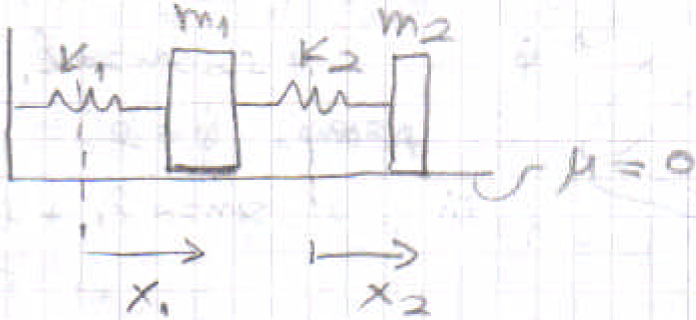
Pauta C2:

P1)



x_1, x_2: medidas desde la posición de equilibrio

i) $L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2$

$$\mathbb{T} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \mathbb{V} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \; (0{,}5)$$

$\det(-\omega^2 \mathbb{T} + \mathbb{V}) = 0 \quad \leftarrow (0{,}3)$

$$\Rightarrow \begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

$$(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2) - k_2^2 = 0$$

$$\omega^4 m_1 m_2 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] + k_2 (k_1 + k_2) - k_2^2 = 0$$

$$\omega^4 m_1 m_2 - \omega^2 [(m_1 + m_2) k_2 + m_2 k_1] + k_1 k_2 = 0$$

$$\omega^2 = \frac{(m_1 + m_2) k_2 + m_2 k_1 \pm \sqrt{[(m_1 + m_2) k_2 + m_2 k_1]^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2} \qquad (1{,}2)$$

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- $(m_1 + m_2)^2 k_2^2 + 2 m_2 k_1 (m_1 + m_2) k_2 + m_2^2 k_1^2 - 4 m_1 m_2 k_1 k_2$

$= (m_1 + m_2)^2 k_2^2 + 2 m_2 k_1 (m_2 - m_1) k_2 + m_2^2 k_1^2$

ii) $k_1 = 2k_2 = 2k \qquad m_1 = m_2 = 2m$

$$\Rightarrow \omega^2 = \frac{4mk + 4mk \pm \sqrt{(4mk+4mk)^2 - 4(4m^2)(2k^2)}}{2(4m^2)}$$

$$= \frac{8mk \pm \sqrt{64m^2k^2 - 32m^2k^2}}{8m^2}$$

$$= \frac{8mk \pm mk\sqrt{32}}{8m^2} = \frac{8k \pm k\,4\sqrt{2}}{8m}$$

$$\boxed{\omega^2 = \frac{k}{m} \pm \frac{k\sqrt{2}}{m\,2}} \; (0,5) \quad \begin{cases} \omega_1^2 = \frac{k}{m}\left(1 + \frac{\sqrt{2}}{2}\right) \\ \omega_2^2 = \frac{k}{m}\left(1 - \frac{\sqrt{2}}{2}\right) \end{cases}$$

Modos normales

ω_1^2)

$$\begin{bmatrix} -2(1+\sqrt{2}/2)k + 3k & -k \\ -k & -2(1+\sqrt{2}/2)k + k \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$k(1-\sqrt{2})a - kb = 0$$

$$b = (1-\sqrt{2})a \Rightarrow \boxed{v_1 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}} \; (0,5)$$

ω_2^2)

$$\begin{bmatrix} -2(1-\sqrt{2}/2)k + 3k & -k \\ -k & -2(1-\sqrt{2}/2)k + k \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$k(1+\sqrt{2})a - kb = 0$$

$$b = a(1+\sqrt{2}) \Rightarrow \boxed{v_2 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}} \; (0,5)$$

Relación de ortonormalización

$$(1 \quad 1-\sqrt{2}) \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \qquad (s \neq l)$$

$$= (1 \quad 1-\sqrt{2}) \begin{bmatrix} 2m \\ 2m(1+\sqrt{2}) \end{bmatrix} = 2m + (1-\sqrt{2})2m(1+\sqrt{2})$$

$$= 2m + 2m\underbrace{(1-2)}_{-1} = 0 \quad (0,5)$$

$(s = l)$

Recordar que $\sum_{\sigma\lambda} \rho_\sigma^{(s)} m_{\sigma\lambda} \rho_\lambda^{(t)} = \delta_{s,t}$

donde $m_{\sigma\lambda} = T_{\sigma\lambda}$ y

$$\vec{V}_1 = \underbrace{\alpha \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}}_{\rho^{(1)}} \qquad \vec{V}_2 = \underbrace{b \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}}_{\rho^{(2)}}$$

$$\Rightarrow \alpha^2 (1 \quad 1-\sqrt{2}) \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

$$= \alpha_1^2 (1 \quad 1-\sqrt{2}) \begin{pmatrix} 2m \\ 2m(1-\sqrt{2}) \end{pmatrix}$$

$$= \alpha^2 (2m + 2m(1-\sqrt{2})^2)$$

$$= \alpha^2 (2m + 2m(1 - 2\sqrt{2} + 2))$$

$$= \alpha^2 (8m - 4\sqrt{2}m)$$

$$= 1 \qquad \text{con} \qquad \alpha^2 = \frac{1}{8m - 4\sqrt{2}m}$$

$$\Rightarrow \beta^2 (1 \quad 1+\sqrt{2}) \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$$= \beta^2 (2m + 2m(1 + 2\sqrt{2} + 2))$$

$$= \beta^2 (8m + 4\sqrt{2}m)$$

$$= 1 \qquad \text{con} \qquad \beta^2 = \frac{1}{8m + 4\sqrt{2}m}$$

(0,5)

iii) <u>Solución general</u>

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{c_1 \alpha}\vec{V}_1 e^{i\omega_1 t} + c_2 \beta \vec{V}_2 e^{i\omega_2 t}$$

$$= \tilde{c}_1 \vec{V}_1 e^{i\omega_1 t} + \tilde{c}_2 \vec{V}_2 e^{i\omega_2 t}$$

$$= \tilde{c}_1 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} e^{i\omega_1 t} + \tilde{c}_2 \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} e^{i\omega_2 t}$$

Tomando la parte real

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tilde{c}_1 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \cos(\omega_1 t + \delta_1) + \tilde{c}_2 \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \cos(\omega_2 t + \delta_2)$$

Condiciones iniciales $(t=0)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ A \end{pmatrix} = \tilde{c}_1 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \cos(\delta_1) + \tilde{c}_2 \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \cos(\delta_2) \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\tilde{c}_1 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \omega_1 \operatorname{sen}(\delta_1) - \tilde{c}_2 \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \omega_2 \operatorname{sen}(\delta_2) \quad \begin{matrix} 3 \\ 4 \end{matrix}$$

$(1) \Rightarrow \quad \tilde{c}_1 \cos\delta_1 = -\tilde{c}_2 \cos\delta_2$

en $(2) \Rightarrow \quad -\tilde{c}_2 \cos\delta_2 (\cancel{1}-\sqrt{2}) + \tilde{c}_2 (\cancel{1}+\sqrt{2}) \cos\delta_2 = A$

$$2\sqrt{2}\, \tilde{c}_2 \cos\delta_2 = A$$

$$\tilde{c}_2 \cos\delta_2 = \frac{A}{2\sqrt{2}}$$

$(3) \Rightarrow \quad \tilde{c}_1 \omega_1 \operatorname{sen}\delta_1 = -\tilde{c}_2 \omega_2 \operatorname{sen}\delta_2 \qquad (5)$

en $(4) \Rightarrow \quad -\tilde{c}_2 \omega_2 \operatorname{sen}\delta_2 (\cancel{1}-\sqrt{2}) + \tilde{c}_2 \omega_2 \operatorname{sen}\delta_2 (\cancel{1}+\sqrt{2}) = 0$

$$2\sqrt{2}\, \omega_2 \tilde{c}_2 \operatorname{sen}\delta_2 = 0$$

$$\Rightarrow \boxed{\delta_2 = 0} \Rightarrow \boxed{\tilde{c}_2 = \frac{A}{2\sqrt{2}}} \quad (1,5)$$

$(5) \Rightarrow \boxed{\delta_1 = 0}$

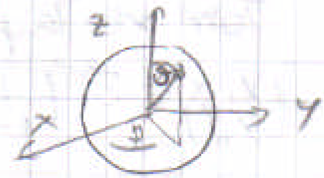
$$\Rightarrow \boxed{\tilde{c}_1 = -\frac{A}{2\sqrt{2}}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{A}{2\sqrt{2}}\left\{ \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}\cos\omega_1 t - \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}\cos\omega_3 t \right\}$$

Para $m_1 \Rightarrow \quad x_1 = \frac{A}{2\sqrt{2}}\left(\cos\omega_2 t - \cos\omega_1 t\right)$

P3)
$$\dot{\vec{r}}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

a) En coordenadas esfericas



$$\dot{\vec{r}}^2 = \dot r^2 + r^2\dot\theta^2 + r^2 \operatorname{sen}^2\theta\,\dot\phi^2$$

$$L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2 + r^2\operatorname{sen}^2\theta\,\dot\phi^2\right) - mgr\cos\theta$$

b) Restriccion $C = r - a = 0$

$$\Rightarrow L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2 + r^2\operatorname{sen}^2\theta\,\dot\phi^2\right) - mgr\cos\theta + \lambda(r-a)$$

$$\frac{\partial L}{\partial\dot\theta} = mr^2\dot\theta \qquad \frac{\partial L}{\partial\theta} = mr^2\operatorname{sen}\theta\cos\theta\,\dot\phi^2 + mg\operatorname{sen}\theta$$

$$\Rightarrow \boxed{mr^2\ddot\theta - mr^2\operatorname{sen}\theta\cos\theta\,\dot\phi^2 - rmg\operatorname{sen}\theta = 0}$$

$$\frac{\partial L}{\partial\dot\phi} = mr^2\operatorname{sen}^2\theta\,\dot\phi \qquad \frac{\partial L}{\partial\phi} = 0$$

$$\Rightarrow mr^2\operatorname{sen}^2\theta\,\dot\phi = cte = P_\phi$$

$$\boxed{\dot\phi = \frac{P_\phi}{mr^2\operatorname{sen}^2\theta}}$$

$$\frac{\partial L}{\partial\dot r} = m\dot r \qquad \frac{\partial L}{\partial r} = mr\dot\theta^2 + mr\operatorname{sen}^2\theta\,\dot\phi^2 - mg\cos\theta + \lambda$$

$$\Rightarrow \boxed{m\ddot r - mr\dot\theta^2 - mr\operatorname{sen}^2\theta\,\dot\phi^2 + mg\cos\theta - \lambda = 0}$$

Imponemos restricción $r = a \quad \dot{r} = 0$

$$-mr\dot{\theta}^2 - mr\,\text{sen}^2\theta\,\dot{\phi}^2 + mg\cos\theta = \lambda$$

$$\lambda = -mr\left[\dot{\theta}^2 + \text{sen}^2\theta\left(\frac{P_\phi}{mr\,\text{sen}^2\theta}\right)^2\right] + mg\cos\theta$$

El multiplicador de Lagrange representa la fuerza de restricción, en este caso la normal.

c) $mr^2\ddot{\theta} - mr^2\,\text{sen}\,\theta\cos\theta\,\dot{\phi}^2 - mgr\,\text{sen}\,\theta = 0$

$$r^2\ddot{\theta} - r^2\,\text{sen}\,\theta\cos\theta\left(\frac{P_\phi}{mr^2\,\text{sen}^2\theta}\right)^2 - gr\,\text{sen}\,\theta = 0$$

$$r^2\ddot{\theta} - \cos\theta\,\frac{P_\phi^2}{m^2r^2\,\text{sen}^3\theta} - rg\,\text{sen}\,\theta = 0$$

$$\ddot{\theta} - \frac{P_\phi^2}{m^2}\,\frac{\cos\theta}{r^4\,\text{sen}^3\theta} - \frac{g}{r}\,\text{sen}\,\theta = 0 \qquad /\dot{\theta}$$

$$\frac{1}{2}\frac{d\dot{\theta}^2}{dt} = \dot{\theta}\ddot{\theta} = \frac{P_\phi^2}{m^2a^4}\,\frac{\cos\theta}{\text{sen}^3\theta}\,\dot{\theta} + \frac{g}{a}\,\text{sen}\,\theta\,\dot{\theta} \qquad /\int$$

$$\frac{1}{2}\dot{\theta}^2 = \underbrace{\int\left(\frac{P_\phi^2}{m^2a^4}\,\frac{\cos\theta\,\dot{\theta}}{\text{sen}^3\theta} + \frac{g}{a}\,\text{sen}\,\theta\,\dot{\theta}\right)dt}_{-\frac{P_\phi^2}{m^2a^4}\frac{1}{2}\frac{1}{\text{sen}^2\theta} - \frac{g}{a}\cos\theta}$$

$$\Rightarrow \quad \dot{\theta}^2 = -\frac{P_\phi^2}{m^2a^4}\,\frac{1}{\text{sen}^2\theta} + \frac{2g}{a}\cos\theta + cte$$

$$E = \frac{1}{2}m\left(a^2\dot{\theta}^2 + a^2\,\text{sen}^2\theta\,\dot{\phi}^2\right) + amg\cos\theta$$

$$\dot{\theta}^2 = \left(E - amg\cos\theta\right)\frac{2}{ma^2} - \frac{a^2\,\text{sen}^2\theta\,\dot{\phi}^2}{a^2}$$

$$= \frac{2E}{ma^2} - \frac{2g\cos\theta}{a} - \text{sen}^2\theta\,\frac{P_\phi^2}{m^2a^4\,\text{sen}^4\theta}$$

$$\Rightarrow cte = \frac{2E}{ma^2} - \cancel{\frac{2g\cos\theta}{a}} - \cancel{\frac{P_\phi^2}{m^2a^4\operatorname{sen}^2\theta}} + \cancel{\frac{P_\phi^2}{m^2a^4} \cdot \frac{1}{\operatorname{sen}^2\theta}} + \cancel{\frac{2g\cos\theta}{a}}$$

$$\boxed{cte = \frac{2E}{ma^2}}$$