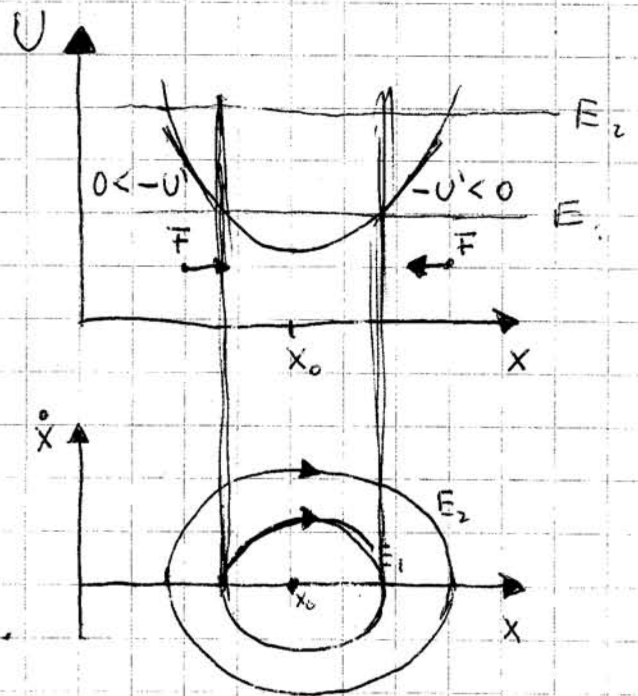
Pequeñas Oscilaciones

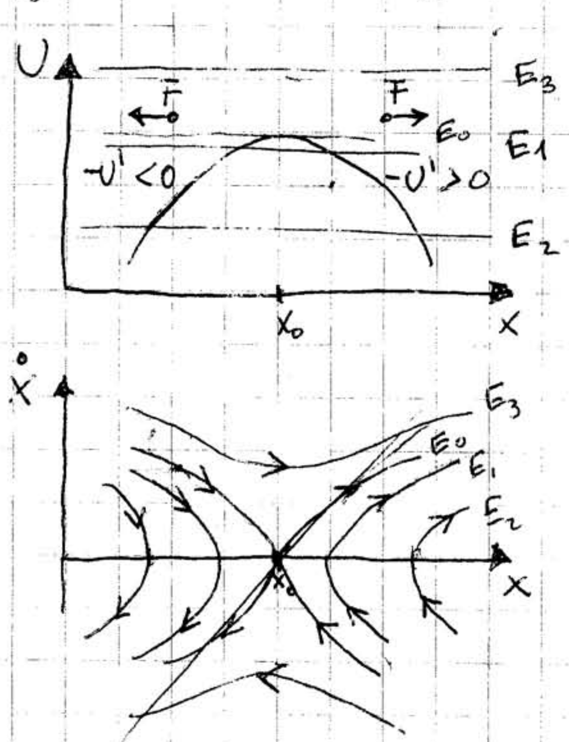


$$F = -\frac{\partial U}{\partial x}$$

Si U mínimo en x_0 $\Rightarrow$ $F = -\frac{dU}{dx}\Big|_{x=x_0} = 0$ y si $x > x_0$

$\Rightarrow$ $F = -\frac{\partial U}{\partial x} < 0$ $\wedge$ si $x < x_0$ $\Rightarrow F = -\frac{\partial U}{\partial x} > 0$

Luego x_0 es un punto de equilibrio estable



Si U máximo en x_0

$\Rightarrow F = -\frac{dU}{dx}\Big|_{x=x_0} = 0 \Rightarrow$ eq inestable!

y si $x > x_0$

$\Rightarrow \overline{F} = -\frac{dU}{dx} > 0$

y si $x < x_0 \Rightarrow \overline{F} = -\frac{dU}{dx} < 0$

25/09/2006

2

$$U(x) = \underbrace{U(x_0)}_{\equiv 0} + \overset{\to 0}{U'(x_0)}(x - x_0) + \frac{1}{2} U''(x_0)(x-x_0)^2 + \dots$$

$$\Rightarrow \quad T = \frac{1}{2} m \dot{x}^2$$

$$\Rightarrow \quad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} U''(x_0)(x - x_0)^2$$

$$k \equiv U''(x_0) \qquad \eta = x - x_0$$

$$\Rightarrow \quad L = \frac{1}{2} m \dot{\eta}^2 - \frac{1}{2} k \eta^2$$

$$E = \frac{1}{2} m \dot{\eta}^2 + \frac{1}{2} k \eta^2$$

$$\Rightarrow \quad \boxed{\ddot{\eta} + \frac{k}{m} \eta = 0}$$

$$\eta = e^{\lambda t}$$

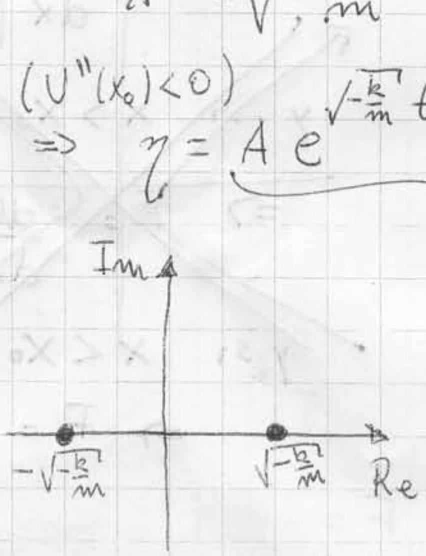
$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda = \pm\sqrt{-\frac{k}{m}}$$

si $k < 0$ $(U''(x_0) < 0)$

$$\Rightarrow \eta = \underbrace{A e^{\sqrt{-\frac{k}{m}}\, t} + B e^{-\sqrt{-\frac{k}{m}}\, t}}\ (*)$$

Espectro



25/09/2006

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Si $k>0$ $\quad (U''(x_0)>0)$ $\quad \lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$

$$\Rightarrow \boxed{\eta = A e^{i\omega t} + B e^{-i\omega t}}$$

$$\eta = (A+B)\cos\omega t + i(A-B)\,\mathrm{sen}\,\omega t$$

$$\boxed{\eta = C_1 \cos\omega t + C_2\,\mathrm{sen}\,\omega t}$$

$$\eta = \sqrt{C_1^2+C_2^2}\left(\frac{C_1}{\sqrt{C_1^2+C_2^2}}\cos\omega t + \frac{C_2}{\sqrt{C_1^2+C_2^2}}\,\mathrm{sen}\,\omega t\right)$$

$$\boxed{A \equiv \sqrt{C_1^2+C_2^2}} \qquad \cos\phi = \frac{C_1}{\sqrt{C_1^2+C_2^2}}\;;\quad \mathrm{sen}\,\phi = \frac{-C_2}{\sqrt{C_1^2+C_2^2}}$$

$$\boxed{\tan\phi = -\frac{C_2}{C_1}}$$

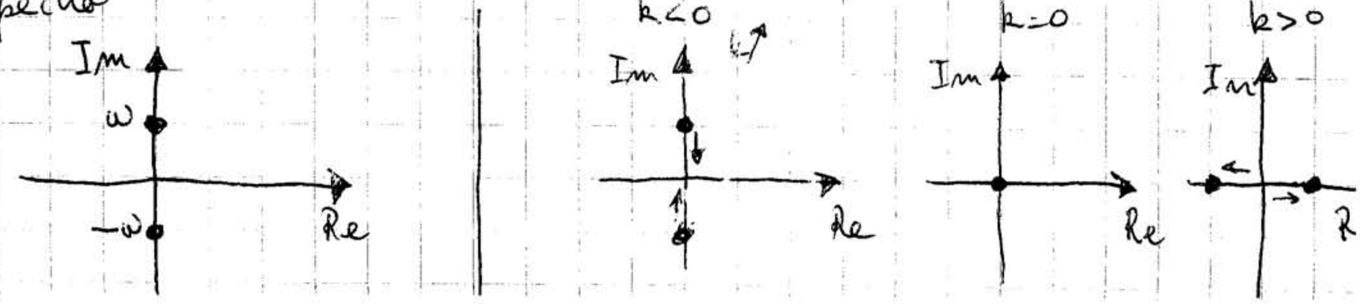
$$\Rightarrow \eta = A(\cos\phi\cos\omega t - \mathrm{sen}\,\phi\,\mathrm{sen}\,\omega t)$$

$$\boxed{\eta = A\cos(\omega t+\phi)}$$

$$E = \frac{1}{2}m\dot{\eta}^2 + \frac{1}{2}k\eta^2 = \frac{1}{2}mA^2\omega^2\,\mathrm{sen}^2(\omega t+\phi) + \frac{1}{2}mA^2\omega^2\cos^2(\omega t+\phi)$$

$$\boxed{E = \frac{1}{2}m\omega^2 a^2} \qquad a \equiv A$$

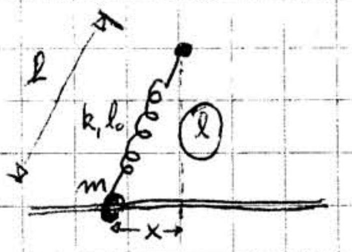
Espectro



25/09/2006

4

Ejemplo:



$$\ell = \sqrt{\ell^2 + x^2} \quad \Rightarrow \quad U = \frac{1}{2} k \left(\sqrt{\ell^2 + x^2} - l_0\right)^2$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$U'(x) = k\left(\sqrt{\ell^2 + x^2} - l_0\right) \cdot \frac{1}{\cancel{2}\sqrt{\ell^2 + x^2}} \cancel{2}x$$

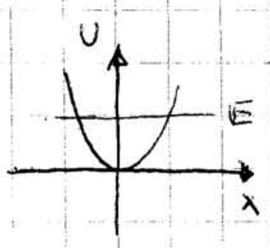
$$U'(x) = kx\left(1 - \frac{l_0}{\sqrt{\ell^2 + x^2}}\right) = 0$$

$$\Rightarrow \quad x_0 = 0 \quad \vee \quad l_0 = \sqrt{\ell^2 + x_0^2} \Rightarrow x_0 = \pm\sqrt{l_0^2 - \ell^2}$$

$\Rightarrow$ 3 casos i) $l_0 < \ell$; ii) $l_0 = \ell$ $\wedge$ iii) $l_0 > \ell$

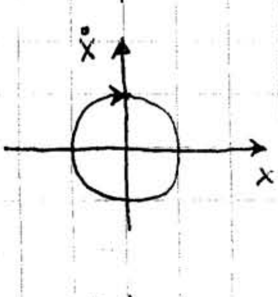
i) $l_0 < \ell$ sólo una solución de equilibrio ($x_0 = 0$)

$$U''(x) = k\left(1 - \frac{l_0}{\sqrt{\ell^2 + x^2}}\right) + kx\left(\frac{x\, l_0}{(\ell^2 + x^2)^{3/2}}\right)$$



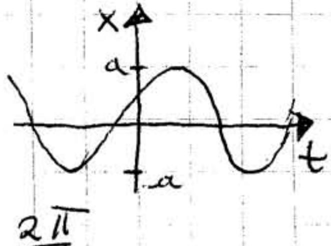
$$\Rightarrow U''(0) = k \; \cancel{x0} \left(1 - \frac{l_0}{\ell}\right) > 0$$

$\Rightarrow$ $x_0 = 0$ punto de equilibrio estable



$$\Rightarrow \quad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k \left(1 - \frac{l_0}{\ell}\right) x^2$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k \left(1 - \frac{l_0}{\ell}\right) x^2$$



$$\Rightarrow \quad \omega^2 = \frac{k(\ell - l_0)}{m\ell}$$

$$\boxed{\omega = \sqrt{\frac{k(\ell - l_0)}{m\ell}}} \quad \wedge \quad T = \frac{2\pi}{\omega}$$

ii) $l = l_0$

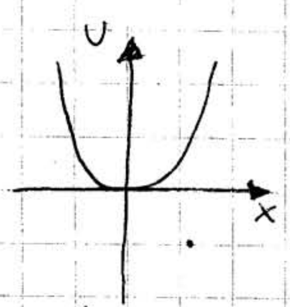
$$\Rightarrow U''(x_0=0) = 0$$

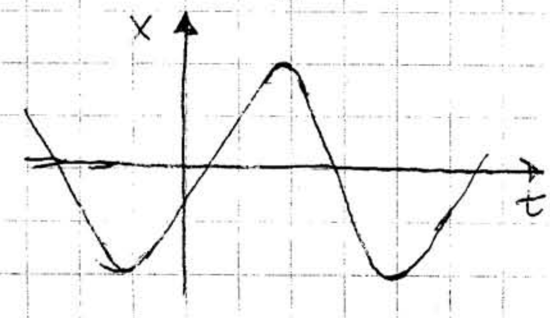
$$U(x) = \frac{1}{2} k \left(\sqrt{l^2+x^2} - l_0\right)^2 \qquad / \ \frac{(\sqrt{l^2+x^2}+l)^2}{(\sqrt{l^2+x^2}+l)^2}$$

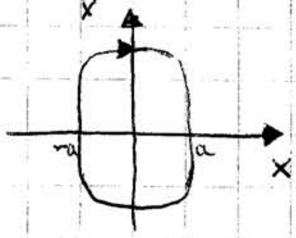
$$= \frac{1}{2} k \frac{(\cancel{l^2}+x^2-\cancel{l^2})^2}{(\sqrt{l^2+x^2}+l)^2}$$

$$= \frac{1}{2} \frac{k\, x^4}{(\sqrt{l^2+x^2}+l)^2}$$

$$\Rightarrow U(x) \simeq \frac{1}{8} \frac{k\, x^4}{l^2}$$







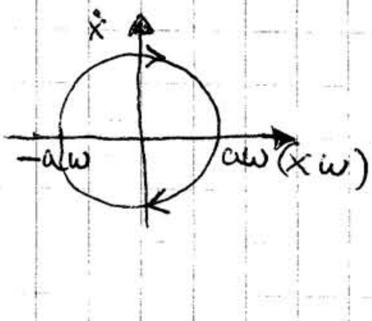
Nota: en el caso i) podemos calcular el periodo fácilmente pues $\omega\tau = 2\pi$ pero hay una forma alternativa

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} m \omega^2 a^2$$

$$\Rightarrow \dot{x}^2 = \omega^2 a^2 - \omega^2 x^2 \qquad / \int_0^{\tau/2} dt$$

$$\Rightarrow \int_{-a}^{a} \frac{dx}{\omega\sqrt{a^2-x^2}} = \int_0^{\tau/2} dt$$

Sea $z = x/a$



$$\Rightarrow \tau = \frac{2}{\omega} \underbrace{\int_{-1}^{1} \frac{dz}{\sqrt{1-z^2}}}_{\pi} = \frac{2\pi}{\omega}$$

Calculemos el período en el caso ii)

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{8} \frac{k}{\ell^2} x^4 = \frac{1}{8} \frac{k}{\ell^2} a^4$$

$$\dot{x}^2 = \frac{1}{4} \frac{k}{m\ell^2} a^4 - \frac{1}{4} \frac{k}{m\ell^2} x^4$$

$$\frac{\dot{x}}{\frac{1}{2}\sqrt{\frac{k}{m\ell^2}}\sqrt{a^4 - x^4}} = 1 \qquad \Big/ \int_0^{\tau/2} dt$$

$$\frac{2}{\sqrt{\frac{k}{m\ell^2}}} \int_{-a}^{a} \frac{dx}{\sqrt{a^4 - x^4}} = \int_0^{\tau/2} dt$$

$z = x/a \qquad dx = a\,dz$

$$\frac{2\ell}{\sqrt{\frac{k}{m}}} \int_{-1}^{1} \frac{a\,dz}{\sqrt{a^4 - a^4 z^4}} = \frac{\tau}{2}$$

$$\Rightarrow \quad \tau = \frac{4\ell}{a} \sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{dz}{\sqrt{1 - z^4}}$$

$$\int_{-1}^{1} \frac{dz}{\sqrt{1-z^4}} = 2 \int_0^1 \frac{dz}{\sqrt{1-z^4}} = \quad \left[\begin{array}{l} \text{sea } t = z^4 \\ dt = 4 z^3 dz \\ \Rightarrow dz = \dfrac{dt}{4 t^{3/4}} \end{array} \right.$$

$$= 2 \int_0^1 \frac{dt}{4 t^{3/4} \sqrt{1-t}}$$

$$= \frac{1}{2} \int_0^1 t^{\frac{1}{4}-1} (1-t)^{\frac{1}{2}-1} dt = \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}$$

25/04/2006

7

$$\tau = \frac{4l}{a}\sqrt{\frac{m}{k}}\cdot\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$\boxed{\tau = \frac{2\sqrt{\pi}}{a}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}\sqrt{\frac{ml^2}{k}}} \simeq \frac{10.4882}{a}\sqrt{\frac{ml^2}{k}}$$

¡Luego el período de oscilación depende de la amplitud de la oscilación!

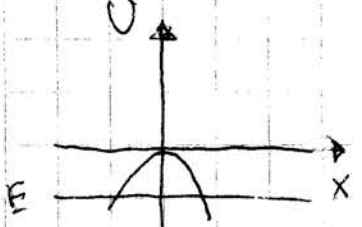
¡más grande la amplitud, más pequeña es el período de oscilación!

iii) $l < l_0$

$$U''(x) = k\left(1 - \frac{l_0}{\sqrt{l^2+x^2}}\right) + \frac{kx^2 l_0}{(l^2+x^2)^{3/2}}$$

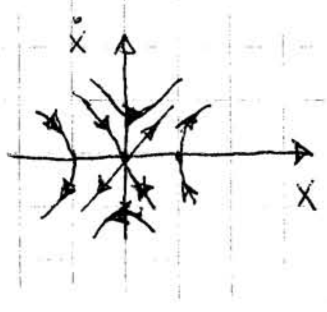
$$\Rightarrow U''(0) = k\left(1 - \frac{l_0}{l}\right) < 0$$

$\Rightarrow x_0 = 0$ punto de equilibrio inestable



$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\left(1 - \frac{l_0}{l}\right)x^2$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k\left(1 - \frac{l_0}{l}\right)x^2$$



$$\lambda^2 = \frac{k(l_0 - l)}{ml}$$

$$x(t) = A e^{\lambda t} + B e^{-\lambda t}$$

25/09/2006

8

En este caso aparecen dos soluciones más

$$x_{1,2} = \pm\sqrt{l_0^2 - l^2}$$

$$x_1 = -\sqrt{l_0^2 - l^2} \quad \wedge \quad x_2 = +\sqrt{l_0^2 - l^2}$$

x_0 es inestable, pero ¿cuál es la naturaleza de estas dos nuevas soluciones?

$$U''(x) = k\left(1 - \frac{l_0}{\sqrt{l^2 + x^2}}\right) + \frac{k\, l_0\, x^2}{(l^2 + x^2)^{3/2}}$$

$$\Rightarrow U''(x_{1,2}) = k\left(1 - \frac{l_0}{\sqrt{\cancel{l^2} + l_0^2 - \cancel{l^2}}}\right) + \frac{k\, l_0\, (l_0^2 - l^2)}{(\cancel{l^2} + l_0^2 - \cancel{l^2})^{3/2}}$$

$$= k(1 - 1) + \frac{k\, l_0\, (l_0^2 - l^2)}{l_0^3}$$

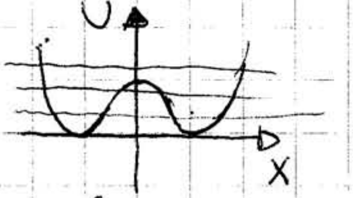
$$U''(x_{1,2}) = k\left(1 - \frac{l^2}{l_0^2}\right) > 0$$

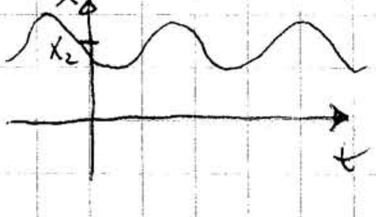
luego $x_{1,2}$ son puntos de equilibrio estables.

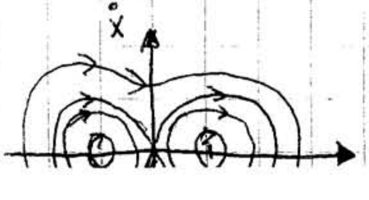
$$\Rightarrow \quad E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k\left(1 - \frac{l^2}{l_0^2}\right) x^2$$

$$\Rightarrow \quad \omega^2 = \frac{k(l_0^2 - l^2)}{m\, l_0^2} \quad \wedge \quad \tau = \frac{2\pi}{\omega}$$

$$\boxed{\omega = \sqrt{\frac{k(l_0^2 - l^2)}{m\, l_0^2}}}$$







25/09/2006

9

Diagrama de Bifurcación