(T)}$ remains small, we do not expect the unstable modes to be present, so that $C(k,m) = 0$ for $k^2 < -m^2$. Within the framework of first order perturbation theory, the $\Pi^{(T)}$ modes satisfy a Minkowski equation of motion and therefore they do not grow exponentially. Hence in this case, the K–mode can only be excited for $\omega^2 = k^2 + m^2 > 0$, *i.e.* $k^2 > -m^2$. However, it is not clear whether this remains true to second order, where the evolution of H feeds back in the equation of motion for $\Pi^{(T)}$. Actually, within a toy model, it has been shown that the full, non-linear evolution can be exponentially unstable even if the linear equations do not excite the unstable mode [21].

4.2.2. Vector perturbations

We now consider vector perturbations only, so that, in generalized longitudinal gauge, the metric takes the form

$$ds^2 \;=\; \left(\frac{\ell}{y}\right)^2 \left(-dt^2 - 4\Sigma_i dt dx^i + \delta_{ij} dx^i dx^j + 4\Xi_i dy dx^i + dy^2\right) \tag{66}$$

The bulk Einstein equations for a mode $\mathbf{k}$ of the vector perturbations Σ and Ξ are

$$\left(\partial_y^2 - \frac{3}{y}\partial_y\right)\Sigma = \left(\partial_t^2 + k^2\right)\Sigma \;, \tag{67}$$

$$\left(\partial_y^2 - \frac{3}{y}\partial_y + \frac{3}{y^2}\right)\Xi = \left(\partial_t^2 + k^2\right)\Xi \;, \tag{68}$$

$$\left(\partial_y - \frac{3}{y}\right)\Xi = -\partial_t \Sigma \;, \tag{69}$$

where Σ and Ξ are transverse vectors, $(\mathbf{k}\cdot\Sigma) = (\mathbf{k}\cdot\Xi) = 0$. The constraint equation (69) fixes the relative amplitudes of Σ and Ξ, showing that there is only one independent vector perturbation in the bulk (the "gravi-photon"). One can check that these equations are consistent, e.g. with the master function approach of Ref. [22].

As in the tensor case, the solutions are Bessel functions of order two (and one). Considering just one component $\Sigma = \Sigma_i$ one obtains the expected oscillatory modes for positive mass-squared, $m^2 > 0$,

$$\Sigma = \exp(\pm i\omega t)(my)^2\left[AJ_2(my) + BY_2(my)\right] \;, \tag{70}$$

$$\Xi = \frac{\pm i\omega}{m}\exp(\pm i\omega t)(my)^2\left[AJ_1(my) + BY_1(my)\right] \;, \tag{71}$$

where $\omega = \sqrt{m^2+k^2}$. These solutions have been found in Ref. [23]. For a negative mass-square, $m^2 < 0$, we obtain again tachyonic solutions. Like in the tensor case, the solution containing the modified Bessel function I_ν cannot be accepted as it is exponentially growing and thus represents a non-normalizable mode. However, the K_ν-solution is exponentially decaying and perfectly acceptable. For tachyonic vector perturbations with $\omega^2 = m^2 + k^2 < 0$ we have

$$\Sigma = C\exp(\pm|\omega|t)(|m|y)^2 K_2(|m|y) \;, \tag{72}$$

$$\Xi = \frac{\pm|\omega|}{|m|}C\exp(\pm|\omega|t)(|m|y)^2 K_1(|m|y) \;. \tag{73}$$

For large enough scales, $-m^2 > k^2$, these solutions again grow exponentially.

The boundary conditions at the brane relate these perturbations to the brane energy momentum tensor. For the energy momentum tensor on the brane, the vector degrees of freedom are defined according to

$$\left(S_{\mu\nu}\right) = \begin{pmatrix} 0 & V_j \\ V_i & \Pi_{ij}^{(V)} \end{pmatrix} - \lambda\left(q_{\mu\nu}\right) \ , \tag{74}$$

where V_i and $\Pi_i^{(V)}$ are divergence-free vector fields and $\Pi_{ij}^{(V)} \equiv [\partial_j \Pi_i^{(V)} + \partial_i \Pi_j^{(V)}]$. The first junction condition simply requires that Σ be continuous at the brane, which it is since the (modified) Bessel functions of even index are even functions. The second junction condition results in (for a detailed derivation, see [24])

$$\partial_t \Xi + \partial_y \Sigma = \kappa_5 V \ , \tag{75}$$

$$\Xi = \kappa_5 \Pi^{(V)} \ , \tag{76}$$

$$\partial_t V = -k^2 \Pi^{(V)} \ . \tag{77}$$

The last equation follows from (75) and (76) and the bulk equations (67)–(69). It represents momentum conservation on the brane, which is guaranteed as long as we have vanishing energy flux off the brane and Z_2–symmetry.

Like for tensor perturbations, we consider homogeneous solutions, setting $\Pi^{(V)} \equiv V \equiv 0$. This requires $\Xi(|m|\ell) = 0$, hence

$$B = -A\frac{J_1(m\ell)}{Y_1(m\ell)} \quad \text{for } m^2 > 0 \ , \tag{78}$$

$$C \equiv 0 \quad \text{for } m^2 < 0 \ . \tag{79}$$

Equation (75) is then identically satisfied.

However, it seems more realistic to allow a small but non-vanishing anisotropic stress contribution $\Pi^{(V)}$ and corresponding vorticity V. In this case, again, we can have solutions with $C \neq 0$ which can grow exponentially in time; hence small initial data can lead to an exponential instability like for tensor perturbations.

Using the normalization condition (57) for the $m = 0$ mode of the variable $\Xi \propto y$ (this is the one which enters as dynamical variable in the perturbed action, see [25]), one finds that $\int |\Xi|^2/y^3 dy$ diverges logarithmically. Contrary to the tensor case, the vector zero-mode is not normalizable. Therefore, on the brane there is only the ordinary massless spin–2 graviton, but there are a continuous infinity of massive spin–2 and spin–1 particles (the modes discussed here, with $m \neq 0$).

4.2.3. Scalar perturbations

We now discuss the most cumbersome, the scalar sector. Scalar–type metric perturbations in the bulk are of the form

$$ds^2 = \frac{\ell^2}{y^2}\left[-(1+2\Psi)dt^2 - 4\mathscr{B}dtdy \right.$$
$$\left. +(1-2\Phi)\delta_{ij}dx^i dx^j + (1+2\mathscr{C})dy^2\right] \ . \tag{80}$$

The bulk Einstein perturbation equations for the mode $\mathbf{k}$ become, after some manipulations and introducing the combination $\Gamma \equiv \Phi + \Psi$ (see Ref. [21]),

$$\Phi - \Psi = \mathscr{C} \; , \tag{81}$$

$$\left(\partial_y^2 - \frac{3}{y}\partial_y\right)\Gamma = \left(\partial_t^2 + k^2\right)\Gamma \; , \tag{82}$$

$$\left(\partial_y^2 - \frac{3}{y}\partial_y + \frac{4}{y^2}\right)\mathscr{C} = \left(\partial_t^2 + k^2\right)\mathscr{C} \; , \tag{83}$$

$$\partial_y\Phi + \left(\partial_y - \frac{3}{y}\right)\mathscr{C} = -\partial_t\mathscr{B} \; , \tag{84}$$

$$\frac{3}{y}\left(\partial_y - \frac{2}{y}\right)\mathscr{C} = 3\partial_t^2\Phi + k^2(\Phi + \mathscr{C}) \; , \tag{85}$$

$$3\partial_t\left(\partial_y\Phi - \frac{1}{y}\mathscr{C}\right) = k^2\mathscr{B} \; , \tag{86}$$

$$\partial_t\left(2\Phi - \mathscr{C}\right) = \left(\partial_y - \frac{3}{y}\right)\mathscr{B} \; . \tag{87}$$

Clearly these equations are not all independent, Eqs. (86)–(87) are identically satisfied if Eqs. (81)–(85) are. The solutions are obtained as for tensor and vector perturbations. For a positive mass-square, $m^2 > 0$, we find ($\omega = \sqrt{m^2 + k^2}$)

$$\Gamma = \exp(\pm i\omega t)(my)^2\left[A'J_2(my) + B'Y_2(my)\right] \; , \tag{88}$$

$$\mathscr{C} = \exp(\pm i\omega t)(my)^2\left[AJ_0(my) + BY_0(my)\right] \; , \tag{89}$$

$$\begin{aligned}\Phi = \frac{1}{2}\exp(\pm i\omega t)(my)^2 & \left[A'J_2(my) + B'Y_2(my)\right. \\ & \left. + AJ_0(my) + BY_0(my)\right] \; ,\end{aligned} \tag{90}$$

$$\begin{aligned}\Psi = \frac{1}{2}\exp(\pm i\omega t)(my)^2 & \left[A'J_2(my) + B'Y_2(my)\right. \\ & \left. - AJ_0(my) - BY_0(my)\right] \; ,\end{aligned} \tag{91}$$

$$\begin{aligned}\mathscr{B} = \frac{\pm i m^3 y^2}{2\omega} & \exp(\pm i\omega t)\times \\ & \left[(A' - 3A)J_1(my) + (B' - 3B)Y_1(my)\right] \; ,\end{aligned} \tag{92}$$

$$\text{with} \quad A' = 3A\frac{m^2}{m^2 + 2\omega^2} \; , \quad \text{and} \quad B' = 3B\frac{m^2}{m^2 + 2\omega^2} \; . \tag{93}$$

For a negative mass-square, $m^2 < 0$, we obtain ($\omega = \sqrt{-m^2 - k^2}$)

$$\Gamma = \exp(\pm\omega t)(|m|y)^2 C'K_2(|m|y) \; , \tag{94}$$

$$\mathscr{C} = \exp(\pm\omega t)(|m|y)^2 CK_0(|m|y) \; , \tag{95}$$

$$\begin{aligned}\Phi = \frac{1}{2} & \exp(\pm\omega t)(|m|y)^2\times \\ & \left[C'K_2(|m|y) + CK_0(|m|y)\right] \; ,\end{aligned} \tag{96}$$

$$\begin{aligned}\Psi = \frac{1}{2} & \exp(\pm\omega t)(|m|y)^2\times \\ & \left[C'K_2(|m|y) - CK_0(|m|y)\right] \; ,\end{aligned} \tag{97}$$

$$\mathscr{B} = \frac{\pm|m|^3 y^2}{2\omega}\exp(\pm\omega t)[C' + 3C]K_1(|m|y) \; , \tag{98}$$

$$\text{with} \quad C' = -3C\frac{|m|^2}{|m|^2 + 2\omega^2} \; , \tag{99}$$

where we have already used that the I–mode is not normalizable and therefore cannot contribute. Like for vector and tensor perturbations, we find again tachyonic solutions with $m^2 < 0$ which represent an exponential instability for sufficiently small wave numbers k (large scales).

Determining the boundary conditions via the first and second junction conditions now requires a bit more care. Since we have already fully specified our coordinate system by the adopted choice of perturbation variables, we must allow for brane bending. We cannot fix the brane at $y_{\mathrm{b}} = \ell$, but we must allow for $y_{\mathrm{b}}^+ = \ell + \mathscr{E}$ and $y_{\mathrm{b}}^- = -\ell - \mathscr{E}$, respectively. Fortunately, $\mathscr{E}$ is a scalar quantity and brane bending therefore does not affect vector and tensor perturbations. The anti-symmetry $y_{\mathrm{b}}^+ = -y_{\mathrm{b}}^-$ is an expression of Z_2–symmetry. The introduction of the new perturbation variable $\mathscr{E}(x^\mu)$ describing brane bending enters the exressions for the first and second fundamental forms. From Eq. (5), we obtain $q_{\mu\nu} = g_{\mu\nu}$ to first order, which implies that Φ and Ψ, hence $\mathscr{C}$, have to be continuous. At the brane position, the perturbed components of the extrinsic curvature (8) are

$$\delta K_{00} = \frac{1}{\ell}\left[\Phi - 3\Psi + 2\frac{\mathscr{E}}{\ell}\right] + \partial_y\Psi - 2\partial_t\mathscr{B} + \partial_t^2\mathscr{E} \ , \tag{100}$$

$$\delta K_{0j} = \partial_j\left(\partial_t\mathscr{E} - \mathscr{B}\right) \ , \tag{101}$$

$$\delta K_{ij} = \left[\frac{1}{\ell}\left(\Psi - 3\Phi - 2\frac{\mathscr{E}}{\ell}\right) + \partial_y\Phi\right]\delta_{ij} + \partial_i\partial_j\mathscr{E} \ . \tag{102}$$

For the energy momentum tensor on the brane, we parameterize the four degrees of freedom according to

$$\left(S_{\mu\nu}\right) = \begin{pmatrix} \rho & v_j \\ v_i & p\delta_{ij} + \Pi_{ij}^{(S)} \end{pmatrix} - \lambda\left(q_{\mu\nu}\right) \ , \tag{103}$$

where $v_i \equiv \partial_i v$ and $\Pi_{ij}^{(S)} \equiv \left(\partial_i\partial_j - \frac{1}{3}\Delta\delta_{ij}\right)\Pi^{(S)}$. With Eqs. (100)–(102), the second junction condition reads

$$\frac{1}{\lambda}\left(2\rho + 3p\right) = \Phi - \Psi + \ell\partial_t\left(\partial_t\mathscr{E} - 2\mathscr{B}\right) + L\partial_y\Psi \ , \tag{104}$$

$$\frac{3}{\lambda\ell}v = \partial_t\mathscr{E} - \mathscr{B} \ , \tag{105}$$

$$\frac{3}{\lambda\ell}\Pi^{(S)} = \mathscr{E} \ , \tag{106}$$

$$\frac{1}{\lambda}\left[\rho - \Delta\Pi^{(S)}\right] = \Psi - \Phi + \ell\partial_y\Phi \ . \tag{107}$$

Combining the time derivative of Eq. (105) with Eqs. (104), (84) and (107), we obtain momentum conservation on the brane,

$$\partial_t v = \frac{2}{3}\Delta\Pi^{(S)} + p \ . \tag{108}$$

Similar manipulations imply energy conservation on the brane,

$$\partial_t\rho = \Delta v \ . \tag{109}$$

Like for tensor and vector perturbations, we look for solutions with vanishing brane matter. Setting $\Pi^{(S)} \equiv \rho \equiv P \equiv v \equiv 0$ forbids brane bending, $\mathscr{E} = 0$. Then Eq. (105) implies $\mathscr{B}(m\ell) = 0$, thus

$$B' - 3B = -(A' - 3A)\frac{J_1(m\ell)}{Y_1(m\ell)} \quad \text{for} \quad m^2 > 0 \ , \tag{110}$$

$$C' + 3C = 0 \quad \text{for} \quad m^2 < 0 \ . \tag{111}$$

The other equations are all satisfied if we require separately

$$\frac{B}{A} = \frac{B'}{A'} = -\frac{J_1(m\ell)}{Y_1(m\ell)} \quad \text{for} \quad m^2 > 0 \ , \tag{112}$$

$$C = C' \equiv 0 \quad \text{for} \quad m^2 < 0 \ . \tag{113}$$

Since $B/A = B'/A'$, equations (110) are (112) are equivalent.

As for vector perturbations, the $m = 0$ scalar mode is not normalizable. Like for tensor and vector perturbations, we have found “scalar gravitons” which appear on the brane as massive particles. If the brane matter is unperturbed, only oscillating $m^2 > 0$ solutions are possible. However, if we allow for non-vanishing matter perturbations on the brane, we can have $C \neq 0$ and the tachyonic modes $m^2 < 0$ can appear exactly like in the tensor and vector sectors.

It is not surprising that the same instability appears in the scalar, vector and tensor sectors, because all modes describe the same bulk particle, the five-dimensional graviton.

4.3. Green’s function, correction to the Newtonian potential

We want to determine the modification to Newton’s law in the RS2 model. Since the extra dimension is not compact and there are massive (homogeneous) modes of all masses $m^2 > 0$, we expect a modification which is not exponentially suppressed.

The 5d Green’s function is defined by

$$\nabla^2 G(x,x) = \delta^5(x-x) ,$$

where ∇^2 is the 5d d’Alembertian in AdS spacetime and we have to glue together a Z_2–symmetric solution on both sides which satisfies the homogeneous junction condition. We can obtain the retarded Green’s function in the standard way from the homogeneous solutions of the equation (see *e.g.* [26]):

$$G_R(x,x') = -\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \left[\frac{y^{-2}y'^{-2}\ell^3}{k^2 - (\omega+i\varepsilon)^2} + \int_0^\infty dm \frac{u_m(y)u_m(y')}{m^2 + k^2 - (\omega+i\varepsilon)^2} \right] .$$

The first term comes from the $m = 0$ solution and the functions u_m are the properly normalized massive modes,

$$u_m(y) = \sqrt{\frac{m\ell}{2}} \frac{J_1(m\ell)Y_2(my) - Y_1(m\ell)J_2(my)}{\sqrt{J_1(m\ell)^2 + J_1(m\ell)^2}} .$$

The general retarded solution for a given energy momentum tensor $\tau_{\mu\nu}$ on the brane is now of the form

$$h_{\mu\nu}(x) = -2\kappa_5 \int d^4x \, G_R(x,x') S_{\mu\nu}(x') .$$

For a stationary matter distribution it is simpler to use the Green’s function of the spatial Laplacian which is related to G_R via integration over time

$$G(\mathbf{x},y,\mathbf{x}',y') = \int_{-\infty}^{\infty} dt' G_R(x,x') = -\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \left[\frac{y^{-2}y'^{-2}\ell^3}{\mathbf{k}^2} + \int_0^\infty dm \frac{u_m(y)u_m(y')}{m^2 + \mathbf{k}^2} \right] , \quad (114)$$

$$= \frac{-y^{-2}y'^{-2}\ell^3}{4\pi r} + \frac{1}{2\pi} \int_0^\infty dm u_m(y) u_m(y') \exp(-mr) . \quad (115)$$

On the brane, $y = y' = \ell$, the first term gives the usual $1/r$ behavior, $r = |\mathbf{x} - \mathbf{x}'|$. Expanding the second term to lowest order in ℓ/r we obtain

$$G(\mathbf{x},\ell,\mathbf{x}',\ell) \simeq \frac{-1}{4\pi\ell r} \left[1 + \frac{\ell^2}{2r^2} + \cdots \right] .$$

This determines the Newtonian potential of a point mass on the brane with mass M,

$$\kappa_5 M G \simeq \frac{-\kappa_4 M}{4\pi r} \left[1 + \frac{\ell^2}{2r^2} + \cdots \right] . \quad (116)$$

Since the extra dimension is non-compact, the correction is not exponentially suppressed but only as a power law. Away from the wall, the potential at large separation is given by

$$G(\mathbf{x},\ell,\mathbf{x}',\ell+y) \simeq \frac{-\ell}{8\pi(\ell+y)^2} \frac{2r^2 + 3y^2}{(r^2+y^2)^{3/2}} \quad (117)$$

The equipotential lines are shown in Fig 6. This formulas have been derived in Ref. [27] from which also Fig. 6 is drawn.

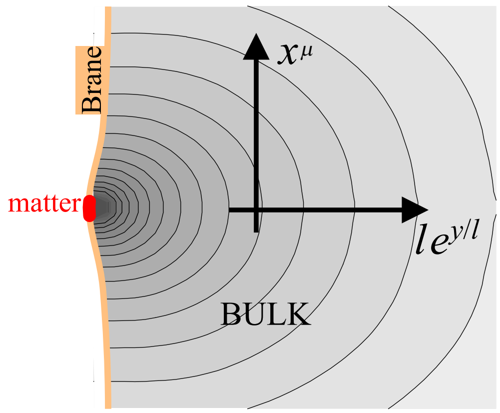

FIGURE 6. The equi-potential lines for the gravitational potential from a point mass on the brane in RS2. Figure from [27].

5. COSMOLOGICAL PERTURBATION THEORY IN 4 DIMENSIONS

Before studying brane cosmology and 5d effects on cosmological perturbations, I present a brief introduction to 4d cosmological perturbation theory and some aspects of 4d cosmology. Much more details can be found *e.g.* in [28, 29]. Some knowledge of 4d cosmology is however assumed (Friedmann equations etc. as they can be found in the first chapter of standard textbooks on cosmology like Refs. [30] or [29]). Students who are familiar with this subject may skip this section.

5.1. Perturbation variables

The observed Universe is not perfectly homogeneous and isotropic. Matter is arranged in galaxies and clusters of galaxies and there are large voids in the distribution of galaxies. Let us assume, however, that these inhomogeneities lead only to small variations of the geometry which we shall treat in first order perturbation theory. For this we define the perturbed geometry by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \ , \qquad \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2 \left(-d\eta^2 + \gamma_{ij} dx^i dx^j\right) \ . \tag{118}$$

Here $\bar{g}_{\mu\nu}$ is the unperturbed Friedmann metric, $a(\eta)$ is the scale factor, η denotes conformal time and γ_{ij} is the 3d metric for a space of constant curvature K. The perturbations are assumed to be small, $|h_{\mu\nu}| \ll 1$. The energy momentum tensor is given by

$$T^\mu_\nu = \overline{T}^\mu_\nu + \theta^\mu_\nu, \qquad \overline{T}^0_0 = -\bar{\rho}, \qquad \overline{T}^i_j = \bar{p}\delta^i_j \qquad |\theta^\mu_\nu|/\bar{\rho} \ll 1. \tag{119}$$

The background energy density ρ and pressure p satisfy the Friedmann equations,

$$\mathscr{H}^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 + K \quad = \quad \frac{8\pi G_4}{3} a^2 \bar{\rho} + \frac{1}{4} a^2 \Lambda_4 \tag{120}$$

$$\dot{\rho} = -3(\bar{\rho} + \bar{p})\mathscr{H} \ , \tag{121}$$

where an over-dot denotes the derivative w.r.t. conformal time η.

Without loss of generality we can choose the so-called longitudinal gauge so that perturbations of the metric are of the form

$$(h_{\mu\nu}) = \begin{pmatrix} -(1+2\Psi) & B_i \\ B_i & (1-2\Phi)\gamma_{ij} + H_{ij} \end{pmatrix} \ . \tag{122}$$

Here B_i is an divergence free vector and H_{ij} is a trace-free, divergence free tensor field. The scalar quantities Ψ and Φ are called the Bardeen potentials. In the Newtonian approximation they are both equal and reduce to the Neewtonian potential.

We also decompose the perturbations into different 'Fourier modes',

$$\Psi(\eta,\mathbf{x}) = Y_{\mathbf{k}}(\mathbf{x})\Psi(\eta,\mathbf{k}), \quad \Phi(\eta,\mathbf{x}) = Y_{\mathbf{k}}(\mathbf{x})\Phi(\eta,\mathbf{k}), \quad B^i(\eta,\mathbf{x}) = Y_{\mathbf{k}}^{(V)i}(\mathbf{x})B(\eta,\mathbf{k}), \quad H_{ij}(\eta,\mathbf{x}) = Y_{\mathbf{k}\,ij}^{(T)}(\mathbf{x})H(\eta,\mathbf{k}) .$$

In a Friedmann Universe with vanishing curvature, these are just ordinary Fourier modes, while in the general case, the functions $Y_{\mathbf{k}}$ are eigenfunctions of the spatial Laplacian with eigenvalue $-k^2$. The functions $Y_{\mathbf{k}}^{(V)i}$ and $Y_{\mathbf{k}\,ij}^{(T)}$ correspondingly are vector- and tensor-type eigenfunctions of the spatial Laplacian with vanishing divergence. For later use we also define a scalar type vector and tensor as well as a vector-type tensor,

$$Y_{\mathbf{k}\,i}^{(S)} = -k^{-1}\nabla_i Y_{\mathbf{k}} , \qquad Y_{\mathbf{k}\,ij}^{(S)} = k^{-2}(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\Delta)Y_{\mathbf{k}} , \tag{123}$$

$$Y_{\mathbf{k}\,ij}^{(V)} = \frac{k^{-1}}{2}(\nabla_i Y_{\mathbf{k}\,j}^{(V)} + \nabla_j Y_{\mathbf{k}\,i}^{(V)}) . \tag{124}$$

Note that contrary to the vector-type vector field $Y_i^{(V)}$, the vector field $Y_i^{(S)}$ is not divergence free. The same is true for the tensor fields $Y_{ij}^{(S)}$ and $Y_{ij}^{(V)}$.

Let $T_\nu^\mu = \overline{T}_\nu^\mu + \theta_\nu^\mu$ be the full energy momentum tensor. We define its energy density ρ and its energy flux 4-vector u as the time-like eigenvalue and eigenvector of T_ν^μ:

$$T_\nu^\mu u^\nu = -\rho u^\mu, \quad u^2 = -1. \tag{125}$$

We then parameterize their perturbations by

$$\rho = \bar{\rho}\,(1+\delta), \quad u = u^0\partial_t + u^i\partial_i. \tag{126}$$

u^0 is fixed by the normalization condition,

$$u^0 = \frac{1}{a}(1-\Psi). \tag{127}$$

We further set

$$u^i = \frac{1}{a}v^i = \frac{1}{a}\left(VY^{(S)i} + V^{(V)}Y^{(V)i}\right) . \tag{128}$$

Here δ is called the density contrast and (v^i) is the peculiar velocity.

We define $P_\nu^\mu \equiv u^\mu u_\nu + \delta_\nu^\mu$, the projection tensor onto the part of tangent space normal to u and the stress tensor

$$\tau^{\mu\nu} = P_\alpha^\mu P_\beta^\nu T^{\alpha\beta}. \tag{129}$$

In the unperturbed case we have $\tau_0^0 = 0, \tau_j^i = \bar{p}\delta_j^i$. Including perturbations, to first order we still obtain

$$\tau_0^0 = \tau_i^0 = \tau_0^i = 0. \tag{130}$$

But τ_j^i contains in general perturbations. We set

$$\tau_j^i = \bar{p}\left[(1+\pi_L)\,\delta_j^i + \Pi_j^i\right], \quad \text{with} \quad \Pi_i^i = 0. \tag{131}$$

We decompose Π_j^i into scalar- vector- and tensor-type contributions,

$$\Pi_j^i = \Pi^{(S)}Y_j^{(S)\,i} + \Pi^{(V)}Y_j^{(V)\,i} + \Pi^{(T)}Y_j^{(T)\,i}. \tag{132}$$

Another important variable is

$$\Gamma = \pi_L - \frac{c_s^2}{w}\delta \tag{133}$$

where $c_s^2 \equiv \dot{p}/\dot{\rho}$ is the adiabatic sound speed and $w \equiv p/\rho$ is the enthalpy. One can show that Γ is proportional to the divergence of the entropy flux of the perturbations. Adiabatic perturbations are characterized by $\Gamma = 0$.

We shall use also other perturbation variables describing the density contrast and peculiar velocity, which actually correspond to these perturbations in different coordinate systems (gauges). One can show that on sub-horizon scales, $k \gg \mathscr{H}$, on which perturbations are actually measurable, they all coincide.

$$
\begin{aligned}
D &\equiv \delta + 3(1+w)\left(\frac{\dot{a}}{a}\right)\frac{V}{k}\,, &(134)\\
D_g &\equiv \delta - 3(1+w)\Phi\,, &(135)\\
\Omega &\equiv V^{(V)} - B^{(V)}\,, &(136)\\
\Omega - V^{(V)} &= -B^{(V)} \equiv \sigma^{(V)}. &(137)
\end{aligned}
$$

Here we use the customary name, $\sigma^{(V)} = -B^{(V)}$, for the vector-type metric perturbation. These variables can be interpreted nicely in terms of gradients of the energy density and the shear and vorticity of the velocity field [31].

5.2. Einstein's equations

We do not derive the first order perturbations of Einstein's equations. This can be done by different methods, for example with Mathematica. We just write down the results.

5.2.1. Constraint equations

$$
\left.\begin{aligned}
4\pi G a^2 \rho D &= -(k^2 - 3K)\Phi \quad (00)\\
4\pi G a^2 (\rho + p) V &= k\left(\mathscr{H}\Psi + \dot{\Phi}\right) \quad (0i)
\end{aligned}\right\} \qquad \text{(scalar)} \qquad (138)
$$

$$
8\pi G a^2(\rho+p)\Omega = \frac{1}{2}\left(2K - k^2\right)\sigma^{(V)} \quad (0i) \qquad \text{(vector)} \qquad (139)
$$

5.2.2. Dynamical equations

$$
k^2(\Phi - \Psi) = 8\pi G a^2 p \Pi^{(S)} \qquad (i \neq j) \qquad \text{(scalar)} \qquad (140)
$$

$$
\mathscr{H}\left[\dot{\Psi} + \left(\frac{\mathscr{H}^2 - \dot{\mathscr{H}}}{\mathscr{H}^2}\Phi + \mathscr{H}^{-1}\dot{\Phi}\right)^{\bullet}\right] + \qquad (141)
$$

$$
\left(\mathscr{H}^2 + 2\dot{\mathscr{H}}\right)\left[\Psi + \frac{\mathscr{H}^2 - \dot{\mathscr{H}}}{\mathscr{H}^2}\Phi + \mathscr{H}^{-1}\dot{\Phi}\right] = 4\pi G a^2\left(c_s^2 D_g + w\Gamma - \frac{2}{3}w\Pi\right) \qquad (i\,i) \quad \text{(scalar)} \qquad (142)
$$

$$
k\left(\dot{\sigma}^{(V)} + 2\left(\frac{\dot{a}}{a}\right)\sigma^{(V)}\right) = 8\pi G a^2 p \Pi^{(V)} \qquad (ij) \qquad \text{(vector)} \qquad (143)
$$

$$
\ddot{H}^{(T)} + 2\left(\frac{\dot{a}}{a}\right)\dot{H}^{(T)} + \left(2K + k^2\right)H^{(T)} = 8\pi G a^2 p \Pi^{(T)} \qquad (ij) \qquad \text{(tensor)} \qquad (144)
$$

For perfect fluids, where $\Pi^i_j \equiv 0$, we have $\Phi = \Psi$, $\sigma^{(V)} \propto 1/a^2$, and $H^{(T)}$ obeys a damped wave equation. The damping term can be neglected on small scales (over short time periods) when $\eta^{-2} \lesssim 2K + k^2$, so that $H^{(T)}$ represents a propagating gravitational wave. For vanishing curvature, the scales $k\eta \gg 1$ are simply the sub-horizon scales. For $K < 0$, waves oscillate with a somewhat smaller frequency, $\omega = \sqrt{2K + k^2}$, while for $K > 0$ the frequency is somewhat higher than k.

5.2.3. Energy momentum conservation

The conservation equations, $\nabla_\nu T^{\mu\nu} \equiv T^{\mu\nu}_{;\nu} = 0$ lead to the following perturbation equations.

$$\left.\begin{array}{r}\dot{D}_g + 3\left(c_s^2 - w\right)\left(\frac{\dot{a}}{a}\right) D_g + (1+w)kV + 3w\left(\frac{\dot{a}}{a}\right)\Gamma = 0 \\ \dot{V} + \left(\frac{\dot{a}}{a}\right)\left(1 - 3c_s^2\right) V = k\left(\Psi + 3c_s^2\Phi\right) + \frac{c_s^2 k}{1+w} D_g \\ + \frac{wk}{1+w}\left[\Gamma - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\Pi\right]\end{array}\right\} \qquad \text{(scalar)}, \tag{145}$$

$$\dot{\Omega} + \left(1 - 3c_s^2\right)\left(\frac{\dot{a}}{a}\right)\Omega = \frac{p}{2(\rho + p)}\left(k - \frac{2K}{k}\right)\Pi^{(V)} \qquad \text{(vector)}. \tag{146}$$

These can of course also be obtained from the Einstein equations since they are equivalent to the contracted Bianchi identities.

For scalar perturbations we have 4 independent equations and 6 variables. For vector perturbations we have 2 equations and 3 variables, while for tensor perturbations we have 1 equation and 2 variables. To close the system we must add matter equations. The simplest prescription is to set $\Gamma = \Pi_{ij} = 0$. These matter equations, which describe adiabatic perturbations of a perfect fluid give us exactly two additional equations for scalar perturbations and one each for vector and tensor perturbations.

Another example is a universe with matter content given by a scalar field. We shall discuss this case in the next section. More complicated examples are those of several interacting particle species of which some have to be described by a Boltzmann equation. This is the actual universe at late times, say $z \lesssim 10^{10}$.

5.2.4. A special case

Here we rewrite the scalar perturbation equations for a simple but important special case. We consider adiabatic perturbations of a perfect fluid. In this case $\Pi = 0$ and $\Gamma = 0$. Eq. (140) implies $\Phi = \Psi$. Using the first equation of (138) and Eqs. (135,134) to replace D_g in the second of Eqs. (145) by Ψ and V, finally replacing V by (138) one can derive a second order equation for Ψ, which is, the only dynamical degree of freedom

$$\ddot{\Psi} + 3\mathscr{H}(1+c_s^2)\dot{\Psi} + [(1+3c_s^2)(\mathscr{H}^2 - K) - (1+3w)(\mathscr{H}^2 + K) + c_s^2 k^2]\Psi = 0\ . \tag{147}$$

Another interesting example (especially when discussing inflation) is the scalar field case. There, as we shall see in Section 5.4, $\Pi = 0$, but in general $\Gamma \neq 0$ since $\delta p/\delta\rho \neq \dot{p}/\dot{\rho}$. Nevertheless, since this case again has only one dynamical degree of freedom, we can express the perturbation equations in terms of one single second order equation for Ψ. In Section 5.4 we shall find the following equation for a perturbed scalar field cosmology

$$\ddot{\Psi} + 3\mathscr{H}(1+c_s^2)\dot{\Psi} + [(1+3c_s^2)(\mathscr{H}^2 - K) - (1+3w)(\mathscr{H}^2 + K) + k^2]\Psi = 0\ . \tag{148}$$

The only difference between the perfect fluid and scalar field perturbation equation is that the latter is missing the factor c_s^2 in front of the oscillatory k^2 term. Note also that for $K = 0$ and $w = c_s^2 =$ constant, the time dependent mass term $m^2(\eta) = -(1+3c_s^2)(\mathscr{H}^2 - K) + (1+3w)(\mathscr{H}^2 + K)$ vanishes.

It is useful to define the variable [32]

$$u = a\left[4\pi G(\mathscr{H}^2 - \dot{\mathscr{H}} + K)\right]^{-1/2}\Psi, \tag{149}$$

which satisfies the equation

$$\ddot{u} + (\Upsilon k^2 - \ddot{\theta}/\theta)u = 0, \tag{150}$$

where $\Upsilon = c_s^2$ or $\Upsilon = 1$ for a perfect fluid or a scalar field background respectively, and

$$\theta = \frac{3\mathscr{H}}{2a\sqrt{\mathscr{H}^2 - \dot{\mathscr{H}} + K}}\ . \tag{151}$$

Another interesting variable is

$$\zeta \equiv \frac{2(\mathscr{H}^{-1}\dot{\Psi} + \Psi)}{3(1+w)} + \Psi\ . \tag{152}$$

For the rest of this section we set $K = 0$ for simplicity. Using Eqs. (147) and (148) respectively one then finds

$$\dot{\zeta} = -k^2 \frac{\Upsilon \mathscr{H}}{\mathscr{H}^2 - \dot{\mathscr{H}}} \Psi \; . \tag{153}$$

On super-horizon scales, $k/\mathscr{H} \ll 1$, this time derivative is suppressed by a factor $\sim (k/\mathscr{H})^2 \simeq (k\eta)^2$ and this variable is (nearly) conserved on large scales.

The evolution of ζ is closely related to the canonical variable v defined by

$$v = -\frac{a\sqrt{\mathscr{H}^2 - \dot{\mathscr{H}}}}{\sqrt{4\pi G}\Upsilon \mathscr{H}} \zeta \; , \tag{154}$$

which satisfies the equation

$$\ddot{v} + (\Upsilon k^2 - \ddot{z}/z)v = 0 \; , \qquad \text{for } z = \frac{a\sqrt{\mathscr{H}^2 - \dot{\mathscr{H}} + \kappa}}{\Upsilon \mathscr{H}} \; . \tag{155}$$

5.3. Dust and radiation

Next we discuss two simple applications which are important to understand the anisotropies in the cosmic microwave background (CMB).

5.3.1. The pure dust fluid for $K = 0, \Lambda = 0$

'Dust' is the cosmological term for non-relativistic particles for which we can neglect the pressure so that $w = c_s^2 = p = 0$ and $\Pi = \Gamma = 0$. The Friedmann equation implies for dust $a \propto \eta^2$ so tha $\mathscr{H} = 2/\eta$. Equation (147) then reduces to

$$\ddot{\Psi} + \frac{6}{\eta}\dot{\Psi} = 0 \; , \tag{156}$$

with the general solution

$$\Psi = \Psi_0 + \Psi_1 \frac{1}{\eta^5} \tag{157}$$

with arbitrary constants Ψ_0 and Ψ_1. Since the perturbations are supposed to be small initially, they cannot diverge for $\eta \to 0$, and we have therefore to keep only the 'growing' mode, $\Psi_1 = 0$. But also the Ψ_0 mode is only constant. This fact led Lifshitz who was the first to analyze cosmological perturbations to the conclusions that linear perturbations do not grow in a Friedman universe and cosmic structure cannot have evolved by gravitational instability [33]. However, the important point to note here is that, even if the gravitational potential remains constant, matter density fluctuations do grow on sub-horizon scales, scales where $k\eta \gg 1$ and hence structure can evolve on scales which are smaller than the Hubble scale.

Defining $x = k\eta$, we obtain for the velocity potential and the density contrast

$$V = \Psi_0 \frac{x}{3} \tag{158}$$

$$D_g = -5\Psi_0 - \frac{1}{6}\Psi_0 x^2 \; , \qquad D = D_g + 3\Psi + \frac{6}{x}V = -\frac{1}{6}\Psi_0 x^2 \; . \tag{159}$$

In the variable D the constant term has disappeared and we have $D \ll \Psi$ on super-horizon scales, $x \ll 1$.

On sub-horizon scales, the density fluctuations grow like the scale factor $a \propto x^2$. Nevertheless, Lifshitz' conclusion [33] that pure gravitational instability cannot be the cause for structure formation has some truth: if we start from tiny thermal fluctuations of the order of 10^{-35}, they can only grow to about 10^{-30} due to this mild, power law instability during the matter dominated regime. Or, to put it differently, if we want to form structure by gravitational instability, we need initial fluctuations of the order of at least 10^{-5}, much larger than thermal fluctuations. According to what we have said here, we need these fluctuations at the beginning of the matter dominated phase, but as we shall see below, perturbations do not grow at all during the radiation dominated era, so that really *initial fluctuations* with amplitudes

$\simeq 10^{-5}$ are needed. One possibility to create such fluctuations is quantum particle production in the classical gravitational field during inflation. The rapid expansion of the universe during inflation quickly expands microscopic scales, at which quantum fluctuations are important, to cosmological scales where these fluctuations are then “frozen in” as classical perturbations in the energy density and the geometry. We will discuss the induced spectrum on fluctuations in Section 5.5.

5.3.2. The pure radiation fluid, $K = 0, \Lambda = 0$

In this limit we set $w = c_s^2 = 1/3$, and $\Pi = \Gamma = 0$ so that $\Phi = -\Psi$. We conclude from $\rho \propto a^{-4}$ and the Friedmann equation that $a \propto \eta$. For radiation, the u–equation (150) becomes

$$\ddot{u} + (\frac{1}{3}k^2 - \frac{2}{\eta^2})u = 0, \tag{160}$$

with general solution

$$u(x) = A\left(\frac{\sin(x)}{x} - \cos(x)\right) + B\left(\frac{\cos(x)}{x} - \sin(x)\right) , \tag{161}$$

where we have set $x = k\eta/\sqrt{3} = c_s k\eta$. For the Bardeen potential we obtain with (149), up to constant factors,

$$\Psi(x) = \frac{u(x)}{x^2} . \tag{162}$$

We must set $B = 0$ for perturbations to remain regular at early times. On super-horizon scales, $x \ll 1$, we then have

$$\Psi(x) \simeq \frac{A}{3} . \tag{163}$$

For the density and velocity perturbations one finds

$$D_g = 2A\left[\cos(x) - \frac{2}{x}\sin(x)\right] , \qquad V = -\frac{\sqrt{3}}{4}D_g' . \tag{164}$$

In the **super-horizon regime**, $x \ll 1$, this yields

$$\Psi = \frac{A}{3}, \quad D_g = -2A - \frac{A}{3\sqrt{3}}x^2, \quad V = \frac{A}{2\sqrt{3}}x . \tag{165}$$

On **sub-horizon scales**, $x \gg 1$, we obtain oscillating solutions with constant amplitude and with frequency $k/\sqrt{3}$:

$$V = \frac{\sqrt{3}A}{2}\sin(x) , \quad D_g = 2A\cos(x) , \quad \Psi = -A\cos(x)/x^2 . \tag{166}$$

Note that also radiation perturbations outside the Hubble horizon are frozen to first order. Once they enter the horizon they start to collapse, but pressure resists the gravitational force and the radiation fluid fluctuations oscillate at constant amplitude. The perturbations of the gravitational potential oscillate and decay like $1/a^2$ inside the horizon.

5.3.3. Adiabatic initial conditions

Adiabaticity requires that the perturbations of all contributions to the energy density are initially in thermal equilibrium. This fixes the ratio of the density perturbations of different components. There is no entropy flux and thus $\Gamma = 0$. Here we consider a mixture of non relativistic matter and radiation. Since the matter and radiation perturbations behave in the same way on super-horizon scales,

$$D_g^{(r)} = A + Bx^2, \quad D_g^{(m)} = A' + B'x^2, \quad V^{(r)} \propto V^{(m)} \propto x, \tag{167}$$

we may require a constant ratio between matter and radiation perturbations. As we have seen in the previous section, inside the horizon ($x > 1$) radiation perturbations start to oscillate while matter perturbations keep following a power law. On sub-horizon scales a constant ratio can thus no longer be maintained. There are two interesting possibilities: adiabatic and isocurvature perturbations. Here we concentrate on adiabatic perturbations which seem to dominate the observed CMB anisotropies.

From $\Gamma = 0$ one easily derives that two components with $p_i/\rho_i = w_i =$constant, $i = 1,2$, are adiabatically coupled if $(1+w_1)D_g^{(2)} = (1+w_2)D_g^{(1)}$. Energy conservation then implies that their velocity fields agree, $V^{(1)} = V^{(2)}$. This result is also a consequence of the Boltzmann equation in the strong coupling regime. We therefore require

$$V^{(r)} = V^{(m)}, \tag{168}$$

so that the energy flux in the two fluids is coupled initially.

We restrict ourselves to a matter dominated background, the situation relevant in the observed universe after equality. We first have to determine the radiation perturbations during a *matter dominated era*. Since Ψ is dominated by the matter contribution (it is proportional to the background density of a given component), we have $\Psi \simeq \text{const} = \Psi_0$. We neglect the contribution from the sub-dominant radiation to Ψ. Energy momentum conservation for radiation then gives, with $x = k\eta$, and $d/dx = \prime$

$$D_g^{(r)\prime} = -\frac{4}{3}V^{(r)} \tag{169}$$

$$V^{(r)\prime} = 2\Psi + \frac{1}{4}D_g^{(r)}. \tag{170}$$

Here Ψ is just a constant given by the matter perturbations, and it acts like a constant source term. The general solution of this system is then

$$D_g^{(r)} = A\cos(c_s x) - \frac{4}{\sqrt{3}}B\sin(c_s x) + 8\Psi\left[\cos(c_s x) - 1\right] \tag{171}$$

$$V^{(r)} = B\cos(c_s x) + \frac{\sqrt{3}}{4}A\sin(c_s x) + 2\sqrt{3}\Psi\sin(c_s x), \tag{172}$$

where $c_s = 1/\sqrt{3}$ is the sound speed of radiation. Our adiabatic initial conditions require

$$\lim_{x\to 0}\frac{V^{(r)}}{x} = V_0 = \lim_{x\to 0}\frac{V^{(m)}}{x} < \infty. \tag{173}$$

Therefore $B = 0$ and $V = V_0 x$ with $V_0 = A/4 - 2\Psi$ on super horizon scales, $x \ll 1$. Using in addition $\Psi = 3V_0$ (see (158)) we obtain

$$D_g^{(r)} = \frac{4}{3}\Psi\cos\left(\frac{x}{\sqrt{3}}\right) - 8\Psi \tag{174}$$

$$V^{(r)} = \frac{1}{\sqrt{3}}\Psi\sin\left(\frac{x}{\sqrt{3}}\right) \tag{175}$$

$$D_g^{(m)} = -\Psi(5 + \frac{1}{6}x^2) \tag{176}$$

$$V^{(m)} = \frac{1}{3}\Psi x. \tag{177}$$

On super-horizon scales, $x \ll 1$ we have

$$D_g^{(r)} \simeq -\frac{20}{3}\Psi \quad \text{and} \quad V^{(r)} \simeq \frac{1}{3}x\Psi\ , \tag{178}$$

note that $D_g^{(r)} = (4/3)D_g^{(m)}$ and $V^{(r)} = V^{(m)}$ as it is required for adiabatic initial conditions.

5.4. Scalar field cosmology

We now consider the special case of a Friedmann universe filled with self interacting scalar field matter. We keep spatial curvature $K = 0$ in this section. The action is given by

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}R + \int d^4x\sqrt{|g|}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - W(\varphi)\right) = S_g + S_m \tag{179}$$

where φ denotes the scalar field and W is the potential. The energy momentum tensor is obtained by varying the matter part of the action, S_m wrt the metric $g^{\mu\nu}$,

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left[\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi + W\right]g_{\mu\nu} \tag{180}$$

The energy density ρ and the energy flux u are defined by

$$T^\mu_\nu u^\nu = -\rho u^\mu \ . \tag{181}$$

For a homogeneous and isotropic universe, $\varphi = \varphi(t)$ and $g_{\mu\nu} = a^2\eta_{\mu\nu}$ we obtain

$$\rho = \frac{1}{2a^2}\dot{\varphi}^2 + W \qquad (u^\mu) = \frac{1}{a}(1, \mathbf{0}) \ . \tag{182}$$

The pressure is given by

$$T^i_j = p\delta^i_j \qquad p = \frac{1}{2a^2}\dot{\varphi}^2 - W \ . \tag{183}$$

We now consider scalar field perturbations,

$$\varphi = \bar{\varphi} + \delta\varphi \ . \tag{184}$$

Clearly, the scalar field only generates scalar-type perturbations (to first order). The perturbed metric is therefore given by $ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(1-2\Phi)\delta_{ij}dx^idx^j$. Inserting Eq. (184) in the definition of the energy velocity perturbation V,

$$(u^\mu) = \frac{1}{a}(1-\Psi, -V_{,i}) \tag{185}$$

and the energy density perturbation $\delta\rho$,

$$\rho = \bar{\rho} + \delta\rho \ , \tag{186}$$

we obtain

$$\delta\rho = \frac{1}{a^2}\bar{\dot{\varphi}}\delta\dot{\varphi} - \frac{1}{a^2}\bar{\dot{\varphi}}^2\Psi + W_{,\varphi}\delta\varphi \tag{187}$$

and

$$V = \frac{k}{\bar{\dot{\varphi}}}\delta\varphi \ . \tag{188}$$

From the stress tensor, $T_{ij} = \varphi_{,i}\varphi_{,j} - \left[\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi + W\right]g_{ij}$ we find

$$p\pi_L = \frac{1}{a^2}\bar{\dot{\varphi}}\delta\dot{\varphi} - \frac{1}{a^2}\bar{\dot{\varphi}}^2\Psi - W_{,\varphi}\delta\varphi \qquad \text{and} \qquad \Pi = 0 \ . \tag{189}$$

Short calculations give

$$\begin{aligned}
D_g &= -(1+w)\left[4\Psi + 2\frac{\dot{a}}{a}k^{-1}V - k^{-1}\dot{V}\right] \ , && (190)\\
D_s &= D_g + 3(1+w)\Psi \ , && (191)\\
\Gamma &= \frac{2W_{,\varphi}}{p\dot{\rho}}\left[\bar{\varphi}\rho D_s - \dot{\rho}\delta\varphi\right] \ , && (192)\\
\Pi &= 0 \ . && (193)
\end{aligned}$$

The Einstein equations then lead to the following second order equation for the Bardeen potential which we have discussed above:

$$\ddot{\Psi} + 2(\mathscr{H} - \ddot{\varphi}/\dot{\varphi})\dot{\Psi} + (2\dot{\mathscr{H}} - 2\mathscr{H}\ddot{\varphi}/\dot{\varphi} + k^2)\Psi = 0 \tag{194}$$

or, using the definition $c_s^2 = \dot{p}/\dot{\rho}$,

$$\ddot{\Psi} + 3\mathscr{H}(1 + c_s^2)\dot{\Psi} + (2\dot{\mathscr{H}} + (1 + 3c_s^2)\mathscr{H}^2 + k^2)\Psi = 0\ . \tag{195}$$

As already mentioned above, this equation differs from the Ψ equation for a perfect fluid only in the last term proportional to k^2. This comes from the fact that the scalar field is not in a thermal state with fixed entropy, but it is in a fully coherent state ($\Gamma \neq 0$) and field fluctuations propagate with the speed of light. On large scales, $k\eta \ll 1$, this difference is not relevant, but on sub–horizon scales it does play a certain role.

5.5. Generation of perturbations during inflation

So far we have simply assumed some initial fluctuation amplitude A, without investigating where it came from or what the k–dependence of A might be. In this section we discuss the most common idea about the generation of cosmological perturbations, namely their production from quantum vacuum fluctuations during an inflationary phase. The treatment here is focused mainly on getting the correct result with as little effort as possible; we ignore several subtleties related, *e.g.* to the transition from quantum fluctuations of the field to classical fluctuations in the energy momentum tensor. The idea is of course that the source for the metric fluctuations are the *expectation values* of the energy momentum tensor operator of the scalar field.

The basic idea is simple: A time dependent gravitational field very generically leads to particle production, analogously to the electron positron production in a classical, time dependent, strong electromagnetic field.

Let us first fix our notation. Inflation is an era during which the expansion of the scale factor is accelerated, $\frac{d^2a}{dt^2} > 0$. In terms of conformal time, $\frac{d}{d\eta} = {}^{\cdot}$, this becomes

$$\frac{d^2a}{dt^2} = \frac{1}{a}\dot{\mathscr{H}} > 0\ .$$

We shall only consider simple power law inflation, where $a = (c\eta)^q$ for some constants c and q. For the scale factor to be positive and real we require $c\eta > 0$. Expansion then happens when $cq > 0$ and accelerated expansion when in addition $q < 0$. Hence for power law inflation, the scale factor behaves like

$$a \propto |\eta|^q$$

and $\eta < 0$ as well as $q < 0$. It is easy to see that de Sitter inflation, $a \propto \exp(Ht)$, corresponds to $q = -1$. In general, for a fluid with $p = w\rho$

$$q = \frac{2}{1 - 3w}\ .$$

Inflation therefore requires $w < -1/3$. During scalar field inflation, the energy density must therefore be dominated by the potential, $W > a^{-2}\dot{\varphi}^2$. We suppose that the field is 'slowly rolling' down the potential until at some later moment the condition $w < -1/3$ breaks down and inflation stops. How far away a given moment is from this end of inflation can be cast in terms of the slow roll parameters ε_1 and ε_2 defined by

$$\varepsilon_1 \;\; = \;\; -\frac{\dot{H}}{aH^2}\ , \quad H = \frac{\mathscr{H}}{a} \quad \text{is the Hubble parameter} \tag{196}$$

$$\varepsilon_2 \;\; = \;\; -\frac{\frac{a^2 d^2\varphi}{dt^2}}{\mathscr{H}\dot{\varphi}} = \left[1 - \frac{\ddot{\varphi}}{\mathscr{H}\dot{\varphi}}\right] = \left[1 + \frac{a^2 W'}{\mathscr{H}\dot{\varphi}}\right]\ . \tag{197}$$

5.5.1. Scalar perturbations

The main result of this subsection is the following: During inflation, the produced particles induce a gravitational field with a (nearly) scale invariant spectrum,

$$k^3|\Psi(k,\eta)|^2 = k^{n-1} \times \text{const.} \quad \text{with} \quad n \simeq 1\ . \tag{198}$$

The quantity $k^3|\Psi(k,\eta)|^2$ is the squared amplitude of the metric perturbation at comoving scale $\lambda = \pi/k$. To ensure that this quantity is small over a broad range of scales, so that neither black holes form on small scales nor large deviation from homogeneity and isotropy on large scales appear, we must require $n \simeq 1$. These arguments have been put forward by Harrison and Zel'dovich [34] (ten years before the advent of inflation), leading to the name 'Harrison-Zel'dovich spectrum' for a scale invariant perturbation spectrum.

To derive the above result we consider a scalar field background dominated by a potential, hence $a \propto |\eta|^q$ with $q \sim -1$. Developing the action of this system,

$$S = \int dx^4 \sqrt{|g|}\left(\frac{R}{16\pi G} + \frac{1}{2}(\nabla\varphi)^2 - W\right) ,$$

to second order in the perturbations (see [32]) around the Friedmann solution one obtains

$$\delta S = \int dx^4 \sqrt{|\bar{g}|}\frac{1}{2}(\partial_\mu v)^2 \tag{199}$$

up to a total differential. Here v is the perturbation variable

$$v = -\frac{a\sqrt{\mathscr{H}^2 - \dot{\mathscr{H}}}}{\sqrt{4\pi G}\mathscr{H}}\zeta \tag{200}$$

introduced in Eq. (154). Via the Einstein equations, this variable can also be interpreted as representing the fluctuations in the scalar field. Therefore, we quantize v and assume that initially, on small scales, $k|\eta| \ll 1$, v is in the (Minkowski) quantum vacuum state of a massless scalar field with mode function

$$v_{\rm in} = \frac{v_0}{\sqrt{k}}\exp(ik\eta) \ . \tag{201}$$

The pre-factor v_0 is a k-independent constant which depends on convention, but is of order unity. From (153) we can derive

$$(v/z)^{\cdot} = \frac{k^2 u}{z} \ ,$$

where $z \propto a$ is defined in Eq. (155) and $u \propto a\eta\Psi$ is given in Eq. (149). On small scales, $k|\eta| \ll 1$, this results in the initial condition for u

$$u_{\rm in} = \frac{-iv_0}{k^{3/2}}\exp(ik\eta) \ . \tag{202}$$

In the case of power law expansion, $a \propto |\eta|^q$, the evolution equation for u, Eq. (150), reduces to

$$\ddot{u} + (k^2 - \frac{q(q+1)}{\eta^2})u = 0. \tag{203}$$

The solutions to this equation are of the form $(k|\eta|)^{1/2}H_\mu^{(i)}(k\eta)$, where $\mu = q + 1/2$ and $H_\mu^{(i)}$ is the Hankel function of the ith kind ($i = 1$ or 2) of order μ. The initial condition (202) requires that only $H_\mu^{(2)}$ appears, so that we obtain

$$u = \frac{\alpha}{k^{3/2}}(k|\eta|)^{1/2}H_\mu^{(2)}(k\eta) \ ,$$

where again α is a constant of order unity. We define the value of the Hubble parameter during inflation, which is nearly constant by H_i. With $H = \mathscr{H}/a \simeq 1/(|\eta|a)$, we then obtain $a \sim 1/(H_i|\eta|)$. With the Planck mass defined by $4\pi G = M_4^{-2}$, Eq. (149) then gives

$$\Psi = \frac{H_i}{2M_4}u \simeq \frac{H_i}{M_4}k^{-3/2}(k|\eta|)^{1/2}H_\mu^{(2)}(k\eta) \ . \tag{204}$$

On small scales this is a simple oscillating function while on large scales $k|\eta| \ll 1$ it can be approximated by a power law,

$$\Psi \simeq \frac{H_i}{M_4}k^{-3/2}(k|\eta|)^{1+q} \ , \ \text{ for } \ k|\eta| \ll 1 \quad . \tag{205}$$

Here we have used $\mu = 1/2 + q < 0$. This yields

$$k^3|\Psi|^2 \simeq \left(\frac{H_i}{M_4}(k|\eta|)^{1+q}\right)^2 \propto k^{n-1} \;, \tag{206}$$

hence $n \simeq 1$ if $q \sim -1$. Detailed studies have shown that the amplitude of Ψ can still be somewhat affected by the transition from inflation to the subsequent radiation era, the spectral index, however, is very stable. Simple deviations from de Sitter inflation, like *e.g.* power law inflation, $q > -1$, lead to slightly blue spectra, $n \gtrsim 1$.

With a somewhat more careful treatment, one finds that both, the amplitude and the spectral index depend on scale via the slow roll parameters ε_1 and ε_2,

$$k^3|\Psi|^2 = \frac{2H_i^2}{M_4^2\varepsilon_1}(k\eta_f)^{n-1} \;, \tag{207}$$

$$n|_{k=a(\eta)H(\eta)} = 1 - 4\varepsilon_1(\eta) - 2\varepsilon_2(\eta) \;. \tag{208}$$

Vector perturbations are not generated during standard inflation; and even if they are generated they only decay during subsequent evolution and we therefore do not discuss them any further. This may change drastically in braneworlds (see [24])!

5.5.2. Tensor perturbations

The situation is different for tensor perturbations. Again we consider the perfect fluid case, $\Pi_{ij}^{(T)} = 0$. Eq. (144) implies

$$\ddot{H}_{ij} + \frac{2\dot{a}}{a}\dot{H}_{ij} + k^2 H_{ij} = 0 \;. \tag{209}$$

If the background has a power law evolution, $a \propto \eta^q$ this equation can be solved in terms of Bessel or Hankel functions. The less decaying mode solution to Eq. (209) is $H_{ij} = e_{ij}x^{1/2-\beta}J_{1/2-q}(x)$, where J_ν denotes the Bessel function of order ν, $x = k\eta$ and e_{ij} is a transverse traceless polarization tensor. This leads to

$$H_{ij} = \text{const} \quad \text{for} \quad x \ll 1 \tag{210}$$

$$H_{ij} = \frac{1}{a} \quad \text{for} \quad x \gtrsim 1 \;. \tag{211}$$

One may also quantize the tensor fluctuations which represent gravitons. Doing this, one obtains (up to log corrections) a scale invariant spectrum of tensor fluctuations from inflation: for tensor perturbations the canonical variable is simple given by $h_{ij} = M_P a H_{ij}$. The evolution equation for $h_{ij} = h e_{ij}$ is of the form

$$\ddot{h} + (k^2 + m^2(\eta))h = 0 \;, \tag{212}$$

where $m^2(\eta) = -\ddot{a}/a$. During inflation $m^2 = -q(q-1)/\eta^2$ is negative, leading to particle creation. Like for scalar perturbations, the vacuum initial conditions are given on scales which are inside the horizon, $k^2 \gg |m^2|$,

$$h_{\rm in} = \frac{1}{\sqrt{k}}\exp(ik\eta) \quad \text{for} \;\; k|\eta| \gg 1.$$

Solving Eq. (212) with this initial condition, gives

$$h = \frac{1}{\sqrt{k}}(k|\eta|)^{1/2} H^{(2)}_{q-1/2}(k\eta) \;,$$

where $H_\nu^{(2)}$ is the Hankel function of degree ν of the second kind. On super-horizon scales where we have $H^{(2)}_{q-1/2}(k\eta) \propto (k|\eta|)^{q-1/2}$, this leads to $|h|^2 \simeq |\eta|(k|\eta|)^{2q-1}$. Using the relation between $h_{ij} = he_{ij}$ and H_{ij} one obtains the spectrum of tensor perturbations generated during inflation. For exponential inflation, $q \simeq -1$ one finds again a scale invariant spectrum for H_{ij} on super-horizon scales,

$$k^3|H_{ij}H^{ij}| \simeq (2H_{\rm in}/M_4)^2(k\eta_f)^{n_T} \quad \text{with} \quad n_T = 2(q+1) \simeq 0 \;. \tag{213}$$

Again, a more careful treatment within the slow roll approximation gives

$$n_T = -2\varepsilon_1 \ . \tag{214}$$

A more detailed analysis also of the amplitudes of the scalar and tensor spectra leads to the consistency relation $n_T = -2A_T^2/A_S^2$ of slow roll inflation. Here A_T and A_S are the amplitudes of tensor and scalar perturbations, respectively.

More details on inflation can be found in many cosmology books, *e.g.* Refs. [29, 35, 36].

5.6. Power spectra

The quantities which we have calculated in the previous subsection are not the precise values of *e.g.* $\Psi(\mathbf{k},\eta)$, but only expectation values $\langle|\Psi(\mathbf{k},\eta)|^2\rangle$. In different realizations of the same inflationary model, the 'phases' $\alpha(\mathbf{k},\eta)$ given by $\Psi(\mathbf{k},\eta) = \exp(i\alpha(\mathbf{k}))|\Psi(k)|$ are different. They are random variables. Since the process which generates the fluctuations Ψ is stochastically homogeneous and isotropic, these phases are uncorrelated (for different values of $\mathbf{k}$). However, the quantity which we can calculate for a given model and which then has to be compared with observation is the power spectrum. Power spectra are the "harmonic transforms" of the two point correlation functions[2]. If the perturbations of the model under consideration are Gaussian, this is a relatively generic prediction from inflationary models, then the two-point functions and therefore the power spectra contain the full statistical information of the model.

Let us first consider the power spectrum of matter,

$$P_D(k) = \left\langle \left|D_g(\mathbf{k},\eta_0)\right|^2 \right\rangle \ . \tag{215}$$

Here $\langle\ \rangle$ indicates a statistical average, ensemble average, over "initial conditions" in a given model. $P_D(k)$ is usually compared with the observed power spectrum of the galaxy distribution.

The spectrum we can both, measure and calculate to the best accuracy is the CMB anisotropy power spectrum. It is defined as follows: The fluctuations of the radiation temperature as observed in the sky, $\Delta T/T$, is a function of our position $\mathbf{x}_0$, time η_0 and the photon direction $\mathbf{n}$. We develop the $\mathbf{n}$–dependence in terms of spherical harmonics. We will suppress the argument η_0 and often also $\mathbf{x}_0$ in the following calculations. All results are for today (η_0) and here ($\mathbf{x}_0$). By statistical homogeneity statistical averages over an ensemble of realizations (expectation values) are supposed to be independent of position. Furthermore, we assume that the process generating the initial perturbations is statistically isotropic. Then, the off-diagonal correlators of the expansion coefficients $a_{\ell m}$ vanish and we have

$$\frac{\Delta T}{T}(\mathbf{x}_0,\eta_0,\mathbf{n}) = \sum_{\ell,m} a_{\ell m}(\mathbf{x}_0,\eta_0)Y_{\ell m}(\mathbf{n}), \quad \langle a_{\ell m}\cdot a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell \ . \tag{216}$$

The C_ℓ's are the CMB power spectrum.

The two point correlation function is related to the C_ℓ's by

$$\begin{aligned}\left\langle \frac{\Delta T}{T}(\mathbf{n})\frac{\Delta T}{T}(\mathbf{n}')\right\rangle_{\mathbf{n}\cdot\mathbf{n}'=\mu} &= \sum_{\ell,\ell',m,m'} \langle a_{\ell m}\cdot a^*_{\ell' m'}\rangle\, Y_{\ell m}(\mathbf{n})Y^*_{\ell' m'}(\mathbf{n}') = \\ \sum_\ell C_\ell \underbrace{\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n})Y^*_{\ell m}(\mathbf{n}')}_{\frac{2\ell+1}{4\pi}P_\ell(\mathbf{n}\cdot\mathbf{n}')} &= \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell P_\ell(\mu),\end{aligned} \tag{217}$$

where we have used the addition theorem of spherical harmonics for the last equality; the P_ℓ's are the Legendre polynomials.

For given metric perturbations and perturbations of the energy momentum tensor of the cosmic fluid, the temperature perturbations can be determined by following the oscillations in the radiation fluid before decoupling (see

[2] The "harmonic transform" in usual flat space is simply the Fourier transform. In curved space it is the expansion in terms of eigenfunctions of the Laplacian on that space, *e.g.* on the sphere it corresponds to the expansion in terms of spherical harmonics

subsection 5.3.2) and by following the propagation of photons along geodesics in the perturbed spacetime after decoupling. Decoupling of photons and matter happens during recombination ($T \simeq 3000$K, $z \simeq 1000$), where electrons and protons recombine to neutral hydrogen. During that process, the number density of free electrons with which the photons can scatter drops drastically and finally becomes so low, that the mean free path of the photons grows larger than the Hubble scale. The surface of constant temperature, $T_{\rm dec} = T(\eta_{\rm dec})$, at which this happens is also called the 'last scattering surface'. After last scattering, the photons effectively cease to interact and move freely along geodesics (more details can be found *e.g.* in [29, 28, 37]).

Clearly the a_{lm}'s from scalar-, vector- and tensor-type perturbations are uncorrelated,

$$\left\langle a^{(S)}_{\ell m} a^{(V)}_{\ell' m'} \right\rangle = \left\langle a^{(S)}_{\ell m} a^{(T)}_{\ell' m'} \right\rangle = \left\langle a^{(V)}_{\ell m} a^{(T)}_{\ell' m'} \right\rangle = 0. \tag{218}$$

Since vector perturbations decay, their contributions, the $C^{(V)}_\ell$, are negligible in models where initial perturbations have been laid down very early, *e.g.* , after an inflationary period. Tensor perturbations are constant on super-horizon scales and perform damped oscillations once they enter the horizon.

Let us first discuss in somewhat more detail scalar perturbations. We restrict ourselves to the case $K = 0$ for simplicity. We suppose the initial perturbations to be given by a spectrum,

$$\left\langle |\Psi|^2 \right\rangle k^3 = A^2 k^{n-1} \eta_0^{n-1}. \tag{219}$$

We multiply by the constant η_0^{n-1}, the actual comoving size of the horizon, in order to keep A dimensionless for all values of n. A then represents the amplitude of metric perturbations at horizon scale today, $k = 1/\eta_0$.

For *adiabatic* perturbations we have obtained on *super-horizon scales*,

$$\frac{1}{4} D_g^{(r)} = -\frac{5}{3}\Psi + \mathscr{O}((k\eta)^2), \quad V^{(b)} = V^{(r)} = \mathscr{O}(k\eta) \ . \tag{220}$$

The dominant contribution to the temperature fluctuations on super-horizon scales (neglecting the integrated Sachs–Wolfe effect $\int \dot\Phi - \dot\Psi$) comes from two terms: the first, 2Ψ, is the change of photon energy due to the gravitational potential at the last scattering surface, $\eta = \eta_{\rm dec}$, and the second, $1/4 D_g^{(r)}$ represents the intrinsic temperature fluctuations (for more details see [38, 30, 37]). With Eq. (220) this yields the famous Sachs-Wolfe formula

$$\frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) = 2\Psi(x_{\rm dec}, \eta_{\rm dec}) + \frac{1}{4} D_g^{(r)}(x_{\rm dec}, \eta_{\rm dec}) = \frac{1}{3}\Psi(x_{\rm dec}, \eta_{\rm dec}). \tag{221}$$

The Fourier transform of (221) gives

$$\frac{\Delta T}{T}(\mathbf{k}, \mathbf{n}, \eta_0) = \frac{1}{3}\Psi(k, \eta_{\rm dec}) \cdot e^{i\mathbf{kn}(\eta_0 - \eta_{\rm dec})} \ . \tag{222}$$

Using the decomposition

$$e^{i\mathbf{kn}(\eta_0 - \eta_{\rm dec})} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(k(\eta_0 - \eta_{\rm dec})) P_\ell(\widehat{\mathbf{k}} \cdot \mathbf{n}) \ ,$$

where j_ℓ are the spherical Bessel functions, we obtain

$$\begin{aligned}
&\left\langle \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}', \eta_0) \right\rangle & (223)\\
&= \frac{1}{V} \int d^3x_0 \left\langle \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}, \eta_0) \frac{\Delta T}{T}(\mathbf{x}_0, \mathbf{n}', \eta_0) \right\rangle \\
&= \frac{1}{(2\pi)^3} \int d^3k \left\langle \frac{\Delta T}{T}(\mathbf{k}, \mathbf{n}, \eta_0) \left(\frac{\Delta T}{T}\right)^* (\mathbf{k}, \mathbf{n}', \eta_0) \right\rangle \\
&= \frac{1}{(2\pi)^3 9} \int d^3k \left\langle |\Psi|^2 \right\rangle \sum_{\ell,\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) i^{\ell-\ell'} \\
&\quad \cdot j_\ell(k(\eta_0 - \eta_{\rm dec})) j_{\ell'}(k(\eta_0 - \eta_{\rm dec})) P_\ell(\hat{\mathbf{k}} \cdot \mathbf{n}) \cdot P_{\ell'}(\hat{\mathbf{k}} \cdot \mathbf{n}') \ . & (224)
\end{aligned}$$

In the second equal sign we have used the unitarity of the Fourier transformation. Inserting $P_\ell(\hat{\mathbf{k}}\mathbf{n}) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\hat{\mathbf{k}})Y_{\ell m}(\mathbf{n})$ and $P_{\ell'}(\hat{\mathbf{k}}\mathbf{n}') = \frac{4\pi}{2\ell'+1}\sum_{m'} Y^*_{\ell' m'}(\hat{\mathbf{k}})Y_{\ell' m'}(\mathbf{n}')$, the integration over directions $d\Omega_{\hat{k}}$ gives $\delta_{\ell\ell'}\delta_{mm'}\sum_m Y^*_{\ell m}(\mathbf{n})Y_{\ell m}(\mathbf{n}')$.
Using as well $\sum_m Y^*_{\ell m}(\mathbf{n})Y_{\ell m}(\mathbf{n}') = \frac{2\ell+1}{4\pi}P_\ell(\mu)$, where $\mu = \mathbf{n}\cdot\mathbf{n}'$, we find

$$\left\langle \frac{\Delta T}{T}(\mathbf{x}_0,\mathbf{n},\eta_0)\frac{\Delta T}{T}(\mathbf{x}_0,\mathbf{n}',\eta_0)\right\rangle_{\mathbf{nn}'=\mu} = \sum_\ell \frac{2\ell+1}{4\pi}P_\ell(\mu)\frac{2}{\pi}\int\frac{dk}{k}\left\langle\frac{1}{9}|\Psi|^2\right\rangle k^3 j_\ell^2(k(\eta_0-\eta_{\rm dec})). \tag{225}$$

Comparing this equation with Eq. (217) we obtain for *adiabatic perturbations* on scales $2 \le \ell \ll (\eta_0-\eta_{\rm dec})/\eta_{\rm dec} \sim 100$

$$C_\ell^{(SW)} \simeq \frac{2}{\pi}\int_0^\infty \frac{dk}{k}\left\langle\left|\frac{1}{3}\Psi\right|^2\right\rangle k^3 j_\ell^2\left(k\left(\eta_0-\eta_{\rm dec}\right)\right). \tag{226}$$

If Ψ is a pure power law as in Eq. (219) and we set $k(\eta_0-\eta_{\rm dec}) \sim k\eta_0$, the integral (226) can be performed analytically. For the ansatz (219) one finds

$$C_\ell^{(SW)} = \frac{A^2}{9}\frac{\Gamma(3-n)\Gamma(\ell-\frac{1}{2}+\frac{n}{2})}{2^{3-n}\Gamma^2(2-\frac{n}{2})\Gamma(\ell+\frac{5}{2}-\frac{n}{2})} \quad \text{for } -3<n<3\,. \tag{227}$$

Of special interest is the *scale invariant* or Harrison–Zel'dovich spectrum, $n=1$ (see Section 5.5). It leads to

$$\ell(\ell+1)C_\ell^{(SW)} = \text{const.} \simeq \left\langle\left(\frac{\Delta T}{T}(\vartheta_\ell)\right)^2\right\rangle, \quad \vartheta_\ell \equiv \pi/\ell\,. \tag{228}$$

This is precisely (within the accuracy of the experiment) the behavior observed by the DMR experiment aboard the satellite COBE [39] and more recently with the WMAP satellite [40].

Inflationary models predict very generically a HZ spectrum (up to small corrections). The DMR discovery has therefore been regarded as a great success, if not a proof, of inflation. There are other models like topological defects [41] or certain string cosmology models [42] which also predict scale–invariant, *i.e.* Harrison Zel'dovich spectra of fluctuations. These models do however not belong to the class investigated here, since in these models perturbations are induced by seeds which evolve non–linearly in time.

For gravitational waves (tensor fluctuations), a formula analogous to (227) can be derived,

$$C_\ell^{(T)} = \frac{2}{\pi}\int dk\, k^2\left\langle\left|\int_{\eta_{\rm dec}}^{\eta_0} d\eta \dot{H}(\eta,k)\frac{j_\ell(k(\eta_0-\eta))}{(k(\eta_0-\eta))^2}\right|^2\right\rangle\frac{(\ell+2)!}{(\ell-2)!}. \tag{229}$$

To a very crude approximation we may assume $\dot{H}=0$ on super-horizon scales and $\int d\eta \dot{H} j_\ell(k(\eta_0-\eta)) \sim H(\eta = 1/k)j_\ell(k\eta_0)$. For a pure power law,

$$k^3\left\langle|H(k,\eta=1/k)|^2\right\rangle \simeq A_T^2 k^{n_T}\eta_0^{-n_T}, \tag{230}$$

one obtains

$$\begin{aligned} C_\ell^{(T)} &\simeq \frac{2}{\pi}\frac{(\ell+2)!}{(\ell-2)!}A_T^2\int\frac{dx}{x}x^{n_T}\frac{j_\ell^2(x)}{x^4} \\ &= \frac{(\ell+2)!}{(\ell-2)!}A_T^2\frac{\Gamma(6-n_T)\Gamma(\ell-2+\frac{n_T}{2})}{2^{6-n_T}\Gamma^2(\frac{7}{2}-n_T)\Gamma(\ell+4-\frac{n_T}{2})}. \end{aligned} \tag{231}$$

For a scale invariant spectrum ($n_T=0$) this results in

$$\ell(\ell+1)C_\ell^{(T)} \simeq \frac{\ell(\ell+1)}{(\ell+3)(\ell-2)}A_T^2\frac{8}{15\pi}\,. \tag{232}$$

The singularity at $\ell = 2$ in this crude approximation is not real, but there is some enhancement of $\ell(\ell+1)C_\ell^{(T)}$ at $\ell \sim 2$ see Fig. 7). Again, inflationary models (and topological defects) predict a scale invariant spectrum of tensor fluctuations ($n_T \sim 0$).

On intermediate scales, $100 < \ell < 1000$, the acoustic oscillations of radiation density fluctuations before decoupling (see subsection 5.3.2) lead to a characteristic series of peaks in the CMB power spectrum which is being measured in great detail and contains very important information on cosmological parameters [43]. On small angular scales, $\ell \gtrsim 800$, fluctuations are damped by collisional damping (Silk damping). This effect has to be discussed with the Boltzmann equation for photons, which goes beyond the scope of this introduction (see [29, 43]).

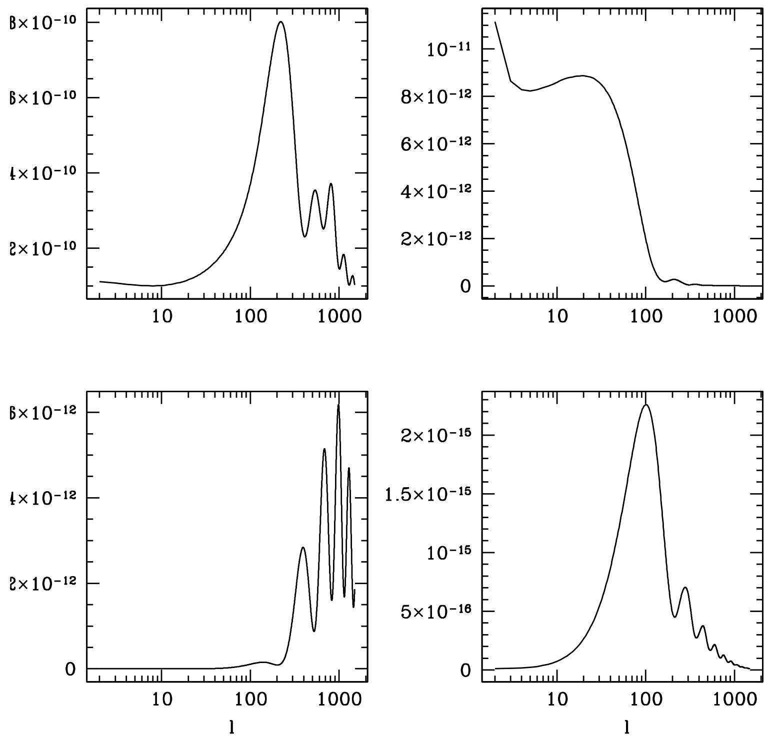

FIGURE 7. Adiabatic scalar (left) and tensor (right) CMB anisotropy spectra are shown. The dimensionless quantity $\ell(\ell+1)C_\ell/(2\pi)$ is plotted. The top panels show the temerature anisotropies while the bottom panels show the corresponding polarization spectra (for an introduction to polarization see *e.g.* [29]).

6. BRANEWORLD COSMOLOGY

We now want to study cosmology of an expanding maximally symmetric braneworld. We still require the bulk to be empty and Z_2–symmetric. One can show that the most general empty bulk allowing for a homogeneous and isotropic brane is Schwarzschild-AdS (Sch-AdS). In the cosmological setting we allow the brane to move in the bulk. As we shall see, this can mimic cosmological expansion. The situation is as depicted in Fig 8.

The 5d metric of Sch-AdS is of the form

$$ds^2 = -F(R)dT^2 + \frac{dR^2}{F(R)} + R^2\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right) \qquad (233)$$

where the function F is determined by the AdS curvature radius ℓ, the 5d mass C and the curvature K of 3d space (on the brane) via

$$F(R) = K + \frac{R^2}{\ell^2} - \frac{C}{R^2} .$$

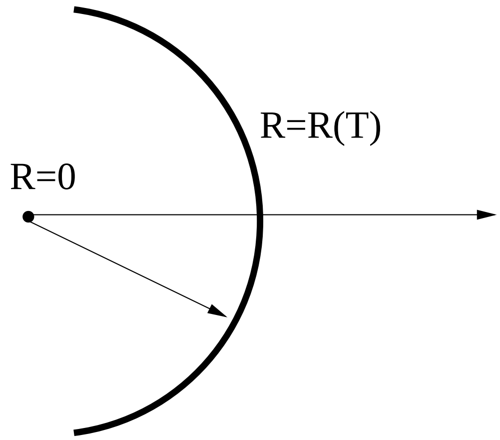

FIGURE 8. A moving brane in AdS-Schwarzschild. In this coordinates the AdS horizon is at $R = \infty$.

From this we can calculate the 5d Weyl tensor, C_{ABCD}, and its 'electric' components defined in (30) with the result

$$E_{\mu\nu} = \rho^E u^E_\mu u^E_\nu + \pi^E_{\mu\nu} \qquad \text{with } u = R^{-1}(1,\mathbf{0}) \;, \tag{234}$$

$$E_{00} = \frac{C}{a^4} = \rho^E \qquad \text{and} \qquad \pi^E_{\mu\nu} = 0 \;. \tag{235}$$

Here $R(T) = a(t)$, where t is cosmic time on the brane and u is the unit normal vector in direction of cosmic time on the brane. $R(T)$ is the brane position at time $t(T)$.

6.1. The modified Friedmann equations

From the brane gravity equations,

$$G_{\mu\nu} = -\Lambda_5 g_{\mu\nu} + \kappa_4 \tau_{\mu\nu} + \kappa_5^2 \sigma_{\mu\nu} - E_{\mu\nu}$$

with $\Lambda_4 = \frac{1}{2}(\Lambda_5 + \kappa_5/6\lambda^2)$, $\kappa_4 = \kappa_5^2\lambda/6$ and

$$\sigma_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} - \frac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}\tau^2$$

with $\tau_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$ we obtain

$$H^2 = \frac{\kappa_4}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{C}{a^4} + \frac{\Lambda_4}{3} + \frac{K}{a^2} \;, \tag{236}$$

where H is the Hubble parameter. In this section we denote the derivative with respect to *cosmological time t* determined by $dt = ad\eta$ by an over-dot, so that $H = \dot{a}/a$ and $\mathscr{H} = \dot{a}$.

The term $\rho^2/(2\lambda)$ in Eq. (236) is a correction to the Friedmann equation which is important only at high energies and $\rho^E = C/a^4$, comes from the Weyl tensor. It is called 'Weyl radiation' since it scales like cosmic radiation, $\propto a^{-4}$. Observations (nucleosynthesis) tell us that latest at the temperature $T \sim 1$MeV these additional terms should be unimportant. More precisely, $\rho^E/\rho\big|_{(nuc)} \lesssim 0.1$ and $\lambda \gtrsim (1\text{MeV})^4$. For the 5d Planck mass this implies $M_5 \gtrsim 3 \times 10^4$GeV. The conservation equation of 4-dimensional cosmology remains unchanged,

$$\dot\rho = -3(\rho + p)\frac{\dot a}{a} \;. \tag{237}$$

Solutions for $C = K = \Lambda_4 = 0$ are readily found. If the equation of state is given by $p = w\rho$ we find

$$a = a_0\left[t(t + t_\lambda)\right]^{\frac{1}{3(1+w)}} \;, \quad t_\lambda = \frac{M_4}{\sqrt{3\pi\lambda}(3+w)} < 1\text{sec} \;, \tag{238}$$

where we have used $\lambda > 1(\text{Mev})^4$ for the inequality. This is to be compared with the usual 4-dimensional behavior from general relativity (GR). There we have $a = a_0 t^{2/3(1+w)}$, which corresponds to the above result in the limit $\lambda \to \infty$. Especially interesting is also the case of de Sitter inflation where $p = -\rho$ and hence $\rho =$constant, so that $a = a_0 e^{Ht}$, with

$$H = \sqrt{\kappa_4 \frac{\rho}{3}\left(1+\frac{\rho}{2\lambda}\right)} > \sqrt{\frac{\kappa_4 \rho}{3}} = H_{GR} \ . \tag{239}$$

At low energies we recover the usual Friedmann equation while at high energies, $\rho \geq \lambda$, the expansion law differs. During a radiation epoch, $p = \rho/3$ with $\rho \gg \lambda$ we have $a \propto t^{1/4}$ (instead of the usual GR behavior, $a \propto t^{1/2}$). This comes from the fact that in braneworlds at high energies $H \propto \rho$, where as in 4d GR we have $H \propto \sqrt{\rho}$. If λ is given by the electroweak scale, $\lambda \geq 1\text{TeV}^4$, the observed low energy cosmology, like nucleosynthesis which starts after $\rho \sim 1\text{MeV}^4$ is not affected (if C is sufficiently small).

However, perturbations will carry 5d effects which should in principle be observable in the fluctuation spectrum of the cosmic microwave background radiation (CMB) and in the large scale distribution of matter. These effects are still under investigation. In Section 6.3 I shall present some preliminary partial results.

6.2. Brane inflation

We now want to study scalar field inflation in braneworlds. Since energy momentum conservation is still valid, the scalar field evolution equation is also not modified,

$$\ddot{\varphi} + 3H\dot{\varphi} + W'(\varphi) = 0 \ .$$

In 4d GR, the condition for inflation, $\ddot{a} > 0$, is equivalent to $\dot{\varphi}^2 < V$ which corresponds to the violation of the strong energy condition

$$p = \frac{1}{2}\dot{\varphi}^2 - W < -\frac{1}{3}\rho = -\frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + W\right) \ , \quad w = \frac{p}{\rho} < -\frac{1}{3} \ , \quad \text{or} \quad \dot{\varphi}^2 < W \ .$$

The braneworld Friedmann equation (239) leads to a stronger condition on w for accelerated expansion. For branewords $0 < \frac{\ddot{a}}{a} = \dot{H} + H^2$ requires

$$w < -\frac{1}{3}\left[\frac{1+2\rho/\lambda}{1+\rho/\lambda}\right] \ , \quad \text{or} \quad \dot{\varphi}^2 < W + \left[\frac{\frac{1}{2}\dot{\varphi}^2 + W}{\lambda}\left(\frac{5}{4}\dot{\varphi}^2 - \frac{1}{2}V\right)\right] \tag{240}$$

for inflation to happen. This becomes the usual $w < -1/3$ at low energy, $\rho << \lambda$, but turns into $w < -2/3$ at high energy.

If the slow roll approximation ($\dot{\varphi}^2 \ll W$) is satisfied we have

$$H^2 \simeq \frac{\kappa_4}{3} W \left[1 + \frac{W}{2\lambda}\right] \ , \quad \dot{\varphi} \simeq -\frac{W'}{3H} \ .$$

Since the Hubble rate is increased, slow roll is maintained *longer* than in usual 4d inflation. Correspondingly, the slow roll parameters are reduced,

$$\varepsilon_1 \equiv -\frac{\dot{H}}{H} = \frac{M_4^2}{16\pi}\left(\frac{W'}{W}\right)^2 \left[\frac{1+W/\lambda}{(1+W/2\lambda)^2}\right] \ , \quad \varepsilon_2 \equiv -\frac{\ddot{\varphi}}{\dot{\varphi}H} = \frac{M_4^2}{8\pi}\left(\frac{W''}{W}\right)\left[\frac{1}{(1+W/2\lambda)}\right] \ . \tag{241}$$

In the high energy regime, $V \gg \lambda$ they are reduced by factors $4\lambda/V$ and $2\lambda/V$, respectively. Hence, the universe may be inflating only because it is in the high energy regime and turn into kinetic dominated expansion, $w \simeq 1$, as soon as $V < \lambda$. Such models are constrained since they induce a blue spectrum of gravity waves [44].

In standard 4d GR the perturbation spectrum induced by inflation is well known and the scalar spectrum agrees extremely well with the observed anisotropies in the CMB. This will most probably lead to the most stringent constraints for braneworlds, which however have not yet been explored in full generality so far. This is still a very active field of research.

6.3. Observable consequences from braneworld cosmology

So far, we have seen that it is conceivable that our Universe is a 3-brane. At least at low energy and disregarding perturbations, we cannot distinguish cosmological evolution due to 4d Einstein equations or the (so different!) brane gravity equations. We finally want to study ways to discover whether the braneworld idea is realized in nature. Are there tell-tale observational signatures which would betray whether we live on a brane?

As we have seen, the gravitational equations for braneworlds differ significantly from Einstein gravity. However, at low energy the Friedmann equations for a homogeneous and isotropic Universe are recovered. Hence there are two regimes in which deviations from Einstein gravity will be found:

- **At high energy.** This is especially interesting for the generation of inflationary perturbations which may be affected by high energy braneworld behavior. As long as we restrict ourselves on background effects the calculations are relatively straight forward and well under control.
- **In the perturbations.** Cosmological brane perturbation theory is not well under control and still a subject of active research. One main point are bulk perturbations which can affect the brane and act there like 'sources'. On the other hand, gravity wave perturbations generated on the brane can be emitted into the bulk.

Here we give examples of both aspects, how effects from braneworlds can enter cosmological perturbations, but we are by no means exhaustive (more details and especially references can be found in [16]).

Let us first consider the high energy universe. During inflation scalar and tensor perturbations are generated. The spectrum of scalar perturbations is $|\Psi|^2k^3 = A_S^2 k^{2q-2} = A_S^2(k/H_0)^{n_s-1}$. The slow roll approximation for braneworlds gives [16]

$$A_S^2 \simeq \frac{512\pi}{75M_4^6}\frac{W^3}{W'^2}\left[\frac{2\lambda+W}{2\lambda}\right]^3 , \quad n_S = 1-4\varepsilon_1-2\varepsilon_2 \, . \tag{242}$$

Similarly, for tensor perturbations, $|H|^2k^3 = A_T^2 k_T^n$ one obtains in the slow roll approximation [45]

$$A_T^2 \simeq \frac{8W}{75M_4^2}F^2(H/\mu) , \quad n_T = -2\varepsilon_1 , \quad \text{where} \quad F(x) = \left[\sqrt{1+x^2} - x^2\sinh^{-1}(1/x)\right]^{-1/2} \tag{243}$$

$$\mu = \sqrt{\frac{4\pi}{3}}\frac{\sqrt{\lambda}}{M_4} \qquad \text{and} \qquad x = H/\mu \simeq \left(\frac{W}{\lambda}\right)^{-1/2} \tag{244}$$

Combining these slow roll equations, at low energy one obtains the same consistency relation as for ordinary inflation,

$$\frac{A_T^2}{A_S^2} \simeq \frac{M_4^2 W'^2}{16\pi W^2} = \varepsilon_1 = -n_T/2 \, . \tag{245}$$

At higher energies, however the relation is different. Furthermore, the tensor to scalar ratio $R = (A_T/A_S)^2$ and the spectral indices, both depend on the energy scale W. In Fig. 9 the behavior of the different quantities is indicated as function of the energy. Of course also the amplitude of the perturbations strongly depends on the parameter W/M_4^4.

In Fig. 10 two models are shown, quartic inflation with $W = \alpha\varphi^4$ and quadratic inflation with $W = m^2\varphi^2$. The lines of these models in the (R, n_s) plane for varying λ are drawn. The parapeters m respectively α are chosen such that the scalar amplitude is compatible with the measured value, $A_S^2 \simeq 10^{-10}$ for each brane tension λ. The observational constraints from WMAP data [40] are also indicated. It is clear, that quartic braneworld inflation fares even worse than ordinary quartic inflation. It is virtually excluded. Also quadratic inflation with strong braneworld effects fits the data somewhat less well than usual quadratic inflation, since it predicts too strong tensor contributions. But clearly, in lack of a concrete model of inflation (e.g., a given potential) there is little which can be said.

Discussing the effects on perturbations from braneworlds is opening Pandora's box. There is a plethora of new phenomena some of which we don't even know the sign. For example: during ordinary inflation, gravitational waves are generated. For a given inflationary potential, their amplitude can be calculated accurately. However, in the braneworld context, a fraction of these waves will be radiated into the bulk and thereby reduce the gravity wave amplitude. On the other hand, there is also gravity wave generation in the bulk, and some of these accumulate on the brane, increasing the amplitude of gravity waves on the brane. Therefore, depending on the precise realization, even the sign of the braneworld effect on a gravity wave background is unknown.

For a more concrete example, let us concentrate on scalar perturbations. We just take into account, that on the perturbative level the Weyl tensor $E_{\mu\nu}$ can no longer be neglected. Its energy density perturbation $\delta\rho_E$ acts like

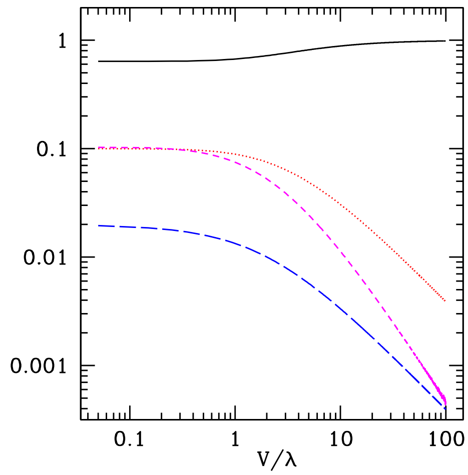

FIGURE 9. We show the slow roll parameters ε_1 (dotted, red) and ε_2 (long dashed, blue) as function of the energy scale of inflation, V/λ. The spectral index n (solid, black) and the tensor to scalar ratio R (dashed, magenta) are also indicated. For $W/\lambda \lesssim 1$ the slow roll parameters stay nearly constant and correspond to their initial values which are chosen $\varepsilon_1(0) = 0.1$ and $\varepsilon_2(0) = 0.02$.

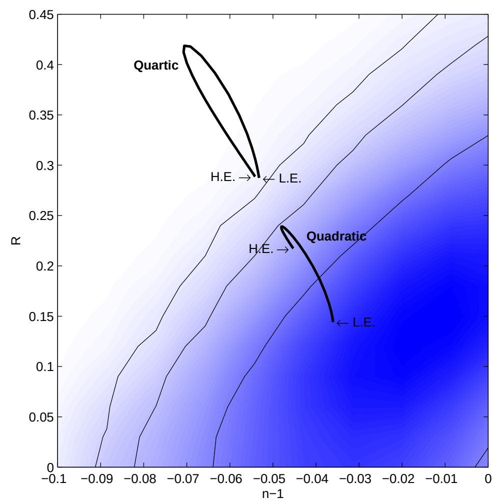

FIGURE 10. The dependence on the brane tension λ of the parameters (R, n_s) is shown for quartic and quadratic inflation. The 1-, 2- and 3-σ contours from the WMAP experiment are also indicated (from [45]).

a radiation perturbation. In addition, however it can have an arbitrary amount of anisotropic stress, Π_E. The latter induces a difference between the two Bardeen potentials, $\Psi - \Phi \propto \Pi_E$. This affects mainly the Sachs–Wolfe term in the CMB fluctuations, hence the low multi-poles up to roughly the first peak. In Fig. 11 we show the effect of a Weyl perturbation as function of an amplitude parameter

$$C_{\text{dark}} \equiv \frac{\delta\rho_E/\rho_r}{4\zeta_m} . \tag{246}$$

Here ζ_m is the ζ variable defined in Eq. (155), due to ordinary matter (without the Weyl component). The anisotropic stress Π_E , on large scales, $\ell \ll 1/k$, can be determined as function of C_{dark} and ζ_m. Confidence plots for the amplitude C_{dark} and several other cosmological parameters from the WMAP data are shown in Fig. 12.

There are many more effects which may come from perturbations in the bulk and the different perturbation equations on the brane which are presently under study. A systematic investigation is still lacking.

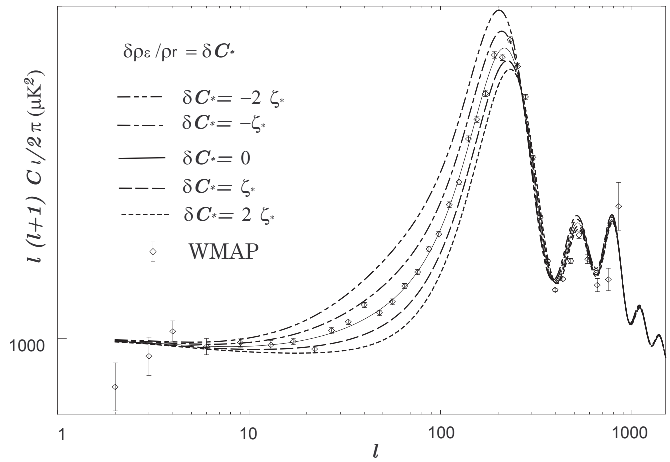

FIGURE 11. The dependence of the CMB power spectrum on the amplitude of the Weyl perturbation, $C_{\rm dark}$ for a fixed set of other cosmological parameters corresponding to the concordance model (from [46]).

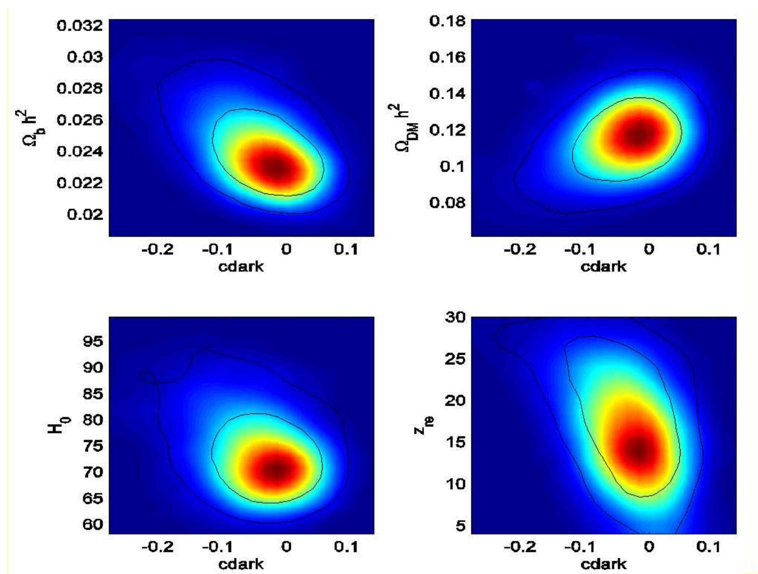

FIGURE 12. Confidence plots from the CMB data for the amplitude $C_{\rm dark}$ and other cosmological parameters. The line contours indicate 2- and 3-σ. Clearly, a Weyl contribution to the perturbations of more than about 10% is strongly disfavored by the data.

7. CONCLUSIONS

In these lectures we have studied the possibility that our Universe may represent a 3-brane in a higher dimensional space. This idea is motivated by string theory. We have especially investigated the case of one large extra dimension where the brane gravitational equations can be obtained purely from the bulk equations. Even though the resulting gravity on the brane differs strongly from Einstein gravity, we have seen that for an Anti–de Sitter bulk, Newton's law is recovered at large distances and the Friedmann equations for the evolution of the Universe are obtained at low energy.

It is, however by no means clear, to which extent the different gravitational equations will spoil the successes of cosmological perturbation theory. This is still an open question and its answer will be crucial for braneworlds.

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