31/7/2006

P1] Sean 3 variables x, y, z que satisfacen una relación del tipo $f(x,y,z)=0$. Demuestre que:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1 \quad , \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

Sol: Como $f(x,y,z)=0$, siempre es posible definir una variable en función de las otras dos:

$$x = x(y,z) \qquad (1)$$

$$y = y(x,z) \qquad (2)$$

$$z = z(x,y) \qquad (3)$$

de (1): $$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \qquad (4)$$

de (2): $$dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz \qquad (5)$$

(5) en (4) para dy: $$dx = \left(\frac{\partial x}{\partial y}\right)_z \left[\left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz\right] + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left[1 - \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z\right] dx = \left[\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x + \left(\frac{\partial x}{\partial z}\right)_y\right] dz \qquad (6)$$

Si tomo a x, z como variables independientes, entonces dx, dz son arbitrarios

Si $dx \neq 0$, $dz = 0$ $\Rightarrow$ $1 - \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 0$

$$\Rightarrow \boxed{\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1}$$

Análogamente se puede demostrar que:

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial y}\right)_x = 1 \quad , \quad \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y = 1$$

Ahora, si en (6) se toma $dx = 0$, $dz \neq 0$

$$\Rightarrow \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x + \left(\frac{\partial x}{\partial z}\right)_y = 0$$

$$\Rightarrow \quad \frac{\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x}{\left(\frac{\partial x}{\partial z}\right)_y} = -1$$

pero $\dfrac{1}{\left(\frac{\partial x}{\partial z}\right)_y} = \left(\dfrac{\partial z}{\partial x}\right)_y$

$$\Rightarrow \quad \boxed{\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1}$$

P2] $dz = (2y^2 + x)\,dx + (y^2 + 4yx)\,dy$

Verifique que dz es un diferencial exacto y determine $z = z(x,y)$

$$dz = \underbrace{(2y^2 + x)}_{M(x,y)}\,dx + \underbrace{(y^2 + 4yx)}_{N(x,y)}\,dy \qquad (*)$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y}(x,y) = \dfrac{\partial}{\partial y}(2y^2 + x) = 4y \\[2ex] \dfrac{\partial N(x,y)}{\partial x} = \dfrac{\partial}{\partial x}(y^2 + 4yx) = 4y \end{array}\right\} \Rightarrow \quad dz \text{ es exacto}$$

Primero integremos $(*)$ con y cte $\Rightarrow dy = 0$

$$\Rightarrow \quad dz = (2y^2 + x)\,dx$$

$$\Rightarrow \quad z(x,y) = \int (2y^2 + x)\,dx + F(y)$$

$$= 2y^2x + \frac{x^2}{2} + F(y) \qquad (**)$$

Ahora hay que determinar $F(y)$

De $(*)$ se sabe que $\dfrac{\partial z}{\partial y} = y^2 + 4yx$

$$(**) \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(2y^2x + \frac{x^2}{2} + F(y)\right)$$

$$= 4yx + F'(y) = y^2 + 4yx$$

se impone

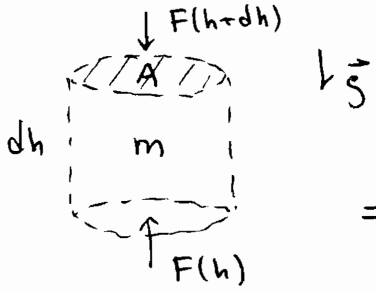
$$\Rightarrow \quad F'(y) = y^2$$

$$\Rightarrow \quad F(y) = \frac{y^3}{3} + c \quad , \; c \text{ cte}$$

$$\therefore \quad \boxed{z(x,y) = 2y^2x + \frac{x^2}{2} + \frac{y^3}{3} + c}$$

P3] Se puede modelar la presión de la atmósfera terrestre como un gas ideal a temp cte T_0. Demuestre que en estas condiciones la presión atmosférica local disminuye exponencialmente con la altura.

Deduzcamos la ley de Pascal. Sea un fluído estático de densidad ρ (no necesariamente cte).



Tomemos un cilindro imaginario y hagamos $F^{total} = m \cdot a = 0$

$$\Rightarrow -F(h+dh) + F(h) - \underbrace{m}_{\rho \cdot A \cdot dh} g = 0$$

$$\Rightarrow \underbrace{P(h+dh)}_{\approx P(h) - dP} - P(h) = -\rho g \, dh \,, \quad P = \frac{F}{A}$$

$$\Rightarrow dP = -\rho g \, dh \quad (*)$$

Gas ideal: $PV = nRT$ donde $n = \frac{M}{m}$ ← masa total [kg] / ← masa molecular [kg/mol]

$$\Rightarrow PV = \frac{M}{m} R T$$

$$\Rightarrow P = \underbrace{\left(\frac{M}{V}\right)}_{\rho} \cdot \underbrace{\left(\frac{R}{m}\right)}_{\equiv \bar{R}} \cdot T \quad \Rightarrow \rho = \frac{P}{\bar{R} \cdot T}$$

Entonces, en (*) $\quad dP = -\frac{P}{\bar{R} \cdot T_0} g \, dh$

$$\Rightarrow \frac{dP}{P} = -\frac{g}{\bar{R} T_0} dh$$

$$\Rightarrow \ln P = -\frac{g}{\bar{R} \cdot T_0} \cdot h + c$$

$$\therefore \boxed{P(h) = P_0 \, e^{-\frac{g}{\bar{R} \cdot T_0} h}}$$